A Coxeter theoretic interpretation of Euler numbers

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The Euler numbers $T_n$ are defined via the generating function

$$\tan(z) + \sec(z) = \sum_{n \geq 0} T_n \frac{z^n}{n!}$$

Désiré André showed that $T_n$ is the number of alternating permutations in $S_n$, i.e. those $w$ such that $w(1) > w(2) < w(3) > \ldots$

Since then, a lot of interest has been given to these numbers and these permutations [Stanley, a survey of alternating permutations (2010)]
Springer showed that, considering alternating permutations as the largest *descent class* in $S_n$, there is an analogue of $T_n$ for other finite irreducible Coxeter groups (he also computed the value in each case of the ABDE... classification).

There is another way to relate $T_n$ with the symmetric group as a Coxeter group, relying on a result of Stanley about orbits of maximal chains in the set partition lattice. We present a method to compute the value in each case of the classification.
Let $\mathcal{P}(n)$ be the lattice of set partitions of $\{1, \ldots, n\}$. It is ordered by refinement: $\mu \leq \pi$ if every block of $\mu$ is contained in a block of $\pi$.

For example: $\{\{1, 2, 4\}, \{3, 6\}, \{5\}\} \leq \{\{1, 2, 4\}, \{3, 5, 6\}\}$.

A maximal chain in $\mathcal{P}(n)$ is a sequence $\pi_1 < \cdots < \pi_n$ where

- $\pi_1 = \{\{1\}, \{2\}, \ldots, \{n\}\}$,
- $\pi_n = \{\{1, \ldots, n\}\}$,
- $\pi_{i+1}$ is obtained by joining two blocks of $\pi_i$.

For example: $\{\{1\}, \{2\}, \{3\}, \{4\}\} < \{\{1, 3\}, \{2\}, \{4\}\} < \{\{1, 3\}, \{2, 4\}\} < \{\{1, 2, 3, 4\}\}$.

$S_n$ acts on these maximal chains in an “obvious way”, and Stanley proved that the number of orbits is $T_{n-1}$. 
In this talk, we see a finite Coxeter group $W$ as a real reflection group, i.e. we have a defining representation $W \subset GL(V)$ for some Euclidian space $V$, such that $W$ is generated by orthogonal reflections.

$H$ is a reflecting hyperplane if the orthogonal reflection through it is in $W$.

**Definition**

The set partition lattice $\mathcal{P}(W)$ is the set of all subspaces $H_1 \cap \cdots \cap H_r$

where each $H_i$ is a reflecting hyperplane. It is ordered by reverse inclusion.
In type $A_{n-1}$, $W = S_n$ acts on $V = \{v \in \mathbb{R}^n : \sum v_i = 0\}$ by permuting coordinates.

$w \in S_n$ is a reflection if it permutes $v_i$ and $v_j$ for some $i < j$, i.e. is the orthogonal reflection through the subspace $H_{ij} = \{v \in V : v_i = v_j\}$.

Then the set of all subspaces

$$H_{i_1j_1} \cap \cdots \cap H_{i_kj_k}$$

is in bijection with set partitions. For example if $n = 7$:

$$H_{1,7} \cap H_{2,4} \cap H_{4,5} = \{v \in V : v_1 = v_7, v_2 = v_4 = v_5\}$$

$$\leftrightarrow \{\{1, 7\}, \{2, 4, 5\}, \{3\}, \{6\}\}.$$ 

And the refinement order on set partitions corresponds to reverse inclusion in subspaces of $V$. 
Let $\mathcal{M}(W)$ the set of maximal chains of the set partition lattice $\mathcal{P}(W)$.

There is an action of $W$ on $\mathcal{M}(W)$, we consider the number of orbits $K(W) = \#(\mathcal{M}(W)/W)$.

So Stanley’s result is $K(A_n) = T_n$.

(Remark: In Springer’s problem of the largest descent class, $T_n$ is the number associated to $S_n$ i.e. type $A_{n-1}$ and not $A_n$.)

What is $K(W)$ for the other cases of the classification?
The general method

$W$ acts on lines (=coatoms) in $\mathcal{P}(W)$. Let $L_1, \ldots, L_k$ be some orbit representatives. For each line $L_i$, let

\[
\text{Fix}(L_i) = \{ w \in W : w(x) = x, \forall x \in L_i \}, \\
\text{Stab}(L_i) = \{ w \in W : w(L_i) = L_i \}.
\]

Proposition

$\text{Fix}(L_i)$ is itself a Coxeter group, so $\mathcal{P}(\text{Fix}(L_i))$ and $\mathcal{M}(\text{Fix}(L_i))$ are defined, they are acted on by $\text{Stab}(L_i)$, and

\[
K(W) = \sum_{i=1}^{k} \#(\mathcal{M}(\text{Fix}(L_i))/\text{Stab}(L_i)).
\]
Remark

If

- $\text{Stab}(L_i) = \text{Fix}(L_i)$, or
- $\text{Stab}(L_i) = \text{Fix}(L_i) \rtimes \{\pm \text{Id}\},$

then we have

$$\#(\mathcal{M}(\text{Fix}(L_i))/\text{Stab}(L_i)) = K(\text{Fix}(L_i)),$$

which we assume we already know by induction.

(note that $\{\pm \text{Id}\}$ acts trivially on $\mathcal{P}(W)$)
Proof.
From each orbit of maximal chains, we can extract an orbit of lines. It suffices to show that the number of orbits of maximal chains associated to the orbit of \( L_i \) is \(#(M(\text{Fix}(L_i))/\text{Stab}(L_i))\).

\( \text{Fix}(L_i) \) is seen as a Coxeter group with defining representation acting on \( L_i^\perp \). The interval \([\hat{0}, L_i] \) in \( \mathcal{P}(W) \) is identified with \( \mathcal{P}(\text{Fix}(L_i)) \) (we use the bijection: subspaces containing \( L_i \) ↔ subspaces in \( L_i^\perp \)).

The number of \( W \)-orbits of chains \( \hat{0} < \cdots < w(L_i) < \hat{1} \) for some \( w \in W \), is also the number of \( \text{Stab}(L_i) \)-orbits of chains \( \hat{0} < \cdots < L_i < \hat{1} \). Hence it is \(#(M(\text{Fix}(L_i))/\text{Stab}(L_i))\).
Another useful result:

**Proposition**

*If* $W_1$, and *$W_2$* *have ranks* $m$ *and* $n$, *we have:*

$$K(W_1 \times W_2) = \binom{m + n}{m} K(W_1)K(W_2).$$

**Proof.**

A maximal chain in $\mathcal{P}(W_1 \times W_2)$ is obtained by “shuffling” two maximal chains in $\mathcal{P}(W_1)$ and $\mathcal{P}(W_2)$. For example from $x_0 < \cdots < x_m$ and $y_0 < \cdots < y_n$ we can form $(x_0, y_0) < (x_0, y_1) < (x_1, y_1) < (x_2, y_1) < \ldots$.

The number of shuffles is the binomial coefficient.

This operation is still well-defined for the orbits of maximal chains.
Case of the symmetric group $S_n$ (type $A_{n-1}$)

Let $V = \{ \mathbf{v} \in \mathbb{R}^n : \sum v_i = 0 \}$. The coatoms are the 2-block set partitions and a set of orbit representatives is:

$$L_i = \{ \mathbf{v} \in V : v_1 = \cdots = v_i, \ v_{i+1} = \cdots = v_n \}$$

with $1 \leq i \leq \frac{n}{2}$.

- If $i < \frac{n}{2}$, $\text{Fix}(L_i) = \text{Stab}(L_i) = S_i \times S_{n-i}$.
- If $i = \frac{n}{2}$, $\text{Fix}(L_i) = S_i \times S_i$ and $\text{Stab}(L_i) = (S_i \times S_i) \rtimes S_2$
  where $S_2$ permutes the two factors in $S_i \times S_i$.

In the second case, we have

$$\mathcal{M}(\text{Fix}(L_i))/\text{Stab}(L_i) = \mathcal{M}(S_i \times S_i)/(S_i \times S_i)/S_2$$

where the $S_2$-action has no fixed point, so

$$\#(\mathcal{M}(\text{Fix}(L_i))/\text{Stab}(L_i)) = \frac{1}{2}K(S_i \times S_i).$$
Case of the symmetric group $S_n$ (type $A_{n-1}$)

Let $a_n = K(A_n)$, we obtain:

$$a_{n-1} = \sum_{1 \leq i < \frac{n}{2}} \binom{n-2}{i-1} a_{i-1} a_{n-i-1} + \chi[n \text{ even}] \frac{1}{2} \binom{n-2}{n/2 - 1} a_{n/2-1}^2.$$

This is equivalent to

$$a_{n-1} = \frac{1}{2} \sum_{i=1}^{n-1} \binom{n-2}{i-1} a_{i-1} a_{n-i-1}.$$

So $A(z) = \sum_{n \geq 0} a_n \frac{z^n}{n!}$ satisfies $A'(z) = \frac{1}{2}(1 + A(z)^2)$ with $A(0) = 1$.

The solution is $A(z) = \tan(z) + \sec(z)$.

So $a_n = T_n$ (number of alternating permutations in $S_n$).
Case of $B_n$

Let $V = \mathbb{R}^n$. In the case $B_n$, the reflecting hyperplanes are:
\[ \{ \mathbf{v} \in V : v_i = 0 \}, \{ \mathbf{v} \in V : v_i = v_j \}, \text{ and } \{ \mathbf{v} \in V : v_i = -v_j \} \]
($i < j$).

As orbit representatives of the lines in $\mathcal{P}(B_n)$, we can take:
\[ L_i = \{ \mathbf{v} \in V : v_1 = \cdots = v_i = 0, \quad v_{i+1} = \cdots = v_n \}, \]
with $0 \leq i \leq n - 1$. We have
\[ \text{Fix}(L_i) = B_i \times A_{n-i-1}, \quad \text{and} \quad \text{Stab}(L_i) = (B_i \times A_{n-i-1}) \rtimes \{ \pm \text{Id} \}. \]

So
\[ b_n = \sum_{i=0}^{n-1} \binom{n-1}{i} b_i a_{n-i-1}. \]
Case of $B_n$

Let $B(z) = \sum_{n \geq 0} b_n \frac{z^n}{n!}$. The recursion

$$b_n = \sum_{i=0}^{n-1} \binom{n-1}{i} b_i a_{n-i-1}, \quad b_0 = 1.$$ 

is equivalent to $B'(z) = B(z)A(z)$ and $B(0) = 1$.

The solution is $B(z) = A'(z) = \frac{1}{1-\sin(z)}$.

So $b_n = T_{n+1}$ (number of alternating permutations in $S_{n+1}$).
Case of $D_n$

Let $V = \mathbb{R}^n$. In type $D_n$, the reflecting hyperplanes are:

\[ \{ v \in V : v_i = v_j \} \quad \text{and} \quad \{ v \in V : v_i = -v_j \} \quad (i < j). \]

As orbit representatives of the lines in $\mathcal{P}(D_n)$, we can take:

- $L_i = \{ v \in V : v_1 = \cdots = v_i = 0, \ v_{i+1} = \cdots = v_n \}$, 
  with $i = 0$ or $2 \leq i \leq n - 1$. We have $\text{Fix}(L_i) = D_i \times A_{n-i-1}$,

\[
\text{Stab}(L_i) = \begin{cases} 
(D_i \times A_{n-i-1}) \rtimes \{ \pm \text{Id} \} & \text{if } n \text{ is even,} \\
(D_i \times A_{n-i-1}) \rtimes \{ \text{Id}, (1, -1, \ldots, -1) \} & \text{if } n \text{ is odd and } i > 0. \\
(D_i \times A_{n-i-1}) \text{ if } n \text{ is odd and } i = 0. 
\end{cases}
\]

- And if $n$ is even, we also include 

  $L'_0 = \{ v \in V : -v_1 = v_2 = \cdots = v_n \}$.

  We have $\text{Fix}(L'_0) = \text{Stab}(L'_0) = A_{n-1}$. 

Case of $D_n$

Let $d_n = K(D_n)$ and $\bar{d}_n = \#(\mathcal{M}(D_n)/B_n)$. The recursion for $d_n$ is:

$$d_n = 2a_{n-1} + \sum_{i=2}^{n-1} \binom{n-1}{i} d_i a_{n-1-i}$$

if $n$ is even, and

$$d_n = a_{n-1} + \sum_{2 \leq i \leq n-1 \atop n-i \text{ even}} \binom{n-1}{i} d_i a_{n-1-i} + \sum_{2 \leq i \leq n-1 \atop n-i \text{ odd}} \binom{n-1}{i} \bar{d}_i a_{n-1-i}$$

if $n$ is odd.

We need to compute $\bar{d}_n = \#(\mathcal{M}(D_n)/B_n)$.

If $n$ is odd, $B_n = D_n \rtimes \{\pm Id\}$ so $\bar{d}_n = d_n$. 

The scheme for computing $K(W)$ works for $\tilde{d}_n$ too and gives:

$$\tilde{d}_n = a_{n-1} + \sum_{i=2}^{n-1} \binom{n-1}{i} \tilde{d}_i a_{n-1-i}.$$

Let $\tilde{D}(z) = \sum_{n \geq 0} \tilde{d}_n \frac{z^n}{n!}$, the recursion is equivalent to:

$$\tilde{D}'(z) = (\tilde{D}(z) - z)A(z), \quad D(0) = 1.$$

This is solved by

$$\tilde{D}(z) = \frac{2 - \cos(z) - z \sin(z)}{1 - \sin(z)}.$$

It follows $\tilde{d}_n = 2T_{n+1} - (n + 1)T_n$ if $n \geq 2$. So for odd $n \geq 2$, we have $d_n = 2T_{n+1} - (n + 1)T_n$. 
From the recursions for $\bar{d}_n$ and $d_n$, we have for even $n$:

$$(d_n - \bar{d}_n) = a_{n-1} + \sum_{i=2}^{n-1} (d_n - \bar{d}_n)a_{n-i-1}.$$

Let $U(z) = 1 + \sum_{n \geq 2} (d_n - \bar{d}_n) \frac{z^n}{n!}$, the recursion is equivalent to:

$$U'(z) = U(z) \tan(z), \quad U(0) = 1.$$

This is solved by $U(z) = \sec(z)$.
So for even $n \geq 2$ we have $d_n = (d_n - \bar{d}_n) + d_n = 2T_{n+1} - nT_n$. 
Remaining cases

Dihedral groups:

\[ K(l_2(m)) = \begin{cases} 
1 & \text{if } n \text{ is odd}, \\
2 & \text{if } n \text{ is even}.
\end{cases} \]

Exceptional groups: one method is to see them as symmetry groups of some semiregular polytopes, and use the geometry of these polytopes.

\[ K(H_3) = 4, \quad K(H_4) = 12, \quad K(F_4) = 16, \]
\[ K(E_6) = 82, \quad K(E_7) = 768, \quad K(E_8) = 4056. \]

In all cases except \( E_6 \), the polytope is centrally symmetric and this ensures we have \( \text{Stab}(L_i) = \text{Fix}(L_i) \rtimes \{ \pm \text{Id} \} \).
The general method to find orbit representatives of lines in $\mathcal{P}(W)$ is the following. Let $H_1, \ldots, H_n$ so that the orthogonal reflections are simple generators and

$$L_i = \bigcap_{j \neq i} H_j.$$ 

Then $L_1, \ldots, L_n$ are representatives, except that if $L_i = w_0(L_j)$ we only take one of $L_i$ and $L_j$ ($w_0$ is the longest element).
thanks for your attention