Cluster algebras and Lie theory, III

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- For $M \in \operatorname{mod} \Lambda$ and $\mathbf{i} = (i_1, \dots, i_d)$ let $\mathscr{F}_{M,\mathbf{i}}$ be the variety of composition series of M of type $\mathbf{i}$:

$$\{0\} = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_d = M$$

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- $\chi_{M,\mathbf{i}} := \chi(\mathscr{F}_{M,\mathbf{i}}) \in \mathbb{Z}$ (Euler characteristic).

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Theorem (Lusztig, Geiss-L-Schröer)

There exits a unique $\varphi_M \in \mathbb{C}[N]$ such that for all $\mathbf{j} = (j_1, \dots, j_k)$

$$\varphi_M(x_{j_1}(t_1) \cdots x_{j_k}(t_k)) = \sum_{\mathbf{a} \in \mathbb{N}^k} \chi_{M,\mathbf{j}^\mathbf{a}} \frac{t_1{}^{a_1} \cdots t_k{}^{a_k}}{a_1! \cdots a_k!}$$

where $\mathbf{j}^\mathbf{a} = (\underbrace{j_1, \dots, j_1}_{a_1}, \dots, \underbrace{j_k, \dots, j_k}_{a_k})$

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- Indecomposable projectives:

$$\begin{tikzcd}[column sep=small, row sep=small] 1 \arrow[dr] & & \\ & 2 \arrow[dr] & \\ & & 3 \end{tikzcd} \quad \mapsto \; D_{234}, \qquad \begin{tikzcd}[column sep=small, row sep=small] & 2 \arrow[dl] \arrow[dr] & \\ 1 \arrow[dr] & & 3 \arrow[dl] \\ & 2 & \end{tikzcd} \quad \mapsto \; D_{34}, \qquad \begin{tikzcd}[column sep=small, row sep=small] & & 3 \arrow[dl] \\ & 2 \arrow[dl] & \\ 1 & & \end{tikzcd} \; \mapsto \; D_4,$$

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- Other indecomposables:

$$\begin{tikzcd}[column sep=small] & 2 \arrow[dl] \arrow[dr] & \\ 1 & & 3 \end{tikzcd} \mapsto D_{13,34}, \qquad \begin{tikzcd}[column sep=small] 1 \arrow[dr] & & 3 \arrow[dl] \\ & 2 & \end{tikzcd} \mapsto D_{24}, \qquad \begin{tikzcd}[column sep=small] 1 \arrow[dr] & \\ & 2 \end{tikzcd} \mapsto D_{23},$$

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$$1 \mapsto D_{2}, \qquad 2 \mapsto D_{13}, \qquad 3 \mapsto D_{124}.$$

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- We have two short exact sequences:

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Theorem (Geiss-L-Schröer)

- for every $M, L \in \operatorname{mod}\Lambda$, $\varphi_M \varphi_L = \varphi_{M\oplus L}$
- if $\dim \operatorname{Ext}^1_\Lambda(M,L) = \dim \operatorname{Ext}^1_\Lambda(L,M) = 1$ then

$$\varphi_M \varphi_L = \varphi_X + \varphi_Y,$$

where $0 \to M \to X \to L \to 0$ and $0 \to L \to Y \to M \to 0$ are the two non-split short exact sequences.

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- Example in type A_3:

$$T = 1 \oplus \begin{tikzcd}[column sep=small, row sep=small] 1 \arrow[dr] & \\ & 2 \end{tikzcd} \oplus \begin{tikzcd}[column sep=small, row sep=small] & 2 \arrow[dl] \\ 1 & \end{tikzcd} \oplus \begin{tikzcd}[column sep=small, row sep=small] 1 \arrow[dr] & & \\ & 2 \arrow[dr] & \\ & & 3 \end{tikzcd} \oplus \begin{tikzcd}[column sep=small, row sep=small] & 2 \arrow[dl] \arrow[dr] & \\ 1 \arrow[dr] & & 3 \arrow[dl] \\ & 2 & \end{tikzcd} \oplus \begin{tikzcd}[column sep=small, row sep=small] & & 3 \arrow[dl] \\ & 2 \arrow[dl] & \\ 1 & & \end{tikzcd}$$

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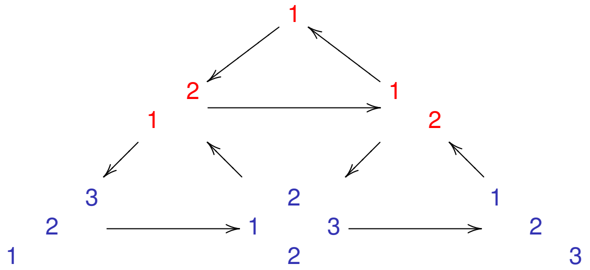

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Define $\mu_k(T) := (T/T_k) \oplus T_k{}^*$, the mutation of T in direction k.

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Does every cluster monomial belong to the dual canonical basis of $\mathbb{C}[N]$?

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Boltzmann weights of integrable models of statistical mechanics

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- Hernandez-L; Nakajima (2009): partial answer for $U_q(L\mathfrak{g})$.

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Theorem (Hernandez-Leclerc 2009, Nakajima 2009)

- There exists a cluster algebra structure on the Grothendieck ring of certain tensor subcategories of mod $U_q(L\mathfrak{g})$.
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$\rightsquigarrow$ new combinatorial and geometric formulas for certain irreducible characters of $U_q(L\mathfrak{g})$.

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