Which algebraic integers are chromatic roots?

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The chromatic polynomial

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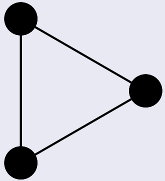
Example: colourings of K_3

K_3 is the complete graph on three vertices.

$P_{K_3}(x) = x(x-1)(x-2)$, so K_3 has six 3-colourings.



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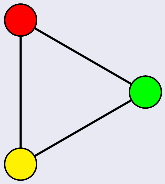

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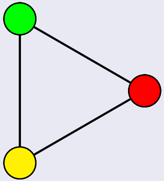

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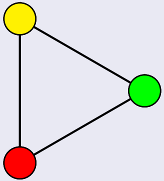

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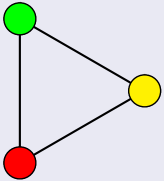

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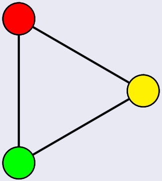

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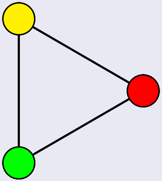

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A *chromatic root* of G is a zero of $P_G(x)$. Much is known about the distribution of chromatic roots in the real line and complex plane.

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But these are analytic results—they tell us almost nothing about which specific complex numbers can be chromatic roots...

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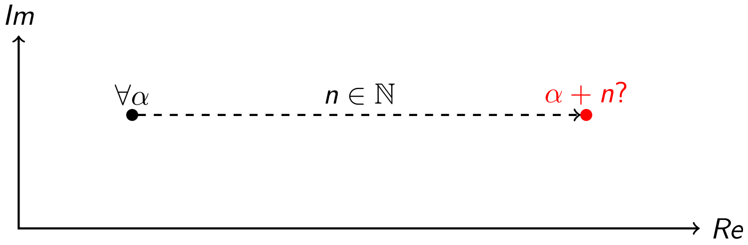
Slightly easier questions

- Is the set of chromatic roots closed under multiplication by positive integers?
- Is every normal extension of $\mathbb{Q}$ generated by a factor of some chromatic polynomial?

Two conjectures on chromatic roots

Conjecture 1 (The $\alpha + n$ Conjecture)

For any algebraic integer α, there is some natural number n such that $\alpha + n$ is a chromatic root.

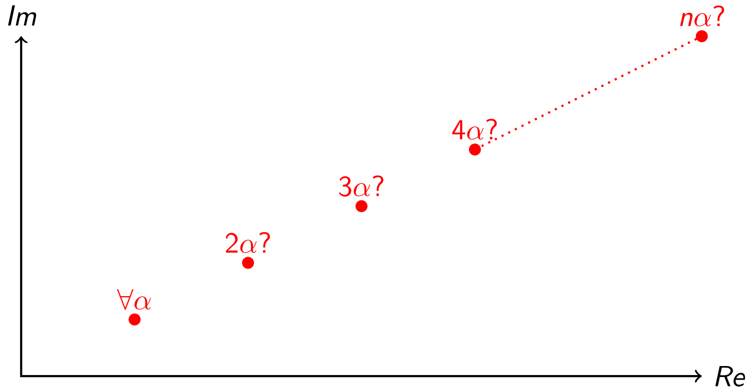


If this were true, it would imply that every number field is contained in the splitting field of a chromatic factor.

Two conjectures on chromatic roots

Conjecture 2 (The $n\alpha$ Conjecture)

If α is a chromatic root, then so too is $n\alpha$ for all natural numbers n.



Evidence?

Other than data on small graphs, the main reason to suspect these conjectures may be true is the following:

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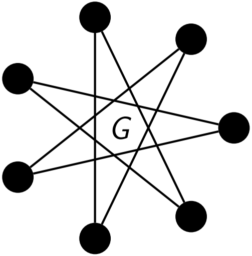

$P_G(\alpha) = 0$

Outline of proof

Let G be a graph with chromatic root α, and let G_i be the join of G with a copy of K_i.

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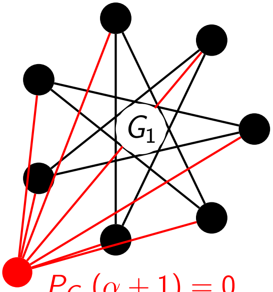

Outline of proof

Let G be a graph with chromatic root α, and let G_n be the join of G with a copy of K_n.

Then: $P_{G_1} = xP_G(x-1)$, and so G_1 has a chromatic root $\alpha + 1$.

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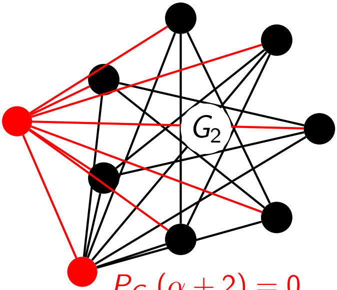

$P_{G_2}(\alpha + 2) = 0$

Outline of proof

Let G be a graph with chromatic root α, and let G_n be the join of G with a copy of K_n.

$P_{G_2} = (x)_2 P_G(x-2)$, and so G_2 has a chromatic root $\alpha + 2$.

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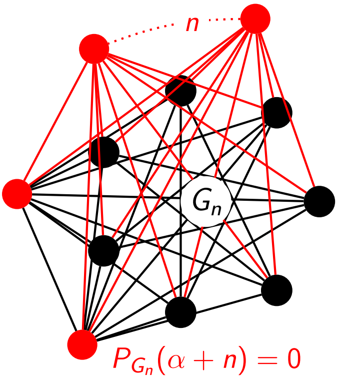

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Let G be a graph with chromatic root α, and let G_n be the join of G with a copy of K_n.

$P_{G_n} = (x)_n P_G(x - n)$, and so G_n has a chromatic root $\alpha + n$.

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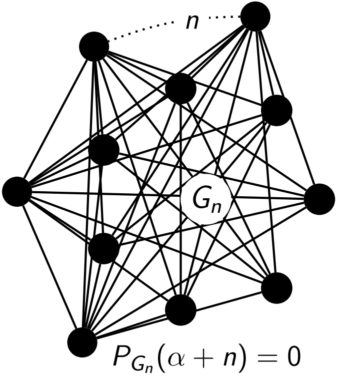
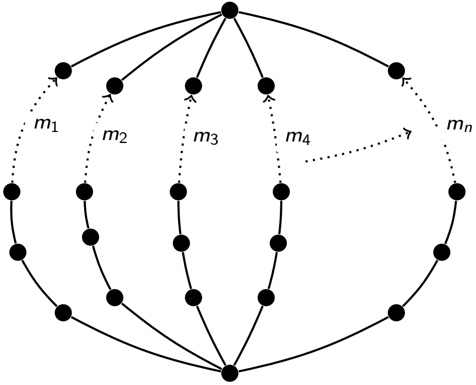


- The $n\alpha$ conjecture proposes a multiplicative analogue of this result.
- The implication that algebraic integers with larger real parts are more likely to be chromatic roots lends credibility to the $\alpha + n$ conjecture.

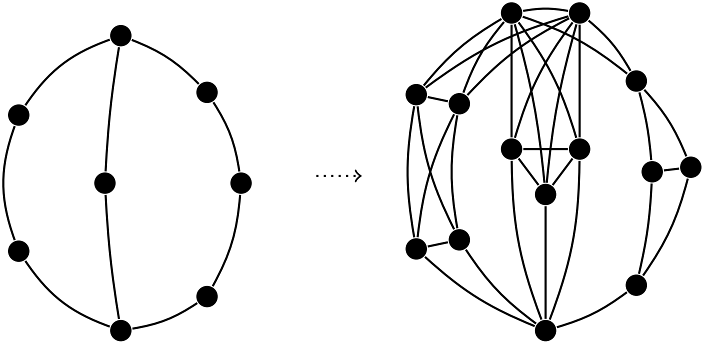
Generalised theta graphs

A *generalised theta graph* $\Theta_{m_1,\ldots,m_n}$ consists of two vertices joined by n otherwise disjoint paths of length $m_1, \ldots, m_n$. Sokal used these graphs in his proof that chromatic roots are dense in the complex plane.



Clique-theta graphs, and the $n\alpha$ Conjecture

A *clique-theta graph* is any graph obtained from a generalised theta graph by "blowing up" vertices into cliques.



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Let G be a clique-theta graph consisting of n disjoint "clique-paths" connecting a single vertex to a k-clique, and let $a_{i(j)}$ be the size of the jth clique in the ith path. Then the chromatic polynomial of G is:

$$\left[k(x-k)^{n-1}\prod_{i=1}^{n}\frac{1}{x}\left(\prod_{l=1}^{m_i}(x-a_{i(l)})-\prod_{l=1}^{m_i}(-a_{i(l)})\right)\right] + \left[\prod_{i=1}^{n}\frac{1}{x}\left((x-k)\prod_{l=1}^{m_i}(x-a_{i(l)})+k\prod_{l=1}^{m_i}(-a_{i(l)})\right)\right].$$

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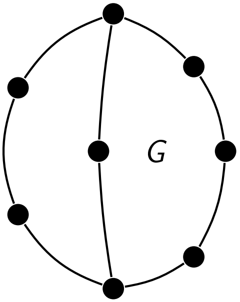
Proposition

There is a dense subset of $\mathbb{C}$ consisting of chromatic roots which remain chromatic roots upon multiplication by positive integers.

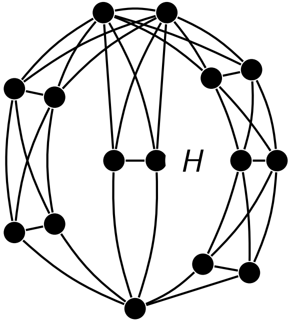
Clique-theta graphs, and the $n\alpha$ Conjecture

Proof

Let α be a non-integer chromatic root of a generalised theta graph G. Then, for any $n \in \mathbb{N}$, $n\alpha$ is a chromatic root of the clique-theta graph H obtained by replacing all but one endpoint vertex of G with n-cliques.



$\cdots\rightarrow$



$$\forall \alpha \notin \mathbb{Z}, P_G(\alpha) = 0 \quad \Rightarrow \quad P_H(2\alpha) = 0$$

A rephrasing of the $\alpha + n$ conjecture

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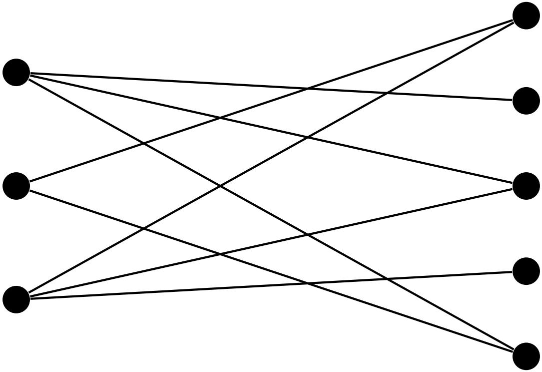
- The $\alpha + n$ conjecture is thus equivalent to the assertion that every reduced polynomial can be obtained from some chromatic factor by translating the domain.

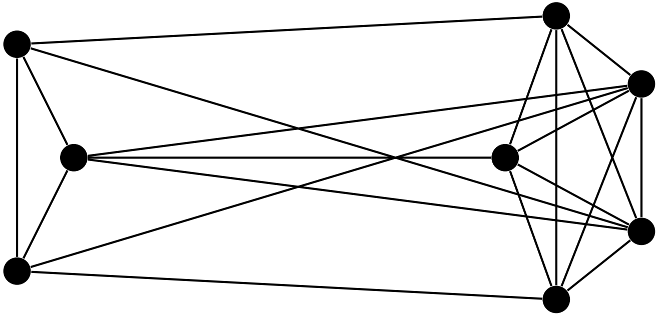
Bicliques

A graph G is *bipartite* if its set of vertices can be partitioned into 2 subsets with no internal edges.



Bicliques

A *biclique* is the complement of a bipartite graph, consisting of two cliques joined by edges. A (j, k)-*biclique* has cliques of size j and k, with $j \leq k$.



A $(3, 5)$-biclique

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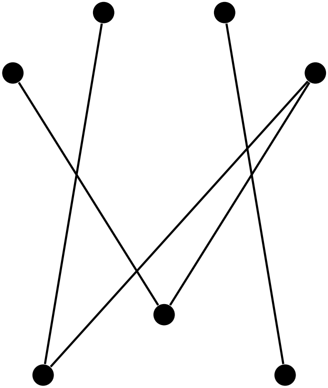
Each colouring of G is thus induced by a *matching* of $\bar{G}$,and:

$$P_G(x) = \sum_M (x)_{|V|-|M|} = \sum_M (x)_{j+k-|M|},$$

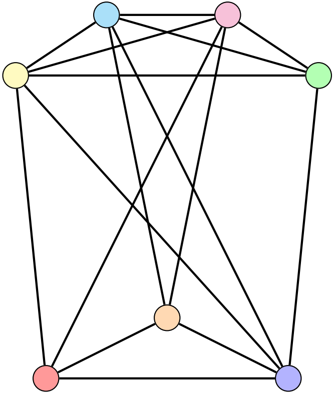
where the sum is over all matchings M of $\bar{G}$.

Example

$P_G(x) = (x)_7 +$



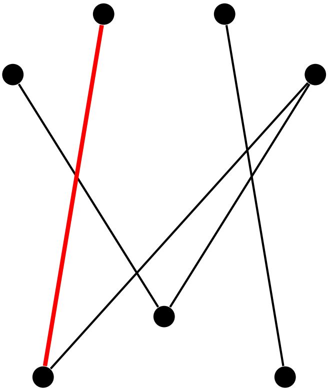
Matching of $\bar{G}$



Possible corresponding colouring of G

Example

$P_G(x) = (x)_7 + (x)_6 +$



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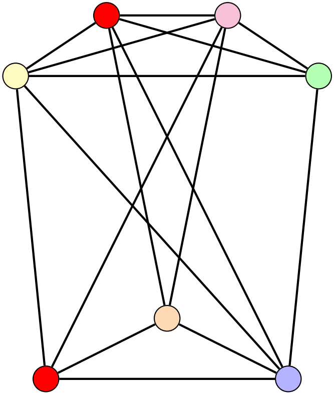

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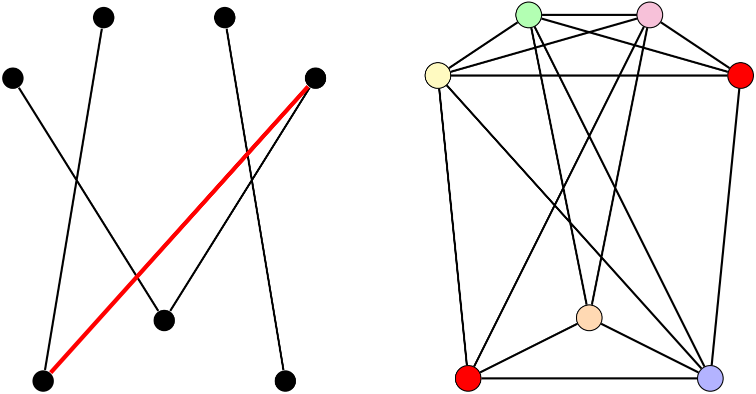

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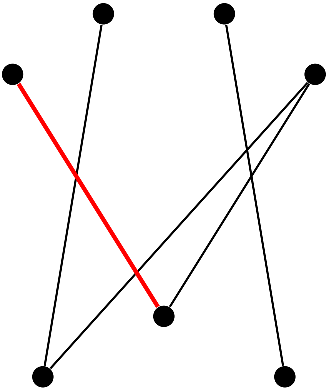

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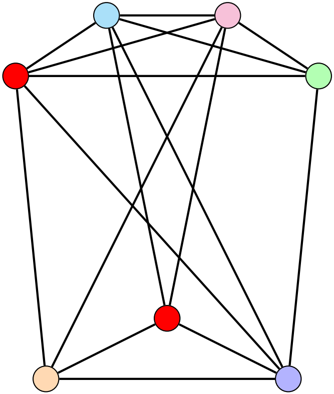

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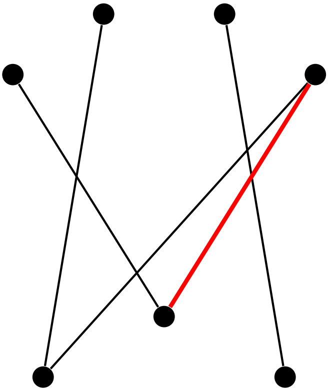

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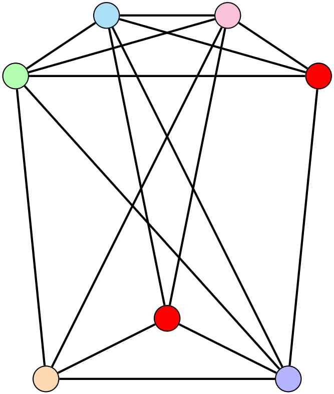

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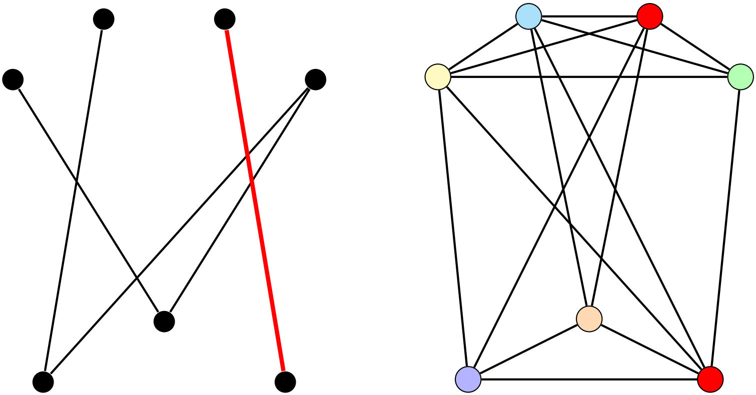
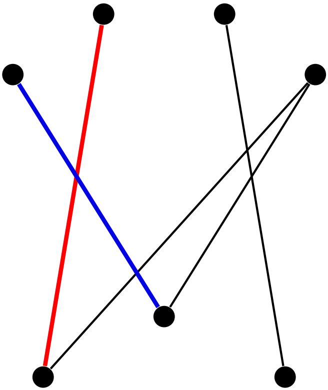


Matching of $\bar{G}$

Possible corresponding colouring of G

$$P_G(x) = (x)_7 + 5(x)_6 + (x)_5 +$$



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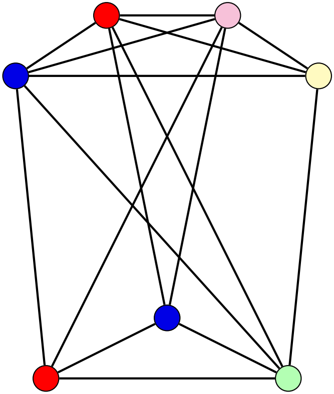

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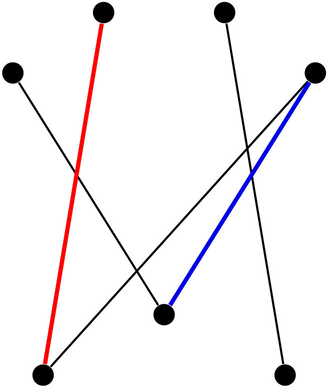

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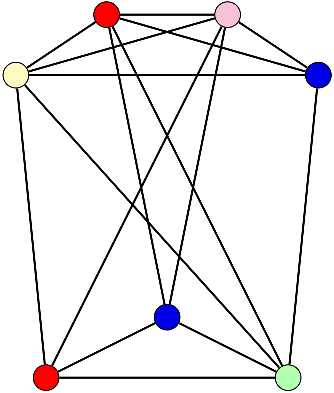

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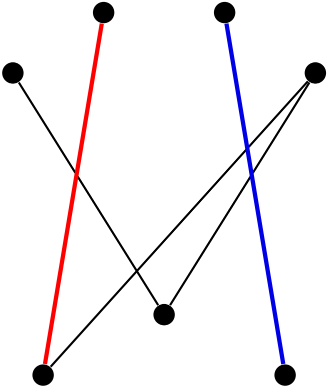

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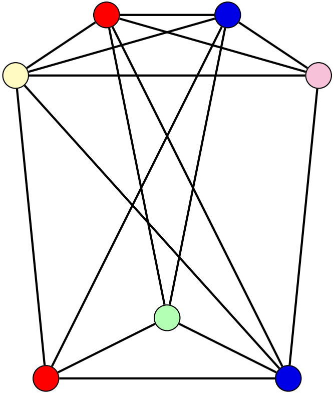

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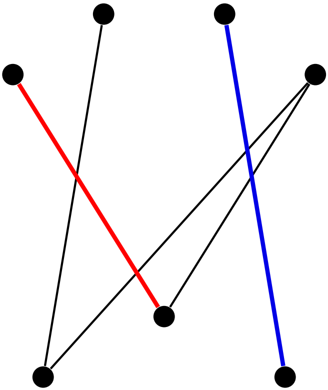
$$P_G(x) = (x)_7 + 5(x)_6 + (x)_5 + (x)_5 + (x)_5 + (x)_5 +$$



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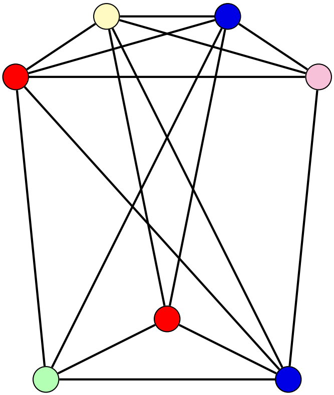

Possible corresponding colouring of G

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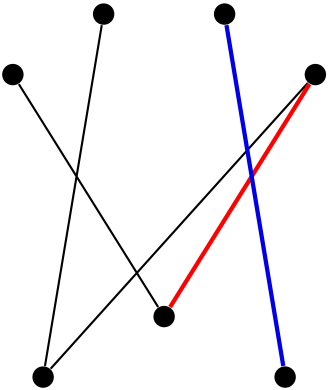
$$P_G(x) = (x)_7 + 5(x)_6 + (x)_5 + (x)_5 + (x)_5 + (x)_5 + {\color{red}(x)_5 +}$$



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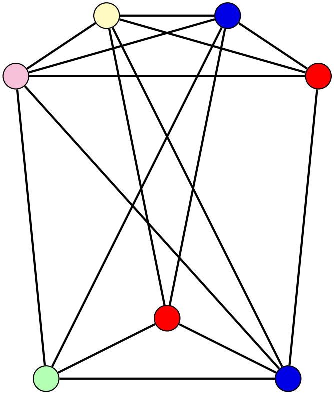

Possible corresponding colouring of G

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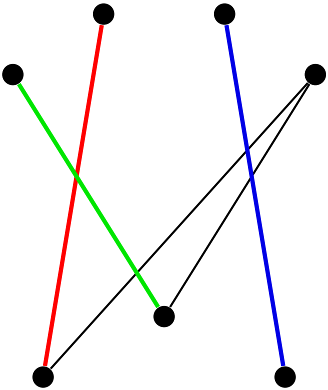
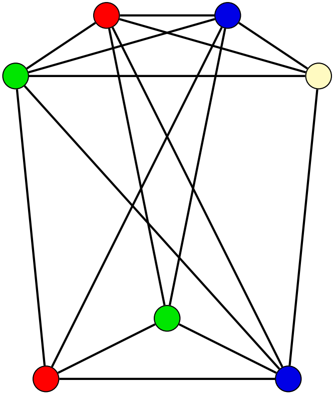
$$P_G(x) = (x)_7 + 5(x)_6 + 5(x)_5 + (x)_4 +$$



Matching of $\bar{G}$



Possible corresponding colouring of G

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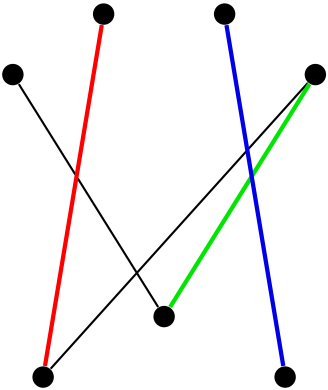

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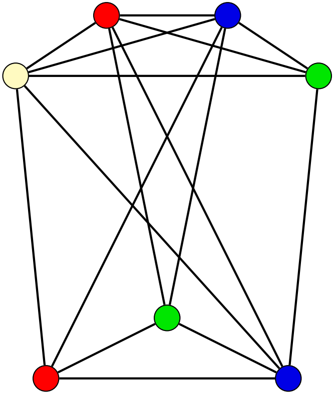

Possible corresponding colouring of G

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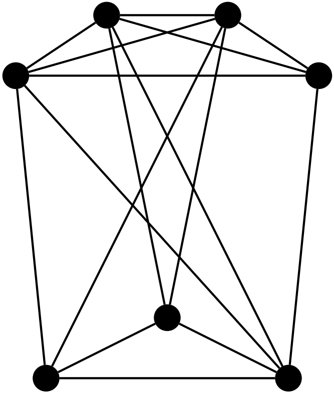

Bicliques and the $\alpha + n$ conjecture

Let G be a (j, k)-biclique, with bipartite complement $\bar{G}$. Then:

$$P_G(x) = (x)_k \sum_{i=0}^{j} m^i_{\bar{G}} (x - k)_{j-i},$$

where $m^i_{\bar{G}}$ is the ith *matching number* of $\bar{G}$.

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Interesting consequence

Any two (j, k)-bicliques whose inter-clique edges complement each other inside $K_{j,k}$ have chromatic polynomials which generate the same splitting field.

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Theorem

For all quadratic (resp. cubic) integers α, there are natural numbers n, k and a $(2,k)$-biclique (resp. $(3,k)$-biclique) G such that $\alpha + n$ is a chromatic root of G.

Matching polynomials & rook polynomials

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The $\alpha + n$ conjecture for bicliques is thus equivalent to a statement about which sequences of coefficients can appear in rook polynomials.

The End

Thanks for listening!