Fuchsian type multipliers and shuffle algebras.

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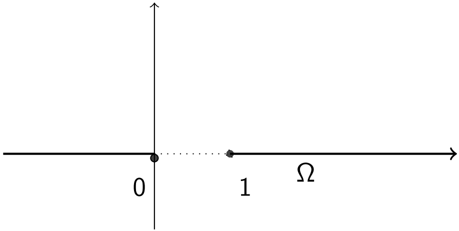
Summary

Introduction

- Many discussions and interactions convinced us that a special care should be payed on the study of *coefficients* about polylogarithms : indeed, *the only known proofs of the linear independance of polylogarithms and hyperlogarithms were through an argument of* monodromy *(i.e. integration along a path with the addition of one, or several, loops containing singularities). A strong need was to examine whether one could prove independance without using* monodromy *and, if so, if one could "push" further the set of coefficients.*
- Today we will show an abstract theorem about differential algebra and some of its applications to polylogarithms.
- The road to polylogarithms is the following
  - Riemann zeta values
  - One wants to produce a tractable $\mathbb{Q}$-algebra where these functions live
  - Euler-Zagier sums
  - Polylogarithms (analytic function)
  - Integral form of polylogarithms (today's concern)

Strings of primitives 1/4

We begin by a classical example, the generation of polylogarithms.
On the following domain,



$$\mathbb{C} - (]-\infty, 0] \cup [1, +\infty[)$$

let's play a game ...

Strings of primitives 2/4

Starting from 1 (the constant function on Ω), we read the words of $\{x_0, x_1\}^*$ and compute w.r.t. the following rule (using any "primitive-making" process).

$$\begin{tikzcd} {} & & {} \\ & {} \arrow[ul, "\int ? \frac{dz}{z}"'] \arrow[ur, "\int ? \frac{dz}{1-z}"] & \end{tikzcd}$$

We then get a collection of analytic functions $(S_w(z))_{w \in \{x_0,x_1\}^*}$ satisfying

$$S'_{x_0 w}(z) = \frac{1}{z} S_w(z) \ ; \ S'_{x_1 w}(z) = \frac{1}{1-z} S_w(z) \qquad (1)$$

Strings of primitives 3/4

A classical method (Chen's iterated integrals) to obtain solutions of (5) is to pick an element $z_0 \in \Omega$ and compute the following iterated integrals

$$S_{x_{i_1} x_{i_2} \cdots x_{i_n}}(z) = \int_{z_0}^{z} d\omega_{i_1} \int_{z_0}^{y_1} \cdots \int_{z_0}^{y_{n-1}} d\omega_{i_n} \; ; \; \omega_0 = \frac{dz}{z} \; \omega_1 = \frac{dz}{1-z} \qquad (2)$$

One can also initialize the process with a chain of well choosen primitives i.e.

$$S_{x_0^n}(z) := \frac{log(z)^n}{n!} \qquad (3)$$

and integrate w.r.t. the other moves.
In these two cases, the solution is a morphism w.r.t. the shuffle product.

We then get a map $w \mapsto S_w$ from $X^* = \{x_0, x_1\}^*$ to $\mathcal{C}^\omega(\Omega, \mathbb{C})$ (a ring) which we note as a sum $S = \sum_{w \in X^*} \langle S|w\rangle w$ (a noncommutative series). We recall that the algebra of noncommutative series is endowed with the convolution product (here also concatenation)

$$ST := \sum_{w \in X^*} \Big(\sum_{uv=w} \langle S|u\rangle \langle T|v\rangle \Big) w \tag{4}$$

Now, with the extension of $\frac{d}{dz}$ to series, one has

$$\frac{d}{dz}(S) = MS \tag{5}$$

with $M = \frac{1}{z}.x_0 + \frac{1}{1-z}.x_1$.
This extension of the derivative to noncommutative series permits a closer study of solutions of equations of type (5).
In particular, with certain initial conditions, solutions of (5) are morphisms of the shuffle algebra (this is the case of the two solutions given above).

Shuffle algebras

There is another product between series which is linked to the Hopf algebra structure of the noncommutative polynomials (viewed as the enveloping algebra of the free Lie algebra)

$$(k\langle X\rangle, conc, 1_{X^*}, \Delta_{\sqcup\!\sqcup}, \epsilon, S)\ . \tag{6}$$

The letters are primitive i.e. for all $x \in X$

$$\Delta_{\sqcup\!\sqcup}(x) = x \otimes 1 + 1 \otimes x\,. \tag{7}$$

Then the shuffle product can be defined either by duality with $\Delta_{\sqcup\!\sqcup}$ or by the following recursion

$$\begin{aligned} 1 \sqcup\!\sqcup w &= w \sqcup\!\sqcup 1 = w\ ; \\ a.u \sqcup\!\sqcup b.v &= a.(u \sqcup\!\sqcup b.v) + b(a.u \sqcup\!\sqcup v) \end{aligned} \tag{8}$$

Duality Series/Polynomials

As $k\langle X\rangle = k^{(X^*)}$ and $k\langle\langle X\rangle\rangle = k^{X^*}$, we get a natural pairing

$$k\langle\langle X\rangle\rangle \otimes k\langle X\rangle \to k \tag{9}$$

by $\langle S|P\rangle = \sum_{w\in X^*}\langle S|w\rangle\langle P|w\rangle$.
Here is the meaning of the statement (see above) that certain series S are morphisms for the shuffle product. We have

$$\langle S|u \sqcup\!\sqcup v\rangle = \langle S|u\rangle\langle S|v\rangle \tag{10}$$

Abstract setting 1/2

Let $(\mathcal{A}, d)$ be a k-commutative associative differential algebra with unit ($ch(k) = 0$ and $ker(d) = k$) and $\mathcal{C}$ be a differential subfield of $\mathcal{A}$ (i.e. $d(\mathcal{C}) \subset \mathcal{C}$). We suppose that $S \in \mathcal{A}\langle\langle X\rangle\rangle$ is a solution of the differential equation

$$\mathbf{d}(S) = MS \ ; \ \langle S|1\rangle = 1 \tag{11}$$

where the multiplier $M = \sum_{x \in X} u_x x \in \mathcal{C}\langle\langle X\rangle\rangle$ is an homogeneous series (a polynomial in case X is finite) of degree 1.

Abstract setting 2/2

Theorem

The following condition are equivalent :
i) The family $(\langle S|w\rangle)_{w\in X^*}$ *of coefficients of* S *is free over* $\mathcal{C}$.
ii) The family of coefficients $(\langle S|y\rangle)_{y\in X\cup\{1_{X^*}\}}$ *is free over* $\mathcal{C}$.
iii) The family $(u_x)_{x\in X}$ *is such that, for* $f\in\mathcal{C}$ *and* $\alpha_x\in k$

$$d(f)=\sum_{x\in X}\alpha_x u_x \Longrightarrow (\forall x\in X)(\alpha_x=0)\ . \tag{12}$$

Remark

(iii) can be rephrased as
iv) The family $(u_x)_{x\in X}$ *is free over* k *and*

$$d(\mathcal{C})\cap span_k\Big((u_x)_{x\in X}\Big)=\{0\}\ . \tag{13}$$

Sketch of the proof 1/2

- (iii)$\Longrightarrow$(i) Suppose a linear dependence relation

$$\sum_{w\in X^*} \alpha_w \langle S|w\rangle = 0 \tag{14}$$

(sum with finite support). With $P=\sum_{w\in X^*}\alpha_w\, w = 0$, (14) can be rewritten

$$\langle S|P\rangle = 0\ ;\ \ P\neq 0\ . \tag{15}$$

Take the polynomial with the least degree (lenght of the leading monomial) such that (15) and least leading word among the non-trivial relations, normalize it (because $\mathcal{C}$ is a field, one can divide by the leading coefficient) and write it

$$P = w + \sum_{u<w}\langle P|u\rangle u \tag{16}$$

Sketch of the proof 2/2

Now

$$0 = \langle S|P\rangle' = \langle S'|P\rangle + \langle S|P'\rangle = \langle MS|P\rangle + \langle S|P'\rangle = \langle S|M^\dagger P\rangle + \langle S|P'\rangle = \langle S|M^\dagger P + P'\rangle \quad (17)$$

which implies that $P' = -M^\dagger P$ hence $\langle P|w\rangle' = -\langle M^\dagger P|w\rangle = -\sum_{x\in X} u_x \langle P|xw\rangle$ From this and the fact that $ker(d) = k$, one obtains a contradiction (if we had $deg(P) \geq 1$).

Sketch of the proof 2/2

- (i)$\Longrightarrow$(ii) obvious
- (ii)$\Longrightarrow$(iii) An equality $d(f) = \sum_{x\in X} \alpha_x u_x$ for a function $f \in \mathcal{C}$ can be rewritten $d\Big(f - \sum_{x\in X} \alpha_x S_x\Big) = 0$ and thus $f - \sum_{x\in X} \alpha_x S_x = c \in k$ which implies that $f = c$ and all the α_x are zero.

Application to polylogs

- In order to get a field of function we have to withdraw the poles. We then consider fields of germs of functions. For simplicity, let's take the field of rational functions $\mathbb{C}(z)$, realized on Ω as functions defined on some $(\Omega - F)_{F\ finite}$. This field fulfils the conditions of the theorem w.r.t. $d = \frac{d}{dz}$ and then we have the independance of the polylogarithms w.r.t. the rational functions.
- One can even go further (larger fields of functions, and/or other "entries" u_x) ... in particular
- all other multipliers of "Fuchsian type" (by this we mean that all entries have distinct singularities of order one) yield the same conclusion.

Concluding remarks and perspectives

1. We did not touch the proof of the "shuffle" condition. It is an easy consequence of the primitivity of the multiplier $(\Delta_{\sqcup\!\sqcup}(M) = M \otimes 1 + 1 \otimes M)$
2. Constants of (alien ?) derivatives
   The operators x_0^{-1} and x_1^{-1} act as (directional ?, alien ?) derivatives on $\mathcal{A}_{\mathcal{C}} = \mathcal{C} \otimes_{\mathbb{C}} \mathcal{A}_{\mathbb{C}}$. They coincide respectively with $z\frac{\mathrm{d}}{\mathrm{d}\, z}$ and $(1-z)\frac{\mathrm{d}}{\mathrm{d}\, z}$ on $\mathcal{A}_{\mathbb{C}}$, but their constant field is much larger.
   What is the maximal constant field for these derivatives ? (presumably the field of germs of analytic functions which are inessential at 0 and 1).
3. Better understanding of the combinatorics behind the differential Galois group.

$$S' = MS \ ; \ (SG)' = S'G = M(SG)$$