Bijective proof of the Postnikov formula

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Postnikov's hook length formula

$$\sum_{\substack{\text{plane binary trees}\\ \text{of order } n}} \prod_{v} \left(1 + \frac{1}{h_v}\right) = (n+1)^{n-1} \frac{2^n}{n!}$$

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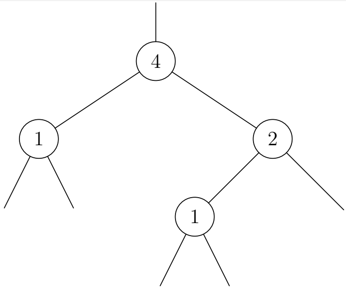

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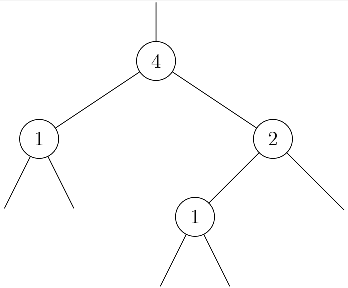

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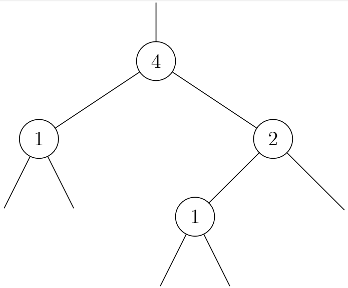

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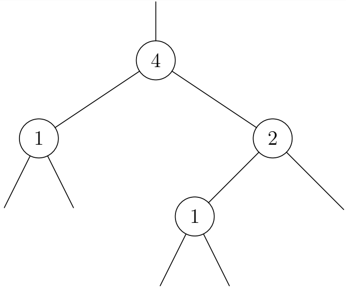

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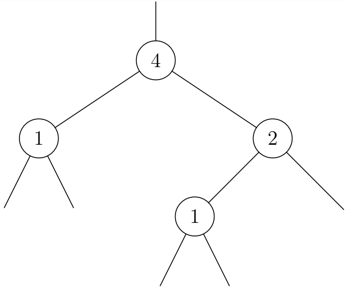

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Hook length formula for trees

$$\sum_{\substack{\text{plane binary trees}\\ \text{of order } n}} \frac{n!}{\prod_{v \text{ vertex}} h_v} \prod_{\substack{v \text{ vertex}\\ \text{but root}}} (1 + h_v) = 2^n (n+1)^{n-2}$$

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- **Oriented trees** with labels on the **edges** (Cayley's formula)

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- Decreasing trees and **permutations** (see next slide)

$$\begin{aligned} \frac{n!}{\prod h_v} &= \#\{\text{decreasing binary trees of order } n\} \\ &= \#\{\text{permutations of } \{1, 2, \ldots, n\}\} \end{aligned}$$

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- Problem: find a combinatorial interpretation of

$$\prod_{\substack{v \text{ vertex}\\ \text{but root}}} (1+h_v)$$

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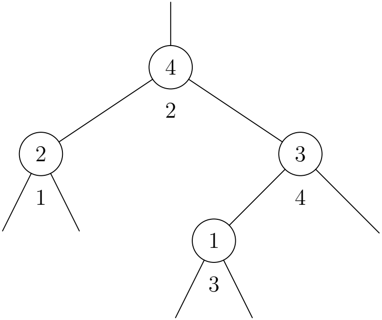
i	1	2	3	4
$\sigma(i)$	2	4	1	3



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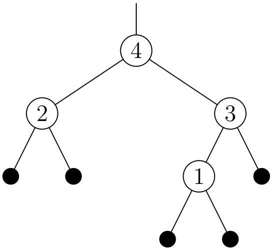

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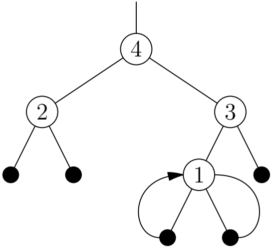

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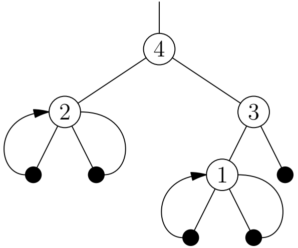

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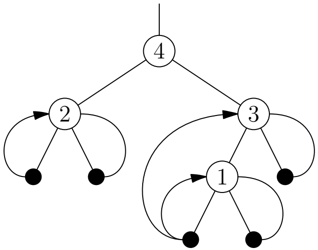

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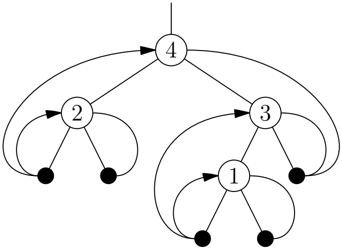

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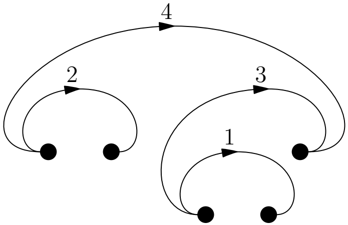
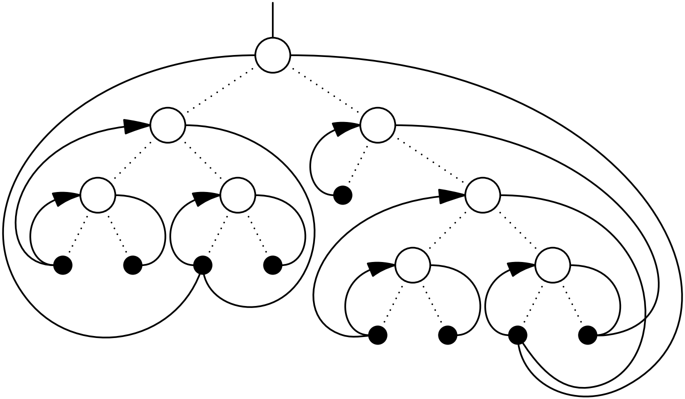


A more complicated example



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permutation

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$\boxed{\begin{array}{c}\text{pair of leaves each in}\\ \text{one of the subtrees}\\ \text{branching at vertex } \sigma(k)\end{array}}$ $\leftrightarrow$

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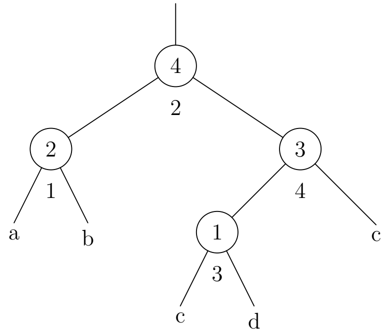

leaves	vertex	interval
(a, d)	4	$\{1, 2, 3\}$
(c, e)	3	$\{3, 4\}$
(c, d)	1	$\{3\}$
(a, e)	4	$\{1, 2, 3, 4\}$

Combinatorial interpretation of the Postnikov hook length formula

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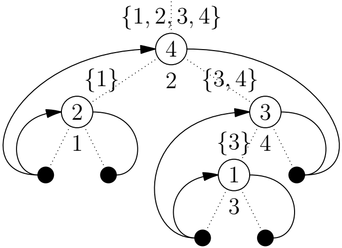

Generating function of connected edge labeled oriented graphs

$$F(s,x,\lambda) = \sum_{\substack{\gamma \text{ connected oriented graph} \\ \text{with labeled edges}}} \frac{s^{e(\gamma)} x^{v(\gamma)} \lambda^{e(\gamma)-v(\gamma)+1}}{e(\gamma)!}$$

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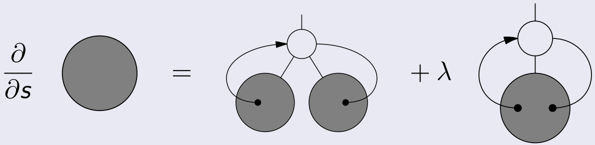
Addition of an edge

Differential equation

$$\frac{\partial F}{\partial s} = \left(x\frac{\partial F}{\partial x}\right)^2 + \lambda x \frac{\partial}{\partial x}\left(x\frac{\partial F}{\partial x}\right)$$

Graphical intepretation



A generalized hook length formula counting connected graphs

$$\sum_{\substack{\text{plane unary-binary trees}\\ \text{of order } n}} \frac{n!}{\prod_{v \text{ vertex}} (k_v + l_v)} \prod_{\substack{v \text{ bivalent vertex}\\ \text{but root}}} (1 + k_v) \prod_{v \text{ univalent vertex}} (1 + k_v)^2$$

$$= \#\Big\{ \text{ connected and oriented graphs with } n \text{ labeled edges} \Big\}$$

with
$$\begin{array}{rcl} k_v & = & \#\{\text{bivalent vertices below } v \text{ (included)}\} \\ l_v & = & \#\{\text{univalent vertices below } v \text{ (included)}\} \end{array}$$

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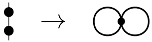

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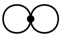
 $\rightarrow$ 

2  $\rightarrow$ 4 

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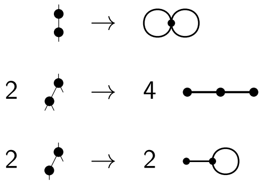

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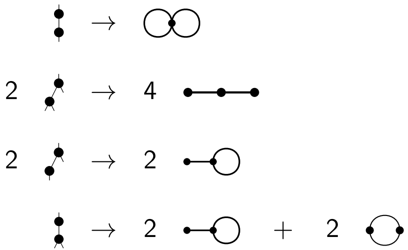

Iterative construction of the graph

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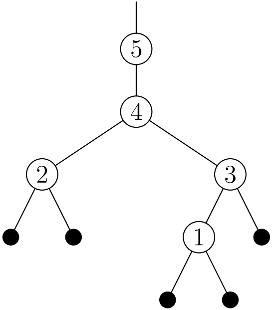

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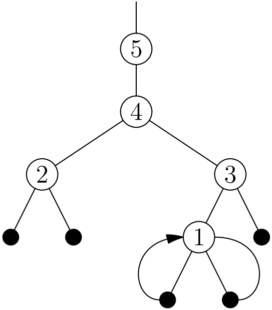

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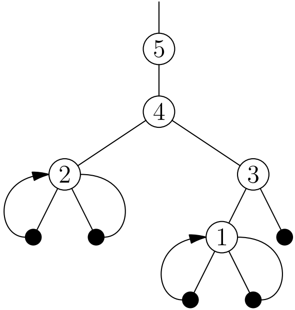

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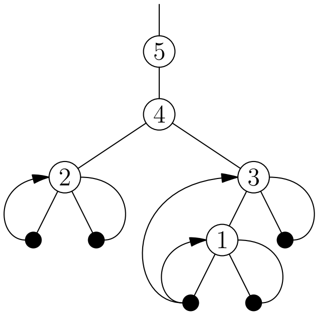

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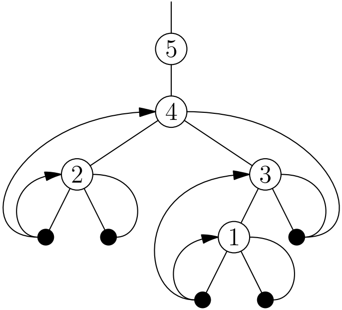
  $$\to \prod_{\text{univalent}} (1 + k_v)^2$$



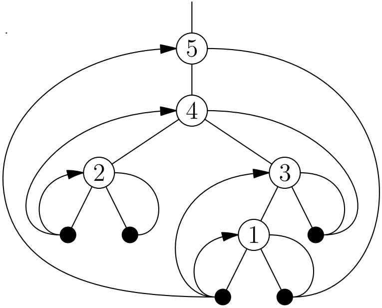
Iterative construction of the graph

- edge attached to different connected components

  $$\rightarrow \prod_{\text{bivalent except root}} (1 + k_v)$$

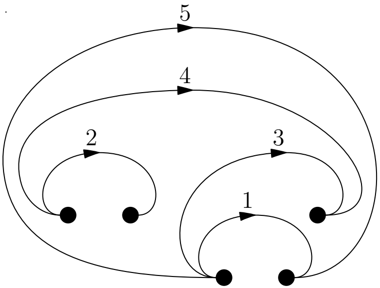
- edge attached to the same connected component

  $$\rightarrow \prod_{\text{univalent}} (1 + k_v)^2$$



Iterative construction of the graph

- edge attached to different connected components
  $\rightarrow \prod_{\text{bivalent except root}} (1 + k_v)$
- edge attached to the same connected component
  $\rightarrow \prod_{\text{univalent}} (1 + k_v)^2$



A more complicated example