Combinatorial Reciprocity for Monotone Triangles

Lukas Riegler
(joint work with Ilse Fischer)

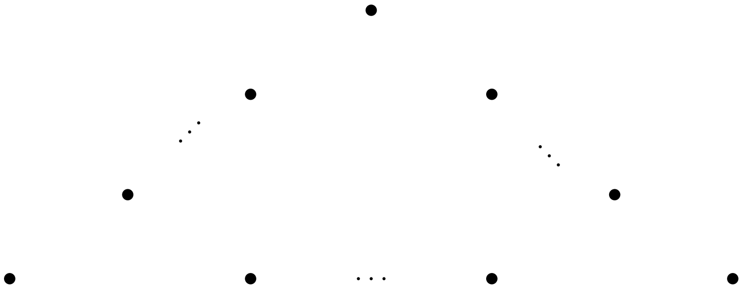
March 26, 2012

Monotone Triangles

Definition (Monotone Triangle)

Triangular array of integers with

- weak increase along North-East diagonals and South-East diagonals
- strict increase along rows

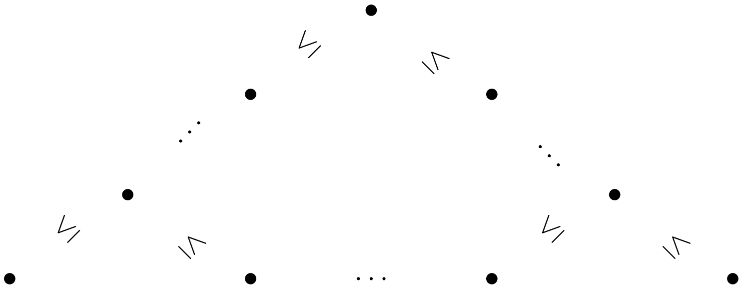


Monotone Triangles

Definition (Monotone Triangle)

Triangular array of integers with

- weak increase along North-East diagonals and South-East diagonals
- strict increase along rows

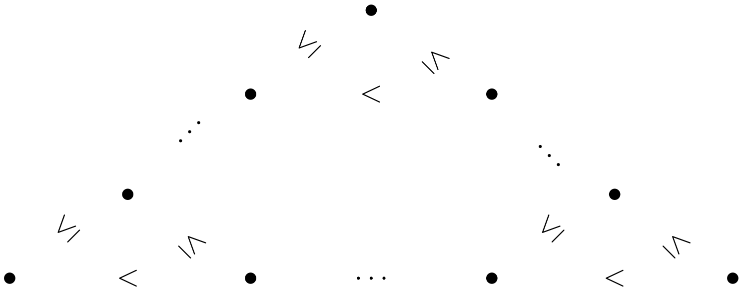


Monotone Triangles

Definition (Monotone Triangle)

Triangular array of integers with

- weak increase along North-East diagonals and South-East diagonals
- strict increase along rows



Monotone Triangles

Example (The seven MTs with bottom row $(1, 2, 3)$)

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 2 & \\ 1 & & 2 & & 3 \end{array} \qquad \begin{array}{ccccc} & & 2 & & \\ & 1 & & 2 & \\ 1 & & 2 & & 3 \end{array} \qquad \begin{array}{ccccc} & & 1 & & \\ & 1 & & 3 & \\ 1 & & 2 & & 3 \end{array}$$

$$\begin{array}{ccccc} & & 2 & & \\ & 1 & & 3 & \\ 1 & & 2 & & 3 \end{array} \qquad \begin{array}{ccccc} & & 3 & & \\ & 1 & & 3 & \\ 1 & & 2 & & 3 \end{array} \qquad \begin{array}{ccccc} & & 2 & & \\ & 2 & & 3 & \\ 1 & & 2 & & 3 \end{array}$$

$$\begin{array}{ccccc} & & 3 & & \\ & 2 & & 3 & \\ 1 & & 2 & & 3 \end{array}$$

MTs with bottom row $k_1 < k_2 < \cdots < k_n$

How many MTs with bottom row $(k_1, k_2, \ldots, k_n)$ are there?

Example

$n = 2$: # MTs with bottom row (k_1, k_2)

$$\begin{array}{ccc} & ? & \\ k_1 & & k_2 \end{array} \quad \rightarrow \quad k_2 - k_1 + 1 \text{ possibilities}$$

Theorem (I. Fischer (2005))

For each $n \geq 1$, there exists a polynomial $\alpha(n; k_1, k_2, \ldots, k_n)$ of degree $n-1$ in each of the n variables satisfying

$$\alpha(n; k_1, k_2, \ldots, k_n) = \#\text{MTs with bottom row } (k_1, k_2, \ldots, k_n),$$

whenever $k_1 < k_2 < \cdots < k_n$.

MTs with bottom row $k_1 < k_2 < \cdots < k_n$

How many MTs with bottom row $(k_1, k_2, \ldots, k_n)$ are there?

Example

$n = 2$: # MTs with bottom row (k_1, k_2)

$$\begin{array}{ccc} & ? & \\ k_1 & & k_2 \end{array} \quad \rightarrow \quad k_2 - k_1 + 1 \text{ possibilities}$$

Theorem (I. Fischer (2005))

For each $n \geq 1$, there exists a polynomial $\alpha(n; k_1, k_2, \ldots, k_n)$ of degree $n-1$ in each of the n variables satisfying

$$\alpha(n; k_1, k_2, \ldots, k_n) = \#MTs \text{ with bottom row } (k_1, k_2, \ldots, k_n),$$

whenever $k_1 < k_2 < \cdots < k_n$.

MTs with bottom row $k_1 < k_2 < \cdots < k_n$

How many MTs with bottom row $(k_1, k_2, \ldots, k_n)$ are there?

Example

$n = 2$: # MTs with bottom row (k_1, k_2)

$$\begin{array}{ccc} & ? & \\ k_1 & & k_2 \end{array} \quad \rightarrow \quad k_2 - k_1 + 1 \text{ possibilities}$$

Theorem (I. Fischer (2005))

For each $n \geq 1$, there exists a polynomial $\alpha(n; k_1, k_2, \ldots, k_n)$ of degree $n-1$ in each of the n variables satisfying

$$\alpha(n; k_1, k_2, \ldots, k_n) = \#MTs \text{ with bottom row } (k_1, k_2, \ldots, k_n),$$

whenever $k_1 < k_2 < \cdots < k_n$.

MTs with bottom row $k_1 < k_2 < \cdots < k_n$

How many MTs with bottom row $(k_1, k_2, \ldots, k_n)$ are there?

Example

$n = 2$: # MTs with bottom row (k_1, k_2)

$$\begin{array}{ccc} & ? & \\ k_1 & & k_2 \end{array} \quad \rightarrow \quad k_2 - k_1 + 1 \text{ possibilities}$$

Theorem (I. Fischer (2005))

For each $n \geq 1$, there exists a polynomial $\alpha(n; k_1, k_2, \ldots, k_n)$ of degree $n-1$ in each of the n variables satisfying

$$\alpha(n; k_1, k_2, \ldots, k_n) = \#\text{MTs with bottom row } (k_1, k_2, \ldots, k_n),$$

whenever $k_1 < k_2 < \cdots < k_n$.

MTs with bottom row $k_1 < k_2 < \cdots < k_n$

How many MTs with bottom row $(k_1, k_2, \ldots, k_n)$ are there?

Example

$n = 2$: # MTs with bottom row (k_1, k_2)

$$\begin{array}{ccc} & ? & \\ k_1 & & k_2 \end{array} \quad \rightarrow \quad k_2 - k_1 + 1 \text{ possibilities}$$

Theorem (I. Fischer (2005))

For each $n \geq 1$, there exists a polynomial $\alpha(n; k_1, k_2, \ldots, k_n)$ of degree $n-1$ in each of the n variables satisfying

$$\alpha(n; k_1, k_2, \ldots, k_n) = \#\textit{MTs with bottom row } (k_1, k_2, \ldots, k_n),$$

whenever $k_1 < k_2 < \cdots < k_n$.

What does $\alpha(n; k_1, k_2, \ldots, k_n)$ count for $k_1 \geq k_2 \geq \cdots \geq k_n$?

Definition (Decreasing Monotone Triangle)

Triangular array of integers with

- weak decrease along NE- and SE-diagonals
- each row contains an entry at most twice
- two consecutive rows do not contain the same entry exactly once

Decreasing Monotone Triangles

What does $\alpha(n; k_1, k_2, \ldots, k_n)$ count for $k_1 \geq k_2 \geq \cdots \geq k_n$?

Definition (Decreasing Monotone Triangle)

Triangular array of integers with

- weak decrease along NE- and SE-diagonals
- each row contains an entry at most twice
- two consecutive rows do not contain the same entry exactly once

Decreasing Monotone Triangles

What does $\alpha(n; k_1, k_2, \ldots, k_n)$ count for $k_1 \geq k_2 \geq \cdots \geq k_n$?

Definition (Decreasing Monotone Triangle)

Triangular array of integers with

- weak decrease along NE- and SE-diagonals
- each row contains an entry at most twice
- two consecutive rows do not contain the same entry exactly once

Decreasing Monotone Triangles

What does $\alpha(n; k_1, k_2, \dots, k_n)$ count for $k_1 \geq k_2 \geq \cdots \geq k_n$?

Definition (Decreasing Monotone Triangle)

Triangular array of integers with

- weak decrease along NE- and SE-diagonals
- each row contains an entry at most twice
- two consecutive rows do not contain the same entry exactly once

Decreasing Monotone Triangles

What does $\alpha(n; k_1, k_2, \dots, k_n)$ count for $k_1 \geq k_2 \geq \cdots \geq k_n$?

Definition (Decreasing Monotone Triangle)

Triangular array of integers with

- weak decrease along NE- and SE-diagonals
- each row contains an entry at most twice
- two consecutive rows do not contain the same entry exactly once

Example (The five DMTs with bottom row $(6, 3, 3, 2, 1)$)

$$\begin{array}{ccccccccc}
 & & & & 2 & & & & \\
 & & & 2 & & 2 & & & \\
 & & 3 & & 2 & & 2 & & \\
 & 3 & & 3 & & 2 & & 2 & \\
6 & & 3 & & 3 & & 2 & & 1
\end{array}
\qquad
\begin{array}{ccccccccc}
 & & & & 3 & & & & \\
 & & & 3 & & 3 & & & \\
 & & 3 & & 3 & & 2 & & \\
 & 3 & & 3 & & 2 & & 2 & \\
6 & & 3 & & 3 & & 2 & & 1
\end{array}$$

$$\begin{array}{ccccccccc}
 & & & & 3 & & & & \\
 & & & 3 & & 3 & & & \\
 & & 3 & & 3 & & 2 & & \\
 & 4 & & 3 & & 2 & & 2 & \\
6 & & 3 & & 3 & & 2 & & 1
\end{array}
\qquad
\begin{array}{ccccccccc}
 & & & & 2 & & & & \\
 & & & 2 & & 2 & & & \\
 & & 4 & & 2 & & 2 & & \\
 & 5 & & 3 & & 2 & & 2 & \\
6 & & 3 & & 3 & & 2 & & 1
\end{array}$$

$$\begin{array}{ccccccccc}
 & & & & 3 & & & & \\
 & & & 3 & & 3 & & & \\
 & & 3 & & 3 & & 2 & & \\
 & 5 & & 3 & & 2 & & 2 & \\
6 & & 3 & & 3 & & 2 & & 1
\end{array}$$

Definition

Two consecutive equal entries (x, x) in a row are called pair.

A duplicate-descendant is a pair (x, x), which is either

- in the bottom row, or
- the row below contains the same pair (x, x).

Theorem 1 (I. Fischer, L. Riegler (2011))

Let $k_1 \geq k_2 \geq \cdots \geq k_n$ and $\mathcal{D}_n(k_1, \ldots, k_n)$ denote the set of DMTs with bottom row $(k_1, \ldots, k_n)$. Then

$$\alpha(n; k_1, \ldots, k_n) = (-1)^{\binom{n}{2}} \sum_{A \in \mathcal{D}_n(k_1, \ldots, k_n)} (-1)^{\mathrm{dd}(A)},$$

where $\mathrm{dd}(A)$ is the number of duplicate-descendants in A.

Definition

Two consecutive equal entries (x, x) in a row are called pair.

A duplicate-descendant is a pair (x, x), which is either

- in the bottom row, or
- the row below contains the same pair (x, x).

Theorem 1 (I. Fischer, L. Riegler (2011))

Let $k_1 \geq k_2 \geq \cdots \geq k_n$ and $\mathcal{D}_n(k_1, \ldots, k_n)$ denote the set of DMTs with bottom row $(k_1, \ldots, k_n)$. Then

$$\alpha(n; k_1, \ldots, k_n) = (-1)^{\binom{n}{2}} \sum_{A \in \mathcal{D}_n(k_1, \ldots, k_n)} (-1)^{\mathrm{dd}(A)},$$

where $\mathrm{dd}(A)$ *is the number of duplicate-descendants in A.*

Definition

Two consecutive equal entries (x, x) in a row are called pair.

A duplicate-descendant is a pair (x, x), which is either

- in the bottom row, or
- the row below contains the same pair (x, x).

Theorem 1 (I. Fischer, L. Riegler (2011))

Let $k_1 \geq k_2 \geq \cdots \geq k_n$ and $\mathcal{D}_n(k_1, \ldots, k_n)$ denote the set of DMTs with bottom row $(k_1, \ldots, k_n)$. Then

$$\alpha(n; k_1, \ldots, k_n) = (-1)^{\binom{n}{2}} \sum_{A \in \mathcal{D}_n(k_1, \ldots, k_n)} (-1)^{\mathrm{dd}(A)},$$

where $\mathrm{dd}(A)$ is the number of duplicate-descendants in A.

Definition

Two consecutive equal entries (x, x) in a row are called pair.

A duplicate-descendant is a pair (x, x), which is either

- in the bottom row, or
- the row below contains the same pair (x, x).

Theorem 1 (I. Fischer, L. Riegler (2011))

Let $k_1 \geq k_2 \geq \cdots \geq k_n$ and $\mathcal{D}_n(k_1, \ldots, k_n)$ denote the set of DMTs with bottom row $(k_1, \ldots, k_n)$. Then

$$\alpha(n; k_1, \ldots, k_n) = (-1)^{\binom{n}{2}} \sum_{A \in \mathcal{D}_n(k_1, \ldots, k_n)} (-1)^{\mathrm{dd}(A)},$$

where $\mathrm{dd}(A)$ *is the number of duplicate-descendants in A.*

Example ($\mathcal{D}_5(6,3,3,2,1)$)

$$\begin{array}{ccccccccc} & & & & 2 & & & & \\ & & & \mathbf{2} & & \mathbf{2} & & & \\ & & 3 & & \mathbf{2} & & \mathbf{2} & & \\ & \mathbf{3} & & \mathbf{3} & & 2 & & 2 & \\ 6 & & \mathbf{3} & & \mathbf{3} & & 2 & & 1 \end{array}$$

$$\begin{array}{ccccccccc} & & & & 3 & & & & \\ & & & \mathbf{3} & & \mathbf{3} & & & \\ & & \mathbf{3} & & \mathbf{3} & & 2 & & \\ & \mathbf{3} & & \mathbf{3} & & 2 & & 2 & \\ 6 & & \mathbf{3} & & \mathbf{3} & & 2 & & 1 \end{array}$$

$$\begin{array}{ccccccccc} & & & & 3 & & & & \\ & & & \mathbf{3} & & \mathbf{3} & & & \\ & & 3 & & 3 & & 2 & & \\ & 4 & & 3 & & 2 & & 2 & \\ 6 & & \mathbf{3} & & \mathbf{3} & & 2 & & 1 \end{array}$$

$$\begin{array}{ccccccccc} & & & & 2 & & & & \\ & & & \mathbf{2} & & \mathbf{2} & & & \\ & & 4 & & \mathbf{2} & & \mathbf{2} & & \\ & 5 & & 3 & & 2 & & 2 & \\ 6 & & \mathbf{3} & & \mathbf{3} & & 2 & & 1 \end{array}$$

$$\begin{array}{ccccccccc} & & & & 3 & & & & \\ & & & \mathbf{3} & & \mathbf{3} & & & \\ & & 3 & & 3 & & 2 & & \\ & 5 & & 3 & & 2 & & 2 & \\ 6 & & \mathbf{3} & & \mathbf{3} & & 2 & & 1 \end{array}$$

$$\alpha(5;6,3,3,2,1) = (-1)^{\binom{5}{2}} \sum_{A \in \mathcal{D}_5(6,3,3,2,1)} (-1)^{\mathrm{dd}(A)} = 3$$

Some motivation. . .

$$\alpha(2n; n, n, n-1, n-1, \ldots, 1, 1) = ?$$

$$\begin{array}{ll} n = 1: & 1 \\ n = 2: & 2 \\ n = 3: & 7 \\ n = 4: & 42 \\ n = 5: & 429 \end{array}$$

Number of Alternating Sign Matrices of size n

Some motivation. . .

$$\alpha(2n; n, n, n-1, n-1, \ldots, 1, 1) = \,?$$

$$\begin{array}{lr} n = 1: & 1 \\ n = 2: & 2 \\ n = 3: & 7 \\ n = 4: & 42 \\ n = 5: & 429 \end{array}$$

Number of Alternating Sign Matrices of size n

Some motivation...

$$\alpha(2n; n, n, n-1, n-1, \ldots, 1, 1) = ?$$

$$\begin{aligned} n = 1 &: \quad 1 \\ n = 2 &: \quad 2 \\ n = 3 &: \quad 7 \\ n = 4 &: \quad 42 \\ n = 5 &: \quad 429 \end{aligned}$$

Number of Alternating Sign Matrices of size n

Some motivation...

$$\alpha(2n; n, n, n-1, n-1, \ldots, 1, 1) = \,?$$

$$\begin{aligned} n = 1 : &\quad 1 \\ n = 2 : &\quad 2 \\ n = 3 : &\quad 7 \\ n = 4 : &\quad 42 \\ n = 5 : &\quad 429 \end{aligned}$$

Number of Alternating Sign Matrices of size n

Some motivation...

$$\alpha(2n; n, n, n-1, n-1, \ldots, 1, 1) = ?$$

$$\begin{aligned} n = 1 &: \quad 1 \\ n = 2 &: \quad 2 \\ n = 3 &: \quad 7 \\ n = 4 &: \quad 42 \\ n = 5 &: \quad 429 \end{aligned}$$

Number of Alternating Sign Matrices of size n

Some motivation...

$$\alpha(2n; n, n, n-1, n-1, \ldots, 1, 1) = ?$$

$$\begin{aligned} n = 1 &: \quad 1 \\ n = 2 &: \quad 2 \\ n = 3 &: \quad 7 \\ n = 4 &: \quad 42 \\ n = 5 &: \quad 429 \end{aligned}$$

Number of Alternating Sign Matrices of size n

Some motivation...

$$\alpha(2n; n, n, n-1, n-1, \ldots, 1, 1) = ?$$

$$\begin{aligned} n = 1: &\quad 1 \\ n = 2: &\quad 2 \\ n = 3: &\quad 7 \\ n = 4: &\quad 42 \\ n = 5: &\quad 429 \end{aligned}$$

Number of Alternating Sign Matrices of size n

Alternating Sign Matrices

Definition (Alternating Sign Matrix of size n)

- $(n \times n)$-matrix
- entries in $\{0, 1, -1\}$
- in each row/column: non-zero entries alternate in sign and sum up to 1

Example (The seven ASMs of size 3)

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Alternating Sign Matrices

Definition (Alternating Sign Matrix of size n)

- $(n \times n)$-matrix
- entries in $\{0, 1, -1\}$
- in each row/column: non-zero entries alternate in sign and sum up to 1

Example (The seven ASMs of size 3)

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Alternating Sign Matrices

Definition (Alternating Sign Matrix of size n)

- $(n \times n)$-matrix
- entries in $\{0, 1, -1\}$
- in each row/column: non-zero entries alternate in sign and sum up to 1

Example (The seven ASMs of size 3)

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Alternating Sign Matrices

Definition (Alternating Sign Matrix of size n)

- $(n \times n)$-matrix
- entries in $\{0, 1, -1\}$
- in each row/column: non-zero entries alternate in sign and sum up to 1

Example (The seven ASMs of size 3)

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Bijection: ASMs of size n $\Leftrightarrow$ MTs with bottom row $(1, 2, \ldots, n)$

Example

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \Longleftrightarrow \begin{array}{ccccccccc} & & & & 2 & & & & \\ & & & 1 & & 4 & & & \\ & & 1 & & 3 & & 5 & & \\ & 1 & & 2 & & 4 & & 5 & \\ 1 & & 2 & & 3 & & 4 & & 5 \end{array}$$

Connection between ASMs and MTs

Bijection: ASMs of size $n \Leftrightarrow$ MTs with bottom row $(1, 2, \ldots, n)$

Example

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \Longleftrightarrow \begin{array}{ccccccccc} & & & & 2 & & & & \\ & & & 1 & & 4 & & & \\ & & 1 & & 3 & & 5 & & \\ & 1 & & 2 & & 4 & & 5 & \\ 1 & & 2 & & 3 & & 4 & & 5 \end{array}$$

Some motivation...

$$\alpha(2n; n, n, n-1, n-1, \ldots, 1, 1) = \,?$$

$$\begin{aligned} n = 1 &: \quad 1 \\ n = 2 &: \quad 2 \\ n = 3 &: \quad 7 \\ n = 4 &: \quad 42 \\ n = 5 &: \quad 429 \end{aligned}$$

$$\boxed{\alpha(2n; n, n, n-1, n-1, \ldots, 1, 1) = \alpha(n; 1, 2, \ldots, n)}$$

Towards a bijective proof

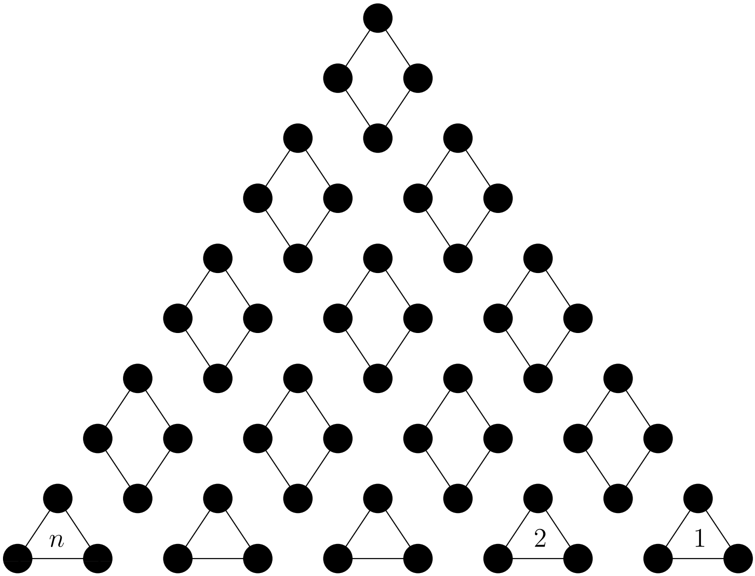
$$\alpha(2n; n, n, n-1, n-1, \ldots, 1, 1) \overset{\text{Th.1}}{=} (-1)^{\binom{2n}{2}} \sum_{A \in \mathcal{D}_{2n}(n,n,n-1,n-1,\ldots,1,1)} (-1)^{\mathrm{dd}(A)} \overset{!}{=} \alpha(n; 1, 2, \ldots, n)$$

$\rightarrow$ find suitable partition of $\mathcal{D}_{2n}(n, n, n-1, n-1, \ldots, 1, 1)$

Towards a bijective proof

$$\alpha(2n; n, n, n-1, n-1, \dots, 1, 1) \stackrel{\text{Th.1}}{=} (-1)^{\binom{2n}{2}} \sum_{A \in \mathcal{D}_{2n}(n,n,n-1,n-1,\dots,1,1)} (-1)^{\mathrm{dd}(A)} \stackrel{!}{=} \alpha(n; 1, 2, \dots, n)$$

$\rightarrow$ find suitable partition of $\mathcal{D}_{2n}(n, n, n-1, n-1, \dots, 1, 1)$



n
2
1

Example

$$\begin{array}{ccccccccccc}
 & & & & & 2 & & & & & \\
 & & & & 2 & & 2 & & & & \\
 & & & 3 & & 2 & & 1 & & & \\
 & & 3 & & 3 & & 1 & & 1 & & \\
 & 3 & & 3 & & 2 & & 1 & & 1 & \\
3 & & 3 & & 2 & & 2 & & 1 & & 1
\end{array}
\quad \Leftrightarrow \quad
\begin{array}{ccccc}
 & & 2 & & \\
 & 1 & & 3 & \\
1 & & 2 & & 3
\end{array}$$

Open problem:

Sign-reversing involution on the remaining set of DMTs?

Example

$$\begin{array}{ccccccccccc}
 & & & & & 2 & & & & & \\
 & & & & 2 & & 2 & & & & \\
 & & & 3 & & 2 & & 1 & & & \\
 & & 3 & & 3 & & 1 & & 1 & & \\
 & 3 & & 3 & & 2 & & 1 & & 1 & \\
3 & & 3 & & 2 & & 2 & & 1 & & 1
\end{array}
\quad \Leftrightarrow \quad
\begin{array}{ccccc}
 & & 2 & & \\
 & 1 & & 3 & \\
1 & & 2 & & 3
\end{array}$$

Open problem:

Sign-reversing involution on the remaining set of DMTs?

Overview of involved combinatorial objects

$\boxed{\text{Monotone Triangles with bottom row } (1, 2, \ldots, n)}$

$\Updownarrow$

$\boxed{(n \times n)\text{-ASMs}}$

$\boxed{\text{DMTs with bottom row } (n, n, n-1, n-1, \ldots, 1, 1)}$

$\Updownarrow$

?

Alternating Sign Matrices

Definition (Alternating Sign Matrix of size n)

- $(n \times n)$-matrix
- entries in $\{0, 1, -1\}$
- in each row/column: non-zero entries alternate in sign and sum up to 1

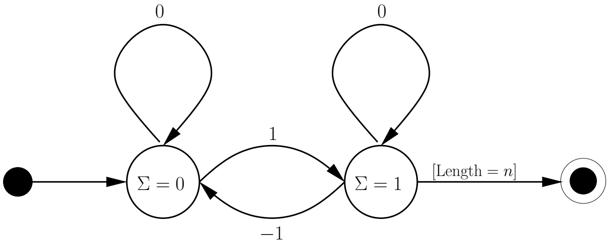


Figure: Machine generating rows and columns of ASMs

2-ASMs

Definition (2-ASM of size n)

- $(2n) \times n$-matrix
- rows generated by ASM-machine
- columns generated by

2-ASMs

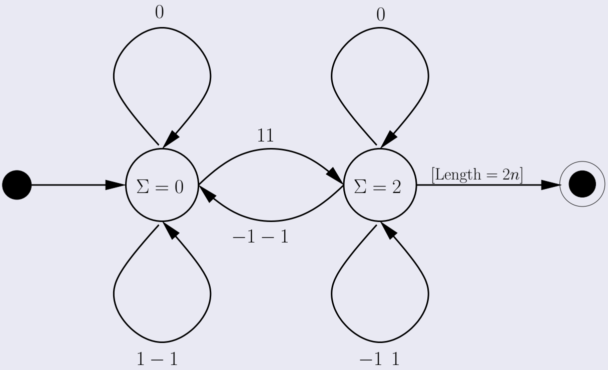
Definition (2-ASM of size n)

- $(2n) \times n$-matrix
- rows generated by ASM-machine
- columns generated by

2-ASMs

Definition (2-ASM of size n)

- $(2n) \times n$-matrix
- rows generated by ASM-machine
- columns generated by



Example (DMT $\Leftrightarrow$ 2-ASM)

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 2 & & & & & & & \\
 & & & & & & 2 & & 2 & & & & & & \\
 & & & & & 3 & & 2 & & 2 & & & & & \\
 & & & & 4 & & 2 & & 2 & & 1 & & & & \\
 & & & 4 & & 4 & & 2 & & 1 & & 1 & & & \\
 & & 4 & & 4 & & 2 & & 2 & & 1 & & 1 & & \\
 & 4 & & 4 & & 3 & & 2 & & 2 & & 1 & & 1 & \\
4 & & 4 & & 3 & & 3 & & 2 & & 2 & & 1 & & 1
\end{array} \Leftrightarrow$$

$$\Leftrightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Theorem

The set $\mathcal{D}_{2n}(n, n, n-1, n-1, \ldots, 1, 1)$ is in bijection with the set of 2-ASMs of size n.

$\boxed{\text{Monotone Triangles with bottom row } (1, 2, \ldots, n)}$

$\Updownarrow$

$\boxed{(n \times n)\text{-ASMs}}$

$\boxed{\text{DMTs with bottom row } (n, n, n-1, n-1, \ldots, 1, 1)}$

$\Updownarrow$

$\boxed{\text{2-ASMs of size } n}$

Theorem

The set $\mathcal{D}_{2n}(n, n, n-1, n-1, \dots, 1, 1)$ is in bijection with the set of 2-ASMs of size n.

$$\boxed{\text{Monotone Triangles with bottom row } (1, 2, \dots, n)}$$

$$\Updownarrow$$

$$\boxed{(n \times n)\text{-ASMs}}$$

$$\boxed{\text{DMTs with bottom row } (n, n, n-1, n-1, \dots, 1, 1)}$$

$$\Updownarrow$$

$$\boxed{\text{2-ASMs of size } n}$$

Theorem 2 (I. Fischer, L. Riegler (2011))

Let $A_{n,i}$ denote the number of ASMs with the first row's unique 1 in column i. Then

$$\alpha(2n-1; n-1+i, n-1, n-1, \dots, 1, 1) = (-1)^{n-1} A_{n,i}$$

holds for $i = 1, \dots, 2n-1$, $n \geq 1$.

Corollary

$$\alpha(2n; n, n, n-1, n-1, \dots, 1, 1) = \alpha(n; 1, 2, \dots, n)$$

Theorem 2 (I. Fischer, L. Riegler (2011))

Let $A_{n,i}$ denote the number of ASMs with the first row's unique 1 in column i. Then

$$\alpha(2n-1; n-1+i, n-1, n-1, \dots, 1, 1) = (-1)^{n-1} A_{n,i}$$

holds for $i = 1, \dots, 2n-1$, $n \geq 1$.

Corollary

$$\alpha(2n; n, n, n-1, n-1, \dots, 1, 1) = \alpha(n; 1, 2, \dots, n)$$

Corollary

$$\alpha(2n; n, n, n-1, n-1, \ldots, 1, 1) = \alpha(n; 1, 2, \ldots, n)$$

Proof.

$$\begin{aligned}
\alpha(n; 1, 2, \ldots, n) &= A_{n+1,1} \\
&\overset{\text{Th.2}}{=} (-1)^n \alpha(2n+1; n+1, n, n, n-1, n-1, \ldots, 1, 1) \\
&\overset{\text{Th.1}}{=} \sum_{A \in \mathcal{D}_{2n+1}(n+1, n, n, \ldots, 1, 1)} (-1)^{\mathrm{dd}(A)} \\
&= \sum_{A \in \mathcal{D}_{2n}(n, n, \ldots, 1, 1)} (-1)^{\mathrm{dd}(A)+n} \\
&\overset{\text{Th.1}}{=} \alpha(2n; n, n, n-1, n-1, \ldots, 1, 1).
\end{aligned}$$

□

Corollary

$$\alpha(2n; n, n, n-1, n-1, \ldots, 1, 1) = \alpha(n; 1, 2, \ldots, n)$$

Proof.

$$\begin{aligned}
\alpha(n; 1, 2, \ldots, n) &= A_{n+1,1} \\
&\stackrel{\text{Th.2}}{=} (-1)^n \alpha(2n+1; n+1, n, n, n-1, n-1, \ldots, 1, 1) \\
&\stackrel{\text{Th.1}}{=} \sum_{A \in \mathcal{D}_{2n+1}(n+1, n, n, \ldots, 1, 1)} (-1)^{\mathrm{dd}(A)} \\
&= \sum_{A \in \mathcal{D}_{2n}(n, n, \ldots, 1, 1)} (-1)^{\mathrm{dd}(A)+n} \\
&\stackrel{\text{Th.1}}{=} \alpha(2n; n, n, n-1, n-1, \ldots, 1, 1).
\end{aligned}$$

$\square$

Corollary

$$\alpha(2n; n, n, n-1, n-1, \ldots, 1, 1) = \alpha(n; 1, 2, \ldots, n)$$

Proof.

$$\alpha(n; 1, 2, \ldots, n) = A_{n+1,1}$$

$$\overset{\text{Th.2}}{=} (-1)^n \alpha(2n+1; n+1, n, n, n-1, n-1, \ldots, 1, 1)$$

$$\overset{\text{Th.1}}{=} \sum_{A \in \mathcal{D}_{2n+1}(n+1,n,n,\ldots,1,1)} (-1)^{\mathrm{dd}(A)}$$

$$= \sum_{A \in \mathcal{D}_{2n}(n,n,\ldots,1,1)} (-1)^{\mathrm{dd}(A)+n}$$

$$\overset{\text{Th.1}}{=} \alpha(2n; n, n, n-1, n-1, \ldots, 1, 1).$$

□

Corollary

$$\alpha(2n; n, n, n-1, n-1, \ldots, 1, 1) = \alpha(n; 1, 2, \ldots, n)$$

Proof.

$$\begin{aligned}
\alpha(n; 1, 2, \ldots, n) &= A_{n+1,1} \\
&\overset{\text{Th.2}}{=} (-1)^n \alpha(2n+1; n+1, n, n, n-1, n-1, \ldots, 1, 1) \\
&\overset{\text{Th.1}}{=} \sum_{A \in \mathcal{D}_{2n+1}(n+1,n,n,\ldots,1,1)} (-1)^{\mathrm{dd}(A)} \\
&= \sum_{A \in \mathcal{D}_{2n}(n,n,\ldots,1,1)} (-1)^{\mathrm{dd}(A)+n} \\
&\overset{\text{Th.1}}{=} \alpha(2n; n, n, n-1, n-1, \ldots, 1, 1).
\end{aligned}$$

□

Corollary

$$\alpha(2n; n, n, n-1, n-1, \dots, 1, 1) = \alpha(n; 1, 2, \dots, n)$$

Proof.

$$\begin{aligned}
\alpha(n; 1, 2, \dots, n) &= A_{n+1,1} \\
&\overset{\text{Th.2}}{=} (-1)^n \alpha(2n+1; n+1, n, n, n-1, n-1, \dots, 1, 1) \\
&\overset{\text{Th.1}}{=} \sum_{A \in \mathcal{D}_{2n+1}(n+1, n, n, \dots, 1, 1)} (-1)^{\mathrm{dd}(A)} \\
&= \sum_{A \in \mathcal{D}_{2n}(n, n, \dots, 1, 1)} (-1)^{\mathrm{dd}(A)+n} \\
&\overset{\text{Th.1}}{=} \alpha(2n; n, n, n-1, n-1, \dots, 1, 1).
\end{aligned}$$

$\square$

By Theorem 1, if n even, then

$$\alpha(n; n, n-1, \ldots, 1) = 0.$$

What about n odd?

Conjecture

For $n = 2m+1$, $m \geq 1$, the equation

$$\begin{aligned}\alpha(n; n, n-1, \ldots, 1) &= (-1)^m \alpha(m; 2, 4, \ldots, 2m) \\ &= (-1)^m \ \#\ \textit{vertically symmetric ASMs of size } 2m+1\end{aligned}$$

seems to hold.

By Theorem 1, if n even, then

$$\alpha(n; n, n-1, \ldots, 1) = 0.$$

What about n odd?

Conjecture

For $n = 2m+1$, $m \geq 1$, the equation

$$\begin{aligned}\alpha(n; n, n-1, \ldots, 1) &= (-1)^m \alpha(m; 2, 4, \ldots, 2m) \\ &= (-1)^m \; \# \text{ vertically symmetric ASMs of size } 2m+1\end{aligned}$$

seems to hold.

R. E. Behrend, P. Di Francesco, and P. Zinn-Justin.
On the weighted enumeration of alternating sign matrices and descending plane partitions.
arXiv:1103.1176v1, 2011.

I. Fischer.
The number of monotone triangles with prescribed bottom row.
Adv. Appl. Math., no.2, 37:249–267, 2006.

I. Fischer.
A new proof of the refined alternating sign matrix theorem.
J. Comb. Theory Ser. A, 114:253–264, 2007.

R.L. Graham, D.E. Knuth, and O. Patashnik.
Concrete Mathematics.
Addison-Wesley, 1989.