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COMBINATORIAL HOPF ALGEBRA FOR THE BEN GELOUN–RIVASSEAU TENSOR FIELD THEORY

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Abstract. The Ben Geloun–Rivasseau quantum field theoretical model is the first tensor model shown to be perturbatively renormalizable. We define here an appropriate Hopf algebra describing the combinatorics of this new tensorial renormalization. The structure we propose is significantly different from the previously defined Connes–Kreimer combinatorial Hopf algebras due to the involved combinatorial and topological properties of the tensorial Feynman graphs. In particular, the 2- and 4-point function insertions must be defined to be non-trivial only if the superficial divergence degree of the associated Feynman integral is conserved.

1. Introduction and Motivation

The present article concerns the combinatorics of renormalization of tensor field theory (TFT) models. TFT models are a path integral formulation of quantum field theories (QFT) whose Feynman graphs correspond combinatorially to discrete geometries. In the perturbative expansion of the partition function of a TFT model, with each such Feynman graph one associates a probability (called the Feynman amplitude in QFT), and accordingly we may understand the TFT model as a quantum model of discrete geometry. Recently, the first renormalizable rank four TFT model with Lie group-valued field variables was formulated and studied by Ben Geloun and Rivasseau [3, 4]. Since then, the renormalizability of several other TFT models has been established [1, 2, 5, 6, 15].

On the combinatorial level of Feynman graphs, the renormalizability property can be phrased in a purely algebraic way, as was shown by Connes and Kreimer [7]. This was followed by the formulation in terms of a Riemann–Hilbert correspondence for categories of differential systems by Connes and Marcolli [8, 9].

The Connes–Kreimer combinatorial approach to renormalization was initially applied to local QFT models. When one generalizes to QFT models on non-commutative spaces, such as the Moyal space, locality is lost due to the quantum uncertainty relations between space coordinates. However, it was shown in [16, 17] that, by replacing the notion of locality with a new concept, the so-called 'Moyality' property for interactions, the Hopf algebraic structure of Feynman graphs allowing for the renormalization of particular non-commutative models can be established. The Connes–Kreimer formalism of matrix field theory, whose two-stranded ribbon Feynman graphs are dual to two-dimensional triangulated surfaces, was also considered in [13]. In addition, Hopf algebraic structures of quantum gravity spin foams, closely related to TFT models, have been considered in [18]. The Hopf algebra of tensorial Feynman graphs of group field theory and the associated Schwinger–Dyson equations have also been considered in [12].

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The goal of the current paper is to continue the above direction of research, by formulating the combinatorial Connes–Kreimer Hopf algebraic structure of the Ben Geloun–Rivasseau (BGR) model. Let us emphasize that this constitutes the first application of the Connes–Kreimer combinatorial renormalization to a TFT model of rank-D for $D > 2$.

We note that a first important difference to the usual case of local QFT is the structure of external constraints; namely, the momentum conservation of local QFT is replaced by the so-called tensorial invariance. Moreover, the implementation of the Connes–Kreimer formalism requires particular attention at the level of the n-point function insertions. This follows from the fact that the divergences of the BGR n-point functions are not only dependent on the external topological data of the Feynman graph, as in ordinary local QFT models. Thus, only when the superficial divergence degree is conserved, a BGR tensor graph insertion is defined to be non-trivial. This is a crucial difference with respect to the previously defined Connes–Kreimer structures for local QFT.

2. Connes–Kreimer algebra for local QFT

2.1. **Feynman graphs of quantum field theory.** The basic feature of any QFT model is that it gives a prescription for associating quantum probability amplitudes (taking values in $\mathbb{C}$) with physical processes. In the path integral formulation of a QFT model, these amplitudes can be extracted from the generating functional $\mathcal{W}$ of the model, that is typically expressed as an infinite dimensional functional integral of the type

$$\mathcal{W}[J] = \int \mathcal{D}\phi \; e^{-S[\phi]+\phi\cdot J} \; ,$$

where ϕ and J denote some fields (usually taken to be smooth sections of a fiber bundle), or collections of fields, on a fixed background manifold X. $S[\phi]$ is called the action functional defining the model, and the integral measure can typically be informally written as an infinite product of Lebesgue measures over the field values at each point in X, $\mathcal{D}\phi := \prod_{x\in X} \mathrm{d}\phi(x)$. Then the correlation functions $\langle\cdots\rangle$ for the monomials in the field variables, which correspond to the basic physical observables of the model, can be written as functional derivatives of the generating functional,

$$\langle\phi(x_1)\cdots\phi(x_n)\rangle \equiv \int \mathcal{D}\phi \; \phi(x_1)\cdots\phi(x_n)e^{-S[\phi]} = \frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)}\mathcal{W}[J]\Big|_{J=0} \; . \tag{2.1}$$

The well-definedness of such an expression is, of course, highly questionable. Indeed, apart from rare exceptions, these expressions are highly divergent.

Nevertheless, one can extract finite quantities from these formal divergent expressions, which have been shown to match experimental results with remarkable accuracy. The main tool in this direction is perturbation theory. Typically the action functional is of the form $S[\phi] = S_{\mathrm{kin}}[\phi] + S_{\mathrm{int}}[\lambda;\phi]$, where $S_{\mathrm{kin}}[\phi]$ is quadratic in the field variables, and the higher order polynomial interaction part $S_{\mathrm{int}}[\lambda;\phi]$ is proportional to a collection of parameters λ, the coupling constants of the model. Thus, the kinetic part $S_{\mathrm{kin}}[\phi]$ by itself gives a simple (infinite dimensional) Gaussian measure, whose correlation functions can be computed explicitly. One can then expand the integrand in Equation (2.1) in powers of the coupling constant(s) λ, and in each finite order one ends up with a sum over products of correlation functions of the Gaussian measure with given weights. It is these terms in the perturbative expansion, and their combinatorial structure in particular, for which Feynman graphs provide a very convenient book-keeping device. The contribution to the full correlation function associated with a particular Feynman graph is called the corresponding Feynman amplitude, and can be obtained from the graph by applying the set of Feynman rules of the particular model. Any finite order contribution is then obtained as a sum over the amplitudes of a finite number of Feynman graphs.

However, individual terms in this perturbative series are generically divergent. This particular problem is addressed by the framework of perturbative renormalization. The Connes–Kreimer combinatorial Hopf algebra of Feynman graphs is aimed at organizing the renormalization of the individual Feynman amplitudes in the perturbative expansion in a coherent unifying algebraic structure.

Let us give a standard example of the general points discussed above about the relationship of Feynman graphs to QFT models: the $\lambda\phi^4$ scalar field model on $\mathbb{R}^4$ (with Euclidean metric). This model is determined by the action functional $S[\phi] = S_{\text{kin}}[\phi] + S_{\text{int}}[\lambda;\phi]$, where

$$S_{\text{kin}}[\phi] = \frac{1}{2}\int_{\mathbb{R}^4} d^4x\ \phi(\Delta + m^2)\phi \quad \text{and} \quad S_{\text{int}}[\lambda;\phi] = \frac{\lambda}{4!}\int_{\mathbb{R}^4} d^4x\ \phi^4\ . \tag{2.2}$$

Here, the single field ϕ takes values in $\mathbb{R}$, and the parameters of the model, $m, \lambda \in \mathbb{R}_+$, correspond to the mass of the particle species modelled and the coupling strength, respectively. Δ denotes the Laplace operator. The interaction is local in the sense that it couples the field values only at the same point in space. It is usual to consider the model in the momentum space, obtained through Fourier transform, instead of the direct space, since due to the translation symmetry of the model the total momentum is conserved. In momentum space, the action (2.2) reads explicitly

$$S_{\text{kin}}[\tilde\phi] = \frac{1}{2}\int_{\mathbb{R}^4} \mathrm{d}^4p\ \tilde\phi(p)(|p|^2 + m^2)\tilde\phi(p)\,, \tag{2.3}$$

$$S_{\text{int}}[\lambda;\tilde\phi] = \frac{\lambda}{4!}\int_{(\mathbb{R}^4)^4} \big[\prod_{i=1}^{4}\mathrm{d}^4p_i\big]\tilde\phi(p_1)\tilde\phi(p_2)\tilde\phi(p_3)\tilde\phi(p_4)\delta^4(p_1+p_2+p_3+p_4)\,, \tag{2.4}$$

where $\tilde\phi(p) := \int_{\mathbb{R}^4}\mathrm{d}^4x\ e^{-ip\cdot x}\phi(x)$ is the Fourier transform of the field ϕ. In the interaction term, the Dirac delta distribution $\delta^4(p_1+p_2+p_3+p_4)$ imposes the so-called *momentum conservation*. The covariance of the kinetic part, i.e., the (free) *propagator*, can be explicitly computed to yield

$$\mathcal{P}(p_1,p_2) := \langle\tilde\phi(p_1)\tilde\phi(p_2)\rangle_{\text{kin}} = \int \mathcal{D}\tilde\phi\ \tilde\phi(p_1)\tilde\phi(p_2)e^{-S_{\text{kin}}[\tilde\phi]} = \frac{1}{|p_1|^2+m^2}\delta^4(p_1-p_2)\,. \tag{2.5}$$

Moreover, due to Wick's theorem, any correlation function of such Gaussian measure can be expressed as a sum of products of propagators. For example, for the so-called four-point function, we have

$$\begin{aligned}\langle\tilde\phi(p_1)\tilde\phi(p_2)\tilde\phi(p_3)\tilde\phi(p_4)\rangle_{\text{kin}} = {}& \langle\tilde\phi(p_1)\tilde\phi(p_2)\rangle_{\text{kin}}\langle\tilde\phi(p_3)\tilde\phi(p_4)\rangle_{\text{kin}} + \langle\tilde\phi(p_1)\tilde\phi(p_3)\rangle_{\text{kin}}\langle\tilde\phi(p_2)\tilde\phi(p_4)\rangle_{\text{kin}}\\ &+ \langle\tilde\phi(p_1)\tilde\phi(p_4)\rangle_{\text{kin}}\langle\tilde\phi(p_2)\tilde\phi(p_3)\rangle_{\text{kin}}.\end{aligned} \tag{2.6}$$

For a monomial of odd degree, the correlation function vanishes, since the Gaussian measure is symmetric. The QFT perturbation expansion is based on this fact. For example, the k-th order contribution to the full correlation function of n field values,

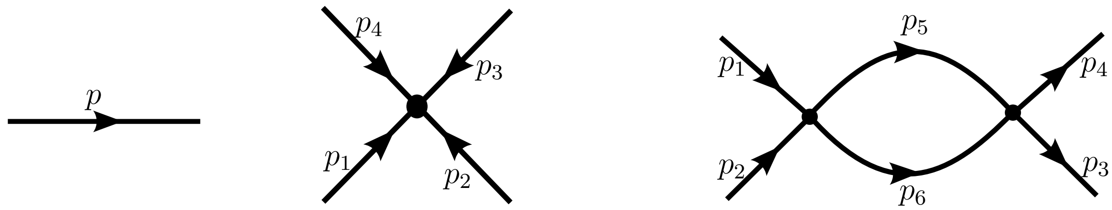

FIGURE 1. On the right: the Feynman graph representation of the propagator and the 4-point-interaction for the $\lambda\phi^4$-model. On the left: an example of a second order Feynman graph with four external edges.

the *n-point function*, is obtained by expanding the interaction part of the exponential integrand, and picking out the k-th order term

$$\langle\tilde{\phi}(p_1)\tilde{\phi}(p_2)\cdots\tilde{\phi}(p_n)\rangle_k = \frac{(-1)^k}{k!}\int \mathcal{D}\tilde{\phi}\ \tilde{\phi}(p_1)\tilde{\phi}(p_2)\cdots\tilde{\phi}(p_2)(S_{\text{int}}[\lambda;\tilde{\phi}])^k e^{-S_{\text{kin}}[\tilde{\phi}]}\,.$$

By substituting the explicit expression for the interaction term (2.4), and, writing all the resulting correlation functions of the free theory in terms of propagators, one can explicitly calculate this k-th order contribution to the n-point function of the theory. In QFT, the physical interpretation of this quantity is that it gives the (non-normalized) probability amplitude for a scattering process from an initial state with m free particles having momenta $(p_i)_{i=1,\dots,m}$ to a final state with $n-m$ free particles having momenta $(p_i)_{i=m+1,\dots,n}$.

We now introduce Feynman graphs in QFT models formally.

Definition 2.1. A *Feynman graph* Γ of a QFT model is a graph that consists of a set of vertices $\Gamma^{[0]}$ and a set of edges $\Gamma^{[1]}$. The edges $\Gamma^{[1]}_{\text{int}} \subseteq \Gamma^{[1]}$ that are connected to vertices (or to the same vertex) at both ends are called *internal*, while the remaining edges $\Gamma^{[1]}_{\text{ext}} \subseteq \Gamma^{[1]}$ are connected to one vertex at one end (the other end of the edge not being connected to any vertex) and they are called *external*. The external edges $e_i^{ext} \in \Gamma^{[1]}_{\text{ext}}$ are labelled by incoming external momenta $e_i^{ext} \mapsto \vec{p}_i$. The external momenta obey the momentum conservation $\sum_i \vec{p}_i = 0$. We denote the space of the external momenta of Γ by P_Γ. We denote the set of all Feynman graphs of a model by $\mathcal{G}$.

Let us briefly describe the Feynman graphs and the associated Feynman rules of the $\lambda\phi^4$-model. Each Feynman graph corresponds to an individual summand in the perturbative expansion of the correlation functions with respect to λ. The Feynman graphs of the $\lambda\phi^4$-model consist of a single type of edges (the propagator) labelled by a momentum variable and four-valent vertices (the interaction), as shown in Figure 1. The number of vertices in a Feynman graph equals the order of the corresponding summand in the perturbative expansion. An arrow on an edge denotes the direction of the momentum flow p, and can be assigned arbitrarily; the opposite orientation corresponds to $-p$. In the following we will often just drop the momentum labels and arrows, since their specific assignment is immaterial — they can always be reintroduced arbitrarily. The Feynman amplitude associated with a particular Feynman graph is constructed as follows:

(1) Let p_l be the momenta labels on the edges $l \in \Gamma$ of the Feynman graph Γ. We form the product of free propagators

$$\prod_{l\in\Gamma} \frac{1}{|p_l|^2 + m^2} \,.$$

(2) For each vertex $v \in \Gamma$, multiply by the coupling constant λ, and impose momentum conservation by multiplying the above product by

$$\prod_{v\in\Gamma} \delta^4\left(\sum_{l\sim v} \sigma_{vl} p_l\right) ,$$

where $l \sim v$ symbolizes that the edge l is connected to the vertex v, and $\sigma_{vl} = \pm 1$ according to whether p_l is incoming or outgoing with respect to the vertex v.

(3) Integrate over the p_l's which are associated with the internal edges of the graph.

The quantity that results from applying these rules to a Feynman graph is the Feynman amplitude of that graph.

For any Feynman graph, the amplitude factorizes into the product of the amplitudes of its connected components, so one can just focus on connected Feynman graphs. On the level of the generating functional, the disconnected graphs can be removed by replacing $\mathcal{W}[J]$ with the logarithm $\log \mathcal{W}[J]$ in (2.1). Moreover, it is common to restrict oneself to consider the so-called one-particle-irreducible (1PI) Feynman graphs, or what are called bridgeless graphs in graph theory language.

Definition 2.2. A Feynman graph Γ is called *one-particle-irreducible* (1PI), if $|\Gamma^{[1]}_{\text{ext}}| > 1$ and it is bridgeless, i.e., it cannot be disconnected by removing a single edge. We also require $|\Gamma^{[1]}_{\text{int}}| \geq 1$. The set of all 1PI Feynman graphs is denoted by $\mathcal{G}_{\text{1PI}}$.

If there would exist a bridge, i.e., an edge connecting two separate components of the graph, the momentum conservation would fix the momentum on this edge. It would then be trivial to consider the amplitude as the product of the amplitudes of the two contributions coming from this two separate components.

For example, by applying the above Feynman rules to the Feynman graph on the right-hand side of Figure 1, we obtain for the corresponding Feynman amplitude the expression

$$\lambda^2 \int_{\mathbb{R}^4} \mathrm{d}^4 p_5 \int_{\mathbb{R}^4} \mathrm{d}^4 p_6 \left[\prod_{l=1}^{6} \frac{1}{|p_l|^2 + m^2}\right] \delta^4(p_1 + p_2 - p_5 - p_6)\delta^4(p_5 + p_6 - p_3 - p_4)$$
$$= \left[\prod_{l=1}^{4} \frac{1}{|p_l|^2 + m^2}\right] \delta^4(p_1 + p_2 - p_3 - p_4) \times \lambda^2 \int_{\mathbb{R}^4} \mathrm{d}^4 p_5 \frac{1}{|p_5|^2 + m^2} \frac{1}{|p_1 + p_2 - p_5|^2 + m^2} \,,$$

where we integrated over p_6. The first part of the bottom line contains the propagators and momentum conservation associated with the external edges of the graph. The second part corresponds to the internal structure, in particular, to the loop formed by edges 5 and 6 that allows for one free momentum integration, unconstrained by the momentum conservation at the vertices. This last integral

$$\lambda^2 \int \mathrm{d}^4 p_5 \frac{1}{|p_5|^2 + m^2} \frac{1}{|p_1 + p_2 - p_5|^2 + m^2}$$

is divergent when one wants to integrate for the different values of p_5. If one integrates only over momenta p_5 with $|p_5| \leq \Lambda$, where $\Lambda > 0$ is a cutoff parameter, one obtains

$$\lambda^2 \int_{|p_5| \leq \Lambda} \frac{\mathrm{d}|p_5|^3 |p_5|}{(|p_5|^2 + m^2)^2}$$

(up to some irrelevant constant coming from integrating over the angular variables of the 4-dimensional volume element in spherical coordinates). Finally, this last integral behaves like $\log \Lambda$, which diverges when taking the limit $\Lambda \to \infty$. Therefore, the momentum integration related to the loop is logarithmically divergent for large momentum — thus it exhibits the so-called ultraviolet divergences of QFT.

This is where renormalization comes in (when possible). In order to establish renormalizability, of paramount importance is the so-called *power counting theorem*. Denote by N_{int} and V the number of internal edges and the number of vertices, respectively. Then, the number of independent momentum integrations is $N_{int} - V + 1$, and thus the superficial divergence degree in this case reads $\omega := 4(N_{int} - V + 1) - 2N_{int} = 2N_{int} - 4(V - 1)$. On the other hand, we have $4V = 2N_{int} + N_{ext}$, where N_{ext} is the number of external edges, so $\omega = 4 - N_{ext}$. This is the power counting theorem for the $\lambda\phi^4$-model. Due to this theorem, only graphs with two or four external edges are proven to be *superficially divergent*.

These considerations prompt us to give the following definitions concerning the analytic aspects of Feynman amplitudes associated with Feynman graphs.

Definition 2.3. The *regularized Feynman rules* of a local QFT model is a map $\Gamma \mapsto \phi_\Lambda(\Gamma) \in E_\Gamma$ for $\Gamma \in \mathcal{G}$, where E_Γ stands for the linear vector space of generalized functions on the space P_Γ of the external parameters $\vec{p}_i$ associated with the external edges $e_i^{ext} \in \Gamma_{\text{ext}}^{[1]}$ of Γ. We call $\phi_\Lambda(\Gamma)$ the *regularized Feynman amplitude* of the graph $\Gamma \in \mathcal{G}$. The parameter $\Lambda \in \mathbb{R}_+$ is the *regularization cut-off*, such that in the limit $\Lambda \to \infty$ we recover the unregularized Feynman rules $\Gamma \mapsto \phi(\Gamma)$.

Remark 2.1. In the following, we will consider the propagators associated with the external edges of Γ not to be included in $\phi(\Gamma)$. Otherwise, one can also simply factor these out of the Feynman amplitudes.

Definition 2.4. A *renormalization scheme* is specified by a linear operator R, which extracts the divergent part of a Feynman amplitude, so that $\lim_{\Lambda \to \infty} \rho_\Lambda(\Gamma) - R(\rho_\Lambda(\Gamma)) < \infty$.

Remark 2.2. There is inherent ambiguity in choosing the renormalization scheme, but all the schemes should lead to the same results, when only differences of amplitudes are considered. Also in practice, there are many different choices for implementing the regularization. In order to define the R-operator, one may consider the regularized Feynman amplitudes to lie in the space of formal Laurent series in a cut-off parameter $\epsilon \sim \Lambda^{-1}$, and then the R-operator projects onto the series with negative powers of ϵ. Nevertheless, most renormalization schemes used in QFT are not specified by a Rota–Baxter operator (for example, this is the case of the celebrated BPHZ scheme). The interested reader is referred to [10] for details.

Definition 2.5. A *superficial divergence degree* is a map $\omega : \mathcal{G} \to \mathbb{Z}$ such that any Feynman graph Γ with $\lim_{\Lambda \to \infty} \rho_\Lambda(\Gamma) < \infty$ satisfies $\omega(\gamma) < 0$ for all $\gamma \subset \Gamma$. Let us call

$\Gamma \in \mathcal{G}_{\mathrm{1PI}}$ with $\omega(\Gamma) \geq 0$ *superficially divergent*, and write $\mathcal{G}_{sd}^{\omega} := \{\Gamma \in \mathcal{G}_{\mathrm{1PI}} : \omega(\Gamma) \geq 0\}$ for the set of superficially divergent 1PI Feynman graphs with respect to ω.

Remark 2.3. Such a superficial divergence degree is usually obtained via a *power counting theorem*, which relates features of the combinatorial and topological structure of Feynman graphs to the divergence of the corresponding Feynman amplitudes. Notice that the superficial divergence degree only sets bounds on the divergence. That is, not all superficially divergent graphs necessarily have divergent Feynman amplitudes. Conversely, a superficially convergent graph may have divergent subgraphs.

2.2. **Hopf algebra of Feynman graphs.** Let us first recall a few definitions of basic algebraic structures.

Definition 2.6. A (unital associative) *algebra* $\mathcal{A}$ over a field $\mathbb{K}$ is a $\mathbb{K}$-linear space endowed with the following two linear homomorphisms:

a product: $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ such that (associativity)

$$m \circ (m \otimes \mathrm{id})(\Gamma) = m \circ (\mathrm{id} \otimes m)(\Gamma) \quad \text{for all } \Gamma \in \mathcal{A}^{\otimes 3}, \tag{2.7}$$

a unit: $u : \mathbb{K} \to \mathcal{A}$ such that

$$m \circ (u \otimes \mathrm{id})(1 \otimes \Gamma) = \Gamma = m \circ (\mathrm{id} \otimes u)(\Gamma \otimes 1) \quad \text{for all } \Gamma \in \mathcal{A}. \tag{2.8}$$

Definition 2.7. A (counital coassociative) *coalgebra* $\mathcal{C}$ over a field $\mathbb{K}$ is a $\mathbb{K}$-linear space endowed with the following two linear homomorphisms:

a coproduct: $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ such that (coassociativity)

$$(\Delta \otimes \mathrm{id}) \circ \Delta(\Gamma) = (\mathrm{id} \otimes \Delta) \circ \Delta(\Gamma) \quad \text{for all } \Gamma \in \mathcal{A}, \tag{2.9}$$

a counit: $\epsilon : \mathcal{A} \to \mathbb{K}$ such that

$$(\epsilon \otimes \mathrm{id}) \circ \Delta(\Gamma) = \Gamma = (\mathrm{id} \otimes \epsilon) \circ \Delta(\Gamma) \quad \text{for all } \Gamma \in \mathcal{A}. \tag{2.10}$$

Definition 2.8. A *bialgebra* $\mathcal{B}$ over a field $\mathbb{K}$ is a $\mathbb{K}$-linear space endowed with both an algebra and a coalgebra structure such that the coproduct and the counit are unital algebra homomorphisms (or, equivalently, the product and the unit are coalgebra homomorphisms), namely,

$$\begin{aligned} \Delta \circ m_{\mathcal{B}} &= m_{\mathcal{B}\otimes\mathcal{B}} \circ (\Delta \otimes \Delta), \quad \text{such that } \Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}, \\ \epsilon \circ m_{\mathcal{B}} &= m_{\mathbb{K}} \circ (\epsilon \otimes \epsilon), \quad \text{such that } \epsilon(\mathbf{1}) = 1, \end{aligned}$$

where we wrote $\mathbf{1} := u(1)$ for the unit in $\mathcal{B}$.

Definition 2.9. A (positively) *graded bialgebra* is a bialgebra $\mathcal{B}$, which is graded as a linear space,

$$\mathcal{B} = \bigoplus_{n=0}^{\infty} \mathcal{B}^{(n)}, \tag{2.11}$$

so that the grading is compatible with the algebra and coalgebra structures, that is,

$$\mathcal{B}^{(n)}\mathcal{B}^{(m)} \subseteq \mathcal{B}^{(n+m)} \quad \text{and} \quad \Delta(B^{(n)}) \subseteq \bigoplus_{k=0}^{n} \mathcal{B}^{k} \otimes \mathcal{B}^{(n-k)}. \tag{2.12}$$

Definition 2.10. A *connected bialgebra* is a graded bialgebra $\mathcal{B}$, for which $\mathcal{B}^{(0)} = u(\mathbb{K})$.

Definition 2.11. A *Hopf algebra* $\mathcal{H}$ over a field $\mathbb{K}$ is a bialgebra over $\mathbb{K}$, which is equipped with an antipode map $S : \mathcal{H} \to \mathcal{H}$ satisfying

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = u \circ \epsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta. \tag{2.13}$$

A fundamental result is the following.

Lemma 2.1 ([14], Corollary II.3.2)**.** *A connected graded bialgebra is a Hopf algebra. The antipode map $S : \mathcal{H} \to \mathcal{H}$ may be obtained through the recursive formula*

$$S(\Gamma) = -\Gamma - m \circ (\mathrm{id} \otimes S) \circ \Delta'(\Gamma), \tag{2.14}$$

or, alternatively,

$$S(\Gamma) = -\Gamma - m \circ (S \otimes \mathrm{id}) \circ \Delta'(\Gamma), \tag{2.15}$$

by setting $S(\mathbf{1}) = \mathbf{1}$.

We now define the corresponding operations on a set of Feynman graphs of a given QFT model, where the $\lambda\phi^4$ scalar field model has already been introduced in the previous subsection.

Definition 2.12. The *residue* $\mathrm{Res}(\Gamma)$ of $\Gamma \in \mathcal{G}$ is the graph obtained by contracting all internal edges of Γ to a point (or points, one for each connected component, if Γ is disconnected). Clearly, $P_{\mathrm{Res}(\Gamma)} = P_\Gamma$, since any Feynman graph of a local QFT model satisfies momentum conservation.

Definition 2.13. A graph $\gamma \in \mathcal{G}$ is a (proper) *subgraph* of a graph $\Gamma \in \mathcal{G}$, denoted by $\gamma \subseteq \Gamma$ ($\gamma \subsetneq \Gamma$), if its internal edges are a (proper) subset of the internal edges of Γ. Two subgraphs are *disjoint*, if they do not share any internal edges.

Let us now introduce an algebraic structure on the set $\mathcal{H}$ of disjoint unions of 1PI Feynman graphs of a local QFT model. Consider the associative commutative product $m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ of graphs as given by the disjoint union $m(\Gamma \otimes \Gamma') = \Gamma \dot{\cup} \Gamma'$. The unit with respect to this product is given by $u : \mathbb{C} \to \mathcal{H}$ such that $u(1) = \mathbf{1}$, where $\mathbf{1} \in \mathcal{H}$ denotes the empty graph. Thus, $\mathcal{H} \equiv \mathbb{C}[\mathbf{1}, \mathcal{G}_{1\mathrm{PI}}]$, the $\mathbb{C}$-module of polynomials generated by the elements of $\mathcal{G}_{1\mathrm{PI}}$ and the empty graph $\mathbf{1}$, equipped with the above operations linearly extended to $\mathcal{H}$, constitutes a unital associative algebra.

Let us next define a coalgebra structure on $\mathcal{H}$. First we need to define the operation of subgraph contraction.

Definition 2.14. Define the operation of a *proper subgraph contraction* as follows: For $\gamma \subsetneq \Gamma$ such that $\mathrm{Res}(\gamma)$ corresponds to an interaction vertex of the model, let $\Gamma/\gamma \in \mathcal{G}$ be the graph that is obtained from Γ by replacing $\gamma \subset \Gamma$ by $\mathrm{Res}(\gamma)$ inside Γ. This definition of proper subgraph contraction has an obvious extension to the case of γ being a disjoint union $\gamma = \bigcup_{i=1}^n \gamma_i$ of disjoint proper subgraphs $\gamma_i \subset \Gamma$ ($|(\gamma_i)^{[1]}_{int} \cap (\gamma_j)^{[1]}_{int}| = 0$), such that $\Gamma/\gamma \equiv (\cdots((\Gamma/\gamma_1)/\gamma_2)\cdots/\gamma_n)$ is obtained from Γ by replacing all γ_i by $\mathrm{Res}(\gamma_i)$ inside Γ.

We now introduce an operation Δ, acting on the generators $\Gamma \in \mathcal{G}$ of $\mathcal{H}$ by

$$\Delta(\Gamma) = \Gamma \otimes \mathbf{1} + \mathbf{1} \otimes \Gamma + \sum_{\substack{\gamma \in \bigcup \mathcal{G}^\omega_{sd} \\ \gamma \subsetneq \Gamma}} \gamma \otimes \Gamma/\gamma\,.$$

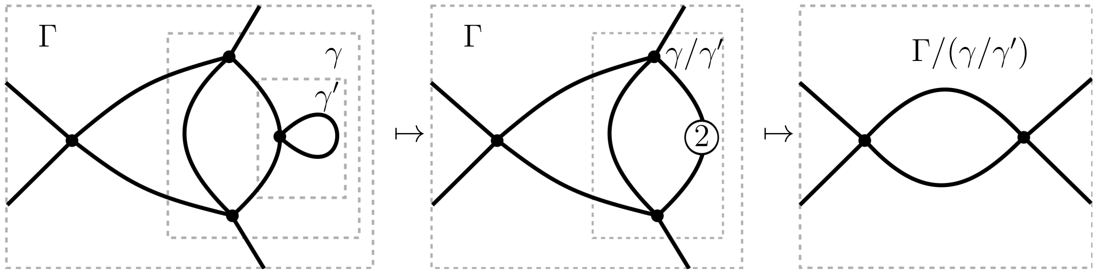

FIGURE 2. An example of subgraph contractions for the $\lambda\phi^4$-model.

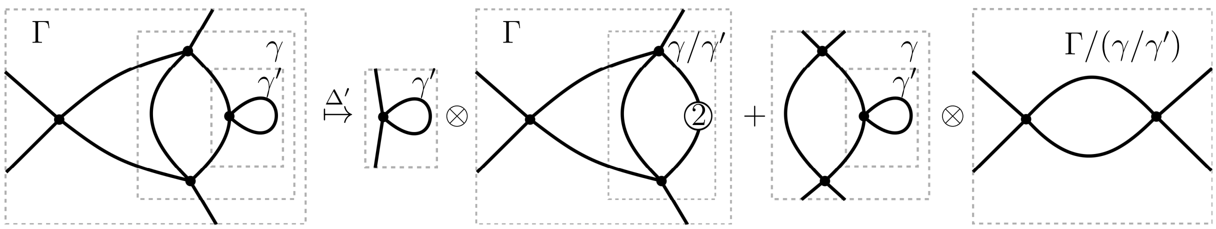

FIGURE 3. An example of the coproduct structure for the $\lambda\phi^4$-model.

Notice that here the sum runs over all disjoint unions $\gamma = \bigcup_i \gamma_i \in \bigcup \mathcal{G}^{\omega}_{sd}$ of superficially divergent disjoint 1PI subgraphs γ_i of Γ. Since Δ is an algebra homomorphism, it may be extended to $\mathcal{H}$. We have the following fact.

Lemma 2.2 ([8], THEOREM 1.27). *$\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ is a coassociative coproduct.*

The counit of $\mathcal{H}$ with respect to Δ is given by the linear extension of $\epsilon : \mathcal{G} \to \mathbb{C}$, where $\epsilon(\mathbf{1}) = 1$ and $\epsilon(\Gamma) = 0$ for all $\Gamma \neq \mathbf{1}$. $\mathcal{H}$ equipped with the above structure thus forms a bialgebra.

Moreover, we have a natural grading for the elements of $\mathcal{H}$ given by the number of internal edges, which is compatible with the coproduct Δ. Clearly, $\mathcal{H}^{(0)} = \mathbb{C}\mathbf{1} = u(\mathbb{C})$. Thus, $\mathcal{H}$ constitutes a graded connected bialgebra. Accordingly, we have the following result.

Theorem 2.1 ([8], THEOREM 1.27). *$(\mathcal{H}, u, m, \epsilon, \Delta, S)$ is a Hopf algebra, where the antipode S is given by the formula* (2.14) *(or* (2.15)*).*

Consequently, the combinatorial algebraic properties of Feynman graphs and amplitudes join their forces in the following theorem by Connes and Kreimer [7].

Theorem 2.2 ([7]). *For a local perturbatively renormalizable QFT model, the renormalized Feynman amplitudes are given by the formula*

$$\phi_R(\Gamma) = S_R^{\phi}(\phi(\Gamma)) \star \phi(\Gamma)\,, \tag{2.16}$$

where S_R^{ϕ} is given recursively by

$$S_R^{\phi}(\phi(\Gamma)) = -R[\phi(\Gamma)] - R\left[\sum_{\substack{\gamma \subsetneq \Gamma \\ \gamma \in \mathcal{G}^{\omega}_{sd}}} S_R^{\phi}(\phi(\gamma))\phi(\Gamma/\gamma)\right]. \tag{2.17}$$

Here, the R-operator defines the corresponding renormalization scheme. Notice that we have $S_R^\phi(\phi(\Gamma)) = -R(\phi(\Gamma))$ *for* Γ *without superficially divergent subgraphs, which facilitates the recursion.*

Finally, we describe *subgraph insertion*, the operation which is dual to subgraph contraction.

Definition 2.15. Let $\Gamma \in \mathcal{G}$ and $v \in \Gamma^{[0]}$. Then, for $\gamma \in \mathcal{G}^\omega_{sd}$ such that $\mathrm{Res}(\gamma) \sim v$ (which means that $\mathrm{Res}(\gamma)$ is of the type of v), we define the subgraph insertion at vertex v, denoted by $\gamma \circ_v \Gamma$, as the graph obtained from Γ by replacing v by γ. We also define

$$\gamma \circ \Gamma = \sum_{\substack{v \in \Gamma^{[0]} \\ v \sim \mathrm{Res}(\gamma)}} \gamma \circ_v \Gamma\,. \tag{2.18}$$

Notice that, in general, there may be several inequivalent ways of inserting γ into a vertex v of Γ. In this case we must provide additional gluing data and sum also over all different ways of insertion in (2.18).

In the $\lambda\phi^4$-model considered above, Feynman graphs consist originally of a single type of edge and a single type of 4-valent vertex. The external edges are labelled by the external 4-momentum variables $\vec{p}_i$, which for any connected component of a Feynman graph satisfy the momentum conservation constraint $\sum_i \vec{p}_i = 0$. It is important in view of subgraph insertions and contractions to note the almost trivial fact that any graph consisting of vertices, which satisfy the momentum conservation constraint, and propagators, which preserve momentum, satisfies itself the momentum conservation constraint.

As we have already seen, the superficial degree of divergence for a Feynman graph Γ of the $\lambda\phi^4$-model is given by the formula $\omega = 4 - N_{ext}$, where $N_{ext} := |\Gamma^{[1]}_{ext}|$ denotes the cardinality of $\Gamma^{[1]}_{ext}$. Thus, only graphs with two or four external edges need to be renormalized. Accordingly, we must consider in addition two types of 2-valent vertices, corresponding to mass and wave function renormalization counter-terms, which arise as residues of superficially divergent graphs. We decorate them with the label "2", as an indication of the quadratic character of the corresponding divergence. The notion of external structure in local QFT is introduced in order to distinguish the contributions of 2-point functions to mass and wave-function renormalization.

Definition 2.16. The *external structure* of a Feynman graph Γ is specified by the action of a set of distributions $\{\sigma_v \in E^*_\Gamma\}$ labelled by the vertices of the model, such that $\langle \sigma_v, \phi(\Gamma)\rangle = \rho_v(\Gamma)\langle \sigma_v, \phi(\mathrm{Res}(\Gamma))\rangle$, where the $\rho_v(\Gamma) \in E_\Gamma$ are characters of $\mathcal{H}$, and $\langle \sigma_v, \phi(\mathrm{Res}(\Gamma))\rangle$ is the kernel of the interaction functional associated with v.

However, we do not make a distinction here at the level of graph drawing. For a graph with two external edges, its contributions to wave function and mass renormalization are given by the external structures for each corresponding vertex. In Figures 2 and 3 we have provided some basic examples of the subgraph contraction operation and the coproduct, respectively, for the $\lambda\phi^4$-model.

Before ending this section, let us emphasize the following fact of particular importance in the sequel. The 2-valent vertices are associated with a quadratic divergence, since the power-counting simply gives $\omega = 4 - N_{ext} = 2$ for them. Due to this simple fact, one may insert any graph with two external edges into a bare propagator of any Feynman graph

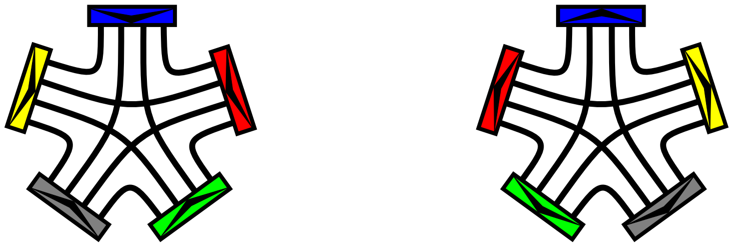

Figure 4. The two orientations of the stranded representation of the combinatorics of a 4-simplex, where we consider the colours to be: blue, red, green, grey and yellow.

without changing the divergence degree of the graph, because one simultaneously needs to add a propagator, which adds a counter-balancing -2 to the divergence degree. Similarly, all 4-valent graphs are associated with a logarithmic divergence, which allows one to insert any graph with four external edges into a vertex without affecting the divergence degree. We already anticipate that this simple behaviour does not hold for models with more complicated power-counting, such as the BGR model, in which case the notions of subgraph contraction insertion must be treated more carefully.

3. The Ben Geloun–Rivasseau tensor field theory model

The form of the BGR model derives from the combinatorics of 4-dimensional simplicial geometry. Consider a 4-simplex. Its boundary is given by five 3-simplices, i.e., tetrahedra. Each pair of these boundary tetrahedra shares one boundary triangle. This combinatorial structure can be illustrated by a stranded graph, as in Figure 4. Here, each individual strand represents a triangle, while the five rectangular boxes, each connected to four strands entering the graph, represent the five boundary tetrahedra. The tetrahedra are considered to be coloured, and we allow for two opposite orientations of the colouring. One considers coloured tetrahedra and only two orientations of the colouring in order to simplify the combinatorics and the topology of this type of models (see, for example, the paper [11], where this type of model was first introduced).

One can then build larger stranded graphs from these building blocks by identifying boundary tetrahedra with matching colours. Moreover, we allow only for identification of tetrahedra belonging to 4-simplices of opposite orientations, taken into account by alignment of arrows in the boxes representing tetrahedra. The resulting stranded graph will then be combinatorially dual to a 4-dimensional simplicial pseudo-manifold.

A (quantum) statistical description of such simplicial geometries can be formulated as a coloured tensor model. This is a QFT model, whose Feynman graphs have exactly such stranded structure. The model has fields of five different colours (each coloured block in Figure 4 being a field), and each such field is a complex-valued function on $U(1)^{\times 4}$. The two stranded graphs in Figure 4 correspond to the two interaction terms of the model, while the propagator simply identifies the strands of the same colour associated with interaction terms with opposite orientations. Such models are sometimes called *combinatorially non-local* due to the combinatorial nature of the non-locality of their interaction terms.

The Feynman graphs of the BGR model are obtained from such stranded representations of simplicial geometry by removing four of the five fields (namely the blue, the

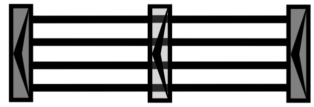

Figure 5. The kinetic term.

red, the green, and the yellow fields in Figure 4 above). Thus, one considers interaction terms such as those illustrated in Figures 6 and 7. These terms correspond to one-to-one relations between particular coloured graphs and new "uncoloured" vertices which, by construction, only have grey fields on the boundaries.

These vertices are dual to four-dimensional polytopes, whose boundaries are composed of six or four tetrahedra. Again, the combinatorial connectivity of the strands represents the identifications of boundary triangles of the boundary tetrahedra.

Note that these are not the only possible "uncoloured" vertices that one can construct. However, these are the only type of vertices considered in the BGR model in order to ensure renormalizability (see again [3, 4] for details).

In addition, a non-trivial kinetic term is added in order to induce dynamics, which allows for renormalization (see Figure 5 for a graphical representation). Note that propagators cannot shuffle strands.

In Figure 8 we illustrate a simple example of a Feynman graph of the BGR model, where the propagators are represented by transparent rectangles, while the external edges end in solid rectangles.

Let us describe in detail the action functional of the BGR model. The fields of the model live on a direct product of four copies of the group $U(1)$. The momentum representation considered in the following is obtained through harmonic analysis, which yields the momentum space $\mathbb{Z}^4$. In the momentum representation the kinetic term of the action functional reads explicitly

$$S_{kin}[\phi] = \sum_{\vec{p}\in\mathbb{Z}^4} \overline{\phi(\vec{p})}(|\vec{p}|^2 + m^2)\phi(\vec{p})\,,$$

where $|p|^2 = \sum_{i=1}^4 |p_i|^2$. This leads to the propagator

$$\mathcal{P}(\vec{p},\vec{q}) = \frac{1}{\sum_i |p_i|^2 + m^2}\prod_{i=1}^4 \delta_{p_i q_i}.$$

In the direct (i.e., $U(1)$) representation, by Fourier transform, the kinetic term reads

$$S_{kin}[\tilde{\phi}] = \int \overline{\tilde{\phi}}_{1,2,3,4}\left(-\sum_{s=1}^4 \Delta_s + m^2\right)\tilde{\phi}_{1,2,3,4},$$

where the integral is performed for the Haar measure over the respective group variables, Δ_s denotes the Laplace operator on $U(1)$ acting on the strand index s. Note that $\tilde{\phi}$ denotes the Fourier transform of the function ϕ.

The interaction terms are based on a completely different principle — instead of locality, they are built on the notion of simplicial combinatorics, as described above. The

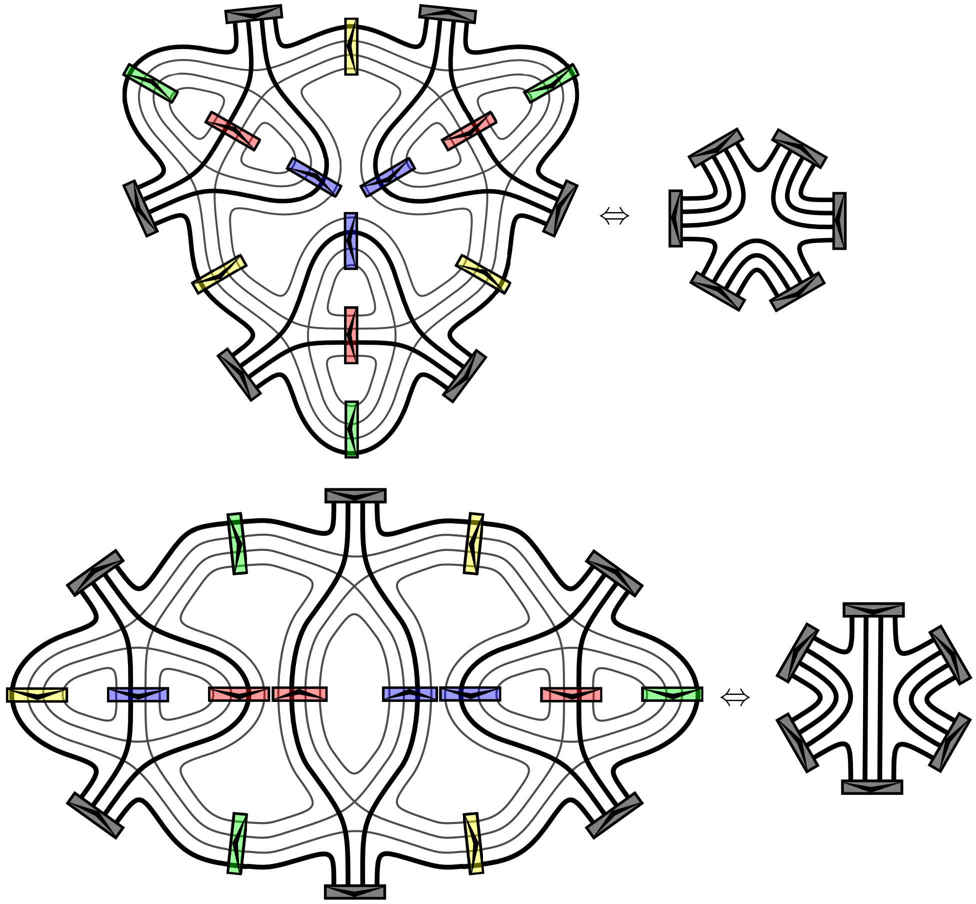

FIGURE 6. Stranded structures of the vertices $V_{6;1}$ (upper) and $V_{6;2}$ (lower) of the BGR model.

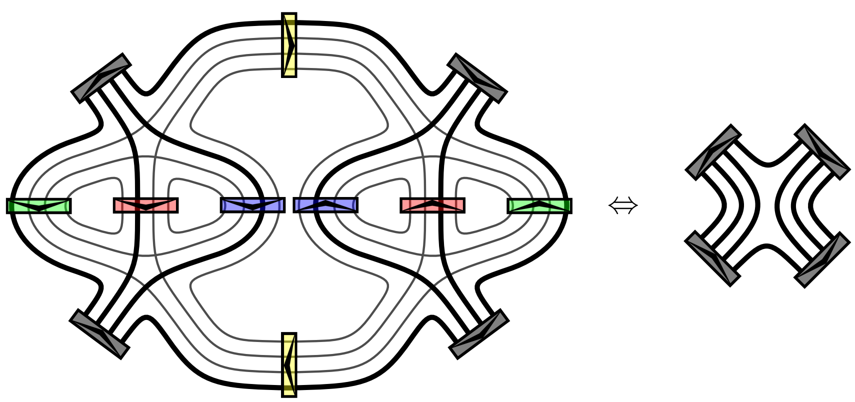

FIGURE 7. Stranded structure of the vertex $V_{4;1}$ of the BGR model.

interaction terms read

$$S^{6;1}_{int}[\phi] = \lambda_{6;1} \sum_{\vec{p},\vec{p}',\vec{p}''\in\mathbb{Z}^4} \sum_{\sigma\in\Sigma_4} \phi_{\sigma_1,\sigma_2,\sigma_3,\sigma_4} \overline{\phi_{\sigma_1',\sigma_2,\sigma_3,\sigma_4}} \, \phi_{\sigma_1',\sigma_2',\sigma_3',\sigma_4'} \overline{\phi_{\sigma_1'',\sigma_2',\sigma_3',\sigma_4'}} \times \phi_{\sigma_1'',\sigma_2'',\sigma_3'',\sigma_4''} \overline{\phi_{\sigma_1,\sigma_2'',\sigma_3'',\sigma_4''}} \,,$$

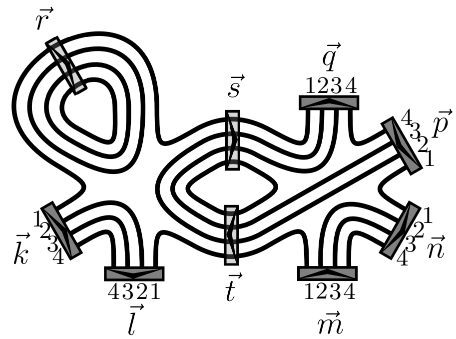

FIGURE 8. A simple example of a Feynman graph of the BGR model. There is a $V_{6;1}$ vertex on the left and a $V_{6;2}$ vertex on the right. $\vec{k}, \vec{l}, \vec{m}, \vec{n}, \vec{p}, \vec{q} \in \mathbb{Z}^4$ label the external momenta, while $\vec{r}, \vec{s}, \vec{t} \in \mathbb{Z}^4$ label momenta running through internal propagators. The momentum components $k_i \in \mathbb{Z}$, $i = 1, 2, 3, 4$, are associated with the strands as indicated by the numbers (and similarly for the other momenta).

and

$$S^{6;2}_{int}[\phi] = \lambda_{6;2} \sum_{\vec{p}, \vec{p}', \vec{p}'' \in \mathbb{Z}^4} \sum_{\sigma \in \Sigma_4} \phi_{\sigma_1,\sigma_2,\sigma_3,\sigma_4} \overline{\phi_{\sigma_1',\sigma_2',\sigma_3',\sigma_4}} \, \phi_{\sigma_1',\sigma_2',\sigma_3',\sigma_4'} \overline{\phi_{\sigma_1'',\sigma_2,\sigma_3,\sigma_4'}} \times \phi_{\sigma_1'',\sigma_2'',\sigma_3'',\sigma_4''} \overline{\phi_{\sigma_1,\sigma_2'',\sigma_3'',\sigma_4''}} \, ,$$

where we wrote

$$\phi(p_1, p_2, p_3, p_4) =: \phi_{1,2,3,4}, \ \ \phi(p_1', p_2, p_3, p_4) =: \phi_{1',2,3,4}, \ \ \phi(p_1, p_2', p_3, p_4) =: \phi_{2',1,3,4}, \ \text{etc.},$$

for simplicity, and we sum over all permutations $\sigma \in \Sigma_4$ of the four colour indices. These interactions lead exactly to the identifications of the field variables represented by the stranded Feynman graphs of the type in Figure 6 modulo permutations of the ordering of the strands.

In addition to the above 6-valent interaction vertices, there are also two 4-valent vertices in this model. The first one is shown in Figure 7, while the second one is a disconnected combination of two propagators, which arises from the vertex $V_{6;2}$ through the insertion of a propagator, as in Figure 9. The corresponding interaction terms of the action functional read

$$S^{4;1}_{int}[\phi] = \lambda_{4;1} \sum_{\vec{p}, \vec{p}' \in \mathbb{Z}^4} \sum_{\sigma \in \Sigma_4} \phi_{\sigma_1,\sigma_2,\sigma_3,\sigma_4} \overline{\phi_{\sigma_1',\sigma_2,\sigma_3,\sigma_4}} \, \phi_{\sigma_1',\sigma_2',\sigma_3',\sigma_4'} \overline{\phi_{\sigma_1,\sigma_2',\sigma_3',\sigma_4'}} \, ,$$

and

$$S^{4;2}_{int}[\phi] = \lambda_{4;2} \sum_{\vec{p}, \vec{p}' \in \mathbb{Z}^4} \sum_{\sigma \in \Sigma_4} \phi_{\sigma_1,\sigma_2,\sigma_3,\sigma_4} \overline{\phi_{\sigma_1,\sigma_2,\sigma_3,\sigma_4}} \, \phi_{\sigma_1',\sigma_2',\sigma_3',\sigma_4} \overline{\phi_{\sigma_1',\sigma_2',\sigma_3',\sigma_4'}} \, .$$

Notice the peculiar disconnected structure of the vertex $V_{4;2}$. This can be interpreted geometrically as a four-sphere with two holes of the form of four-simplices.

The Feynman amplitude associated with a particular Feynman graph is constructed as follows:

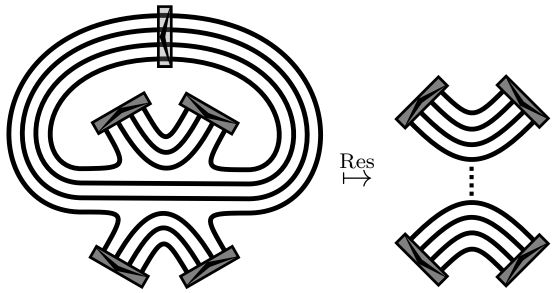

FIGURE 9. Vertex $V_{4;2}$ of the BGR model obtained through the contraction of a self-loop from the vertex $V_{6;2}$.

(1) Let $\vec{p}_e \in \mathbb{Z}^4$ be the momenta labels on the edges $e \in \Gamma^{[1]}$ of the Feynman graph Γ. We construct the product of free propagators

$$\prod_{e \in \Gamma^{[1]}} \frac{1}{|\vec{p}_e|^2 + m^2}\,.$$

(2) For each vertex $v_t \in \Gamma^{[0]}$, multiply by the coupling constant λ_t, where t labels the type of the vertex, and impose identifications of strands (momentum variables) by multiplying the above product by

$$\prod_{v \in \Gamma^{[0]}} \prod_k \delta(c_k^{t(v)}(\vec{p}_l))\,,$$

where $\{c^k_{t(v)}(\vec{p}_l) = 0\}$ is the set of constraints imposed on the incoming momenta by the vertex type $t(v)$ of vertex $v \in \Gamma^{[0]}$. Explicitly, these are given by the following
- $\prod_k \delta(c_k^{6;1}(\vec{p}_l)) = \delta_{p_1n_1}\delta_{p_2q_2}\delta_{p_3q_3}\delta_{p_4q_4}\delta_{q_1k_1}\delta_{k_2l_2}\delta_{k_3l_3}\delta_{k_4l_4}\delta_{l_1m_1}\delta_{m_2n_2}\delta_{m_3n_3}\delta_{m_4n_4}$,
- $\prod_k \delta(c_k^{6;2}(\vec{p}_l)) = \delta_{p_1n_1}\delta_{p_2l_2}\delta_{p_3l_3}\delta_{p_4q_4}\delta_{q_1k_1}\delta_{q_2k_2}\delta_{q_3k_3}\delta_{k_4l_4}\delta_{l_1m_1}\delta_{m_2n_2}\delta_{m_3n_3}\delta_{m_4n_4}$,
- $\prod_k \delta(c_k^{4;1}(\vec{p}_l)) = \delta_{p_1l_1}\delta_{p_2q_2}\delta_{p_3q_3}\delta_{p_4q_4}\delta_{q_1k_1}\delta_{k_2l_2}\delta_{k_3l_3}\delta_{k_4l_4}$,
- $\prod_k \delta(c_k^{4;2}(\vec{p}_l)) = \delta_{p_1q_1}\delta_{p_2q_2}\delta_{p_3q_3}\delta_{p_4q_4}\delta_{k_1l_1}\delta_{k_2l_2}\delta_{k_3l_3}\delta_{k_4l_4}$,

where $\vec{p}, \vec{k}, \vec{m}$ are incoming and $\vec{q}, \vec{l}, \vec{n}$ outgoing momenta. Notice that the vertex functions are not symmetric with respect to all permutations of the edges, so in addition to specifying the type of vertices, one must also specify the orientations. (This is naturally taken care of in the stranded representation.)

(3) Sum over $\vec{p}_e \in \mathbb{Z}^4$ associated with the internal edges $e \in \Gamma^{[1]}_{int}$ of the graph.

The quantity that results from applying these rules to a Feynman graph is the Feynman amplitude of that graph.

Since the momenta are conserved by a simple identification of variables in the propagator and the vertices, as mentioned above, the momentum vector components may be associated with the individual strands $s \in \Gamma^s$ of a Feynman graph in the stranded representation. We emphasize that this combinatorial identification of field variables is, in fact, what allows for the stranded representation in the first place. As a result, the Feynman amplitude of a stranded graph consists of a product over Kronecker deltas identifying the external momentum vector components through the strands, multiplied by the product over internal propagators of the graph, which depend on the momenta associated with the strands that pass through them, summed over the momenta associated with internal (looped) strands. For example, for the Feynman graph in Figure 8, the amplitude reads

$$\begin{aligned}\phi(\Gamma_{\text{Fig. 8}}) := \lambda_{6;1}\lambda_{6;2}\ &\delta_{k_1q_1}\delta_{k_2l_2}\delta_{k_3l_3}\delta_{k_4l_4}\delta_{l_1m_1}\delta_{m_2n_2}\delta_{m_3n_3}\delta_{m_4n_4}\delta_{n_1p_1}\delta_{p_2q_2}\delta_{p_3q_3}\delta_{p_4q_4}\\ &\times\sum_{\substack{r_2,r_3,\\ r_4\in\mathbb{Z}}}\left(\tfrac{1}{k_1^2+r_2^2+r_3^2+r_4^2+m^2}\right)\sum_{s_4\in\mathbb{Z}}\left(\tfrac{1}{l_1^2+p_2^2+p_3^2+s_4^2+m^2}\right)\left(\tfrac{1}{k_1^2+p_2^2+p_3^2+s_4^2+m^2}\right), \quad (3.1)\end{aligned}$$

where $\vec{k},\vec{l},\vec{m},\vec{n},\vec{p},\vec{q}\in\mathbb{Z}^4$ are the momenta associated with the external edges and r_i ($i=2,3,4$) and s_4 are the momentum vector components associated with the internal strands. We observe that, for large absolute values of the internal momenta r_2,r_3,r_4,s_4, the sums behave asymptotically as

$$\phi(\Gamma_{\text{Fig. 8}})\propto\sum_{\substack{r_2,r_3,\\ r_4\in\mathbb{Z}}}\frac{1}{r_2^2+r_3^2+r_4^2}\ \sum_{s_4\in\mathbb{Z}}\frac{1}{s_4^4}\approx\int_{\mathbb{R}^3}\frac{\mathrm{d}^3r}{r^2}\int_{\mathbb{R}}\frac{\mathrm{d}s}{s^4}\propto\Lambda\times\Lambda^{-3}=\Lambda^{-2},$$

where Λ is a cut-off on the momentum summation. Here, the first factor, leading to a linear divergence in the limit $\Lambda\to\infty$, corresponds to the three strand loops in the upper left part of the graph, and the second convergent factor corresponds to the single strand loop in the middle. Although the first factor leads to a subdivergence, the superficial divergence degree of the total graph $\Gamma_{\text{Fig. 8}}$ is -2, and therefore the graph is superficially convergent.

Similarly, the Feynman amplitude for the graph in Figure 10 reads

$$\begin{aligned}\phi(\Gamma_{\text{Fig. 10}}) := \lambda_{6;1}\lambda_{6;2}\ &\delta_{k_1l_1}\delta_{k_2l_2}\delta_{k_3l_3}\delta_{k_4l_4}\delta_{m_1n_1}\delta_{m_2n_2}\delta_{m_3n_3}\delta_{m_4q_4}\delta_{n_4p_4}\delta_{p_1q_1}\delta_{p_2q_2}\delta_{p_3q_3}\\ &\times\sum_{\substack{r_2,r_3,\\ r_4\in\mathbb{Z}}}\left(\tfrac{1}{k_1^2+r_2^2+r_3^2+r_4^2+m^2}\right)\left(\tfrac{1}{l_1^2+p_2^2+p_3^2+m_4^2+m^2}\right)\left(\tfrac{1}{k_1^2+p_2^2+p_3^2+m_4^2+m^2}\right), \quad (3.2)\end{aligned}$$

with the same notation for the momenta as in Figure 8. The difference to the previous graph is that now there are only three internal strands corresponding to r_i, $i=2,3,4$. The superficial divergence degree of this graph is -3 despite the subdivergence, since the propagators contribute negatively to the power counting. In the following we state the power counting more explicitly.

The Feynman amplitude of any other BGR tensor graph may be computed in an analogous way. We leave the calculation of amplitudes for the rest of the graphs of this section as an exercise to the interested reader.

We may also draw the Feynman graphs as ordinary graphs, but then the combinatorics of the internal structure must be taken into account by additional labelling of vertices. We have illustrated the correspondence between the ordinary and the stranded representation of Feynman graphs of the BGR model in Figure 10. The vertices are labelled, for example, as $V_{6;1}^{1234}$, where the lower index indicates the vertex type and the upper indices indicate the permutation of field variables. Since the vertex is not symmetric under permutations of edges, the edges meeting at a vertex must be distinguished. This can be accomplished, for example, by placing the vertex label between the first and the last edge, where the edges are ordered in a right-handed orientation around the vertex with respect to the normal pointing upwards from the surface, on which the graph is drawn.

Definition 3.1. A *face* of a stranded Feynman graph is a strand that forms a loop.

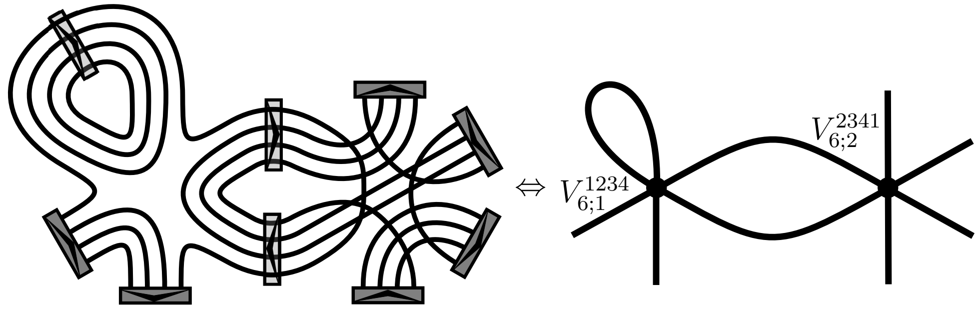

Figure 10. An example of the correspondence between stranded and non-stranded representation of a Feynman graph of the BGR model.

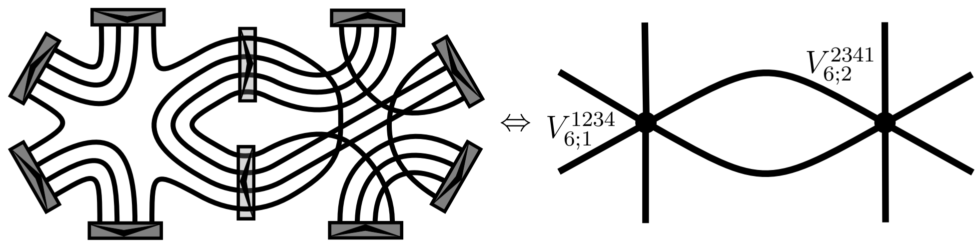

Figure 11. An example of a graph without a face but with a loop in the non-stranded representation.

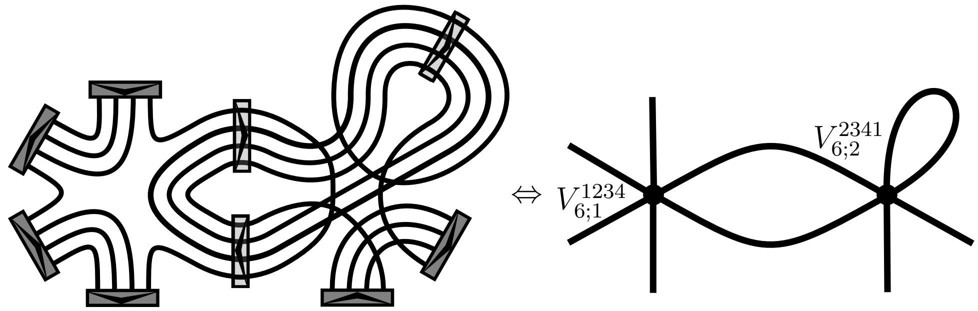

Figure 12. An example of a graph with faces, which extend around several loops of the non-stranded representation.

Note that a face of a stranded graph is dual to the corresponding triangle of the tetrahedra of the four-dimensional triangulation.

The integrations over face momenta may give rise to divergences, as we have already observed. In order for a strand to form a loop, the non-stranded representation must clearly have a loop as well. However, a loop in the non-stranded representation does not imply the existence of a face as can for instance be seen in Figure 11. Moreover, a face may extend around several loops of the non-stranded graph, as in Figure 12. Therefore, the faces can be highly non-local objects on the graph. Fortunately, the power counting restricts the non-locality of divergent faces significantly, as we will see in the following. Notice also that faces cannot extend over bridges of the non-stranded Feynman graph, since due to the colouring a single strand can pass the same propagator only once. Therefore, 1PI Feynman graphs are still sufficient to resolve the divergence structure of Feynman amplitudes of the BGR model.

We have the following auxiliary result.

Lemma 3.1 ([3], Lemma 2). *A superficial divergence degree for the BGR model is given by $\omega(\Gamma) = F_\Gamma - 2P_\Gamma$, where F_Γ and P_Γ are the numbers of faces and propagators in Γ, respectively.*

Thus, we see that not all faces lead to divergences, but there is a balance between the number of propagators and the number of strands. It was further proved that the superficial divergence degree ω for connected graphs, where $V_{4;2}$ is considered disconnected, can be written in terms of the combinatorial and topological structure of the stranded Feynman graphs as follows.

Definition 3.2 ([3], Definition 1). Let Γ be a Feynman graph of the BGR model. Then

- the *coloured extension* Γ_{col} of Γ is the corresponding unique stranded graph obtained by replacing the vertices of Γ by the corresponding graphs of the simplicial coloured model from Figures 6 and 7.
- a *jacket* J of Γ_{col} is a ribbon graph obtained from Γ_{col} by removing a subset of the strands. Specifically, a jacket J is determined by a permutation $(0abcd)$ of colours (up to cyclic permutations), and the strands that are included in J are the ones that connect the tetrahedra of colours $(0a), (ab), (bc), (cd), (d0)$, where $a, b, c, d \in \{R, G, B, Y\}$ are all different. Here, we have chosen 0 to represent the grey colour, while the other colours are labelled by their initials.
- the *pinched jacket* $\tilde{J}$ is the vacuum ribbon graph obtained from the jacket J by closing all external ribbon edges of J. (See Figure 13 for an example.)
- the *boundary graph* $\partial\Gamma$ of Γ is the 3-dimensional tensor graph, which corresponds to the boundary of Γ. It is obtained by first removing all internal faces of Γ, closing the external edges by connecting their strands to a vertex per external edge, replacing the strands by bundles of three strands, and finally replacing the vertices of external edges by tensorial simplicial vertices. (See Figure 14 for an example.)
- the *boundary jacket* J_∂ is a jacket of the boundary graph $\partial\Gamma$.

(See [3, 4] and references therein for further details.) Ben Geloun and Rivasseau proved the following result.

Theorem 3.1. *The superficial divergence degree ω_{BGR} of a connected Feynman graph Γ can be written as*

$$\omega_{BGR}(\Gamma) = -\frac{1}{3}\left[\sum_J g_{\tilde{J}} - \sum_{J_\partial} g_{J_\partial}\right] - (C_\partial - 1) - V_4 - 2V_2'' - \frac{1}{2}(N_{ext} - 6)\,,$$

where g_J denotes the genus of the ribbon graph J, C_∂ is the number of disconnected components of $\partial\Gamma$, V_4 is the number of vertices of type $V_{4;1}$, $V_2''/2$ is the number of vertices of type $V_{4;2}$, and N_{ext} is the number of external edges of Γ.

Moreover, they classified all superficially divergent Feynman graphs in Table 1, which is the departure point for our subsequent analysis.

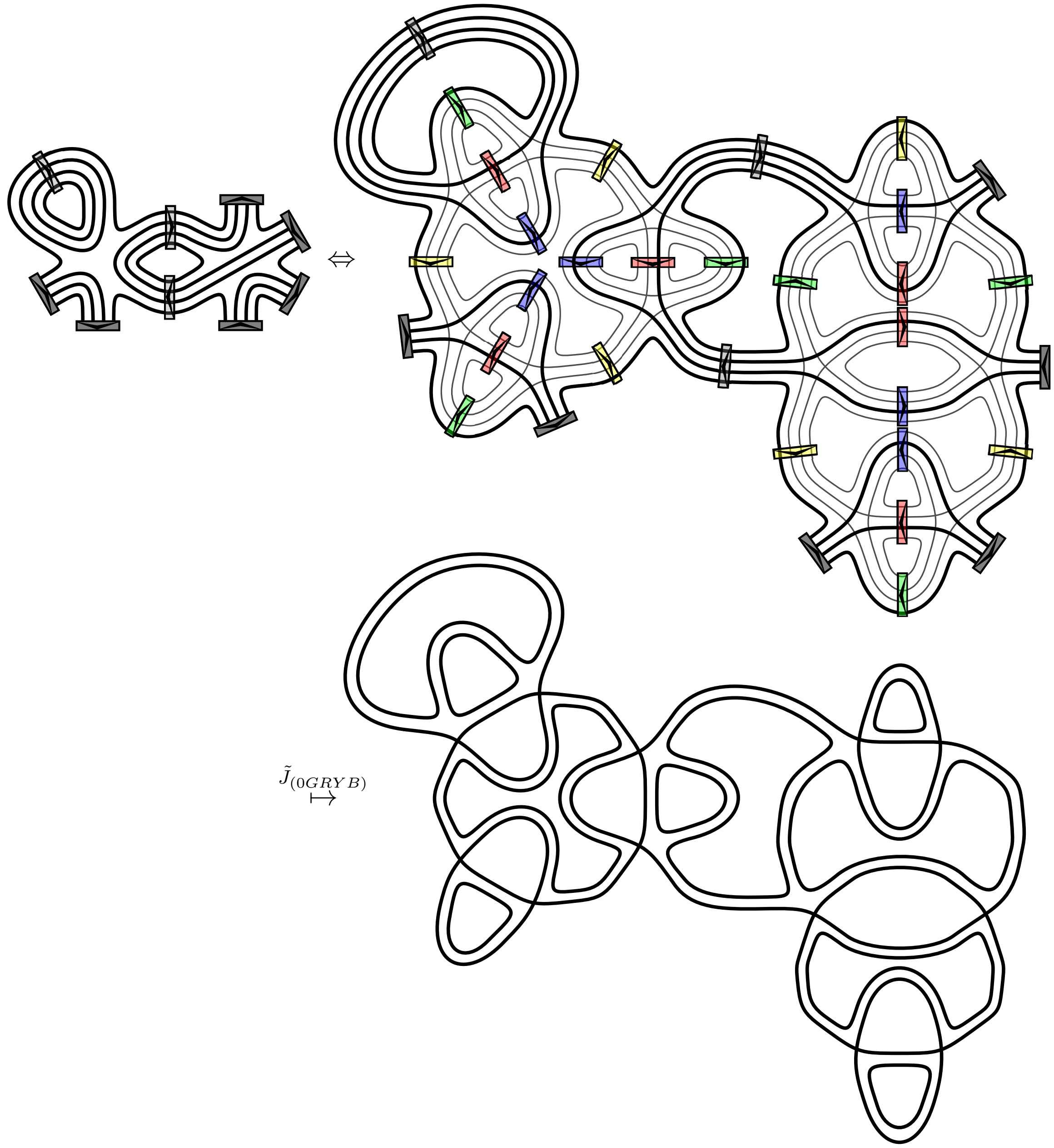

FIGURE 13. An example of a pinched jacket.

Notice that all superficially divergent graphs have a combinatorial structure that corresponds to one of the vertices of the original model, which facilitates perturbative renormalizability. Ben Geloun and Rivasseau [3, 4] showed that the BGR model is perturbatively renormalizable.

4. HOPF ALGEBRAIC DESCRIPTION OF THE COMBINATORICS OF THE RENORMALIZABILITY OF THE BGR MODEL

The formal definitions for local QFT models of Subsection 2.2 also apply to the case of the BGR model, with four important exceptions:

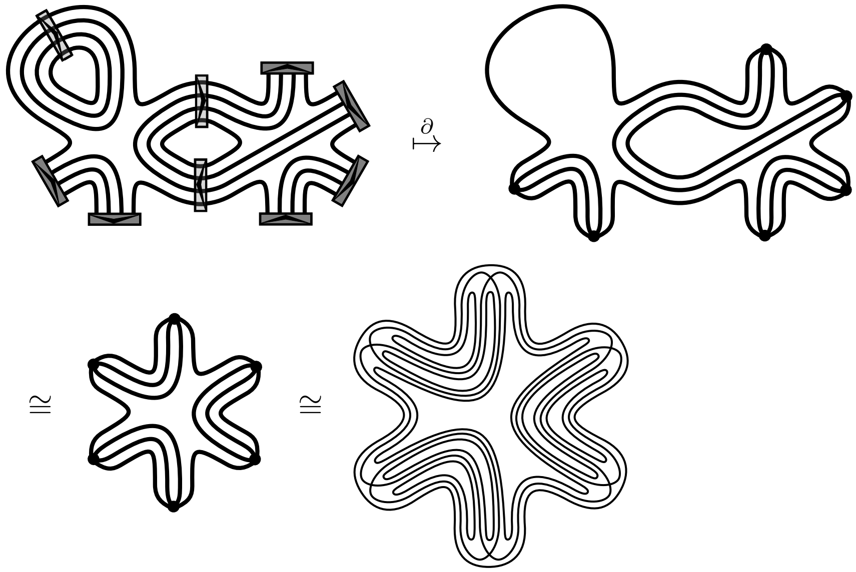

FIGURE 14. An example of a boundary graph.

class	N_{ext}	V_4	$\sum_{J_\partial} g_{J_\partial}$	$C_\partial - 1$	$\sum_{\tilde{J}} g_{\tilde{J}}$	ω_{BGR}	Res
6_1	6	0	0	0	0	0	$V_{6;i}$
4_1	4	0	0	0	0	1	$V^*_{4;1}$
4_2	4	1	0	0	0	0	$V_{4;1}$
4_3	4	0	0	1	0	0	$V_{4;2}$
2_1	2	0	0	0	0	2	$V_{2;1}$
2_2	2	1	0	0	0	1	$V_{2;2}$
2_3	2	2	0	0	0	0	$V_{2;3}$
2_4	2	0	0	0	6	0	$V_{2;3}$

TABLE 1. Classification and residues of superficially divergent Feynman graphs of the BGR model.

(i) The constraints on the external parameters of a Feynman graph Γ are more envolved. One does not need to impose only momentum conservation. Instead, one needs to identify the variables represented by the stranded structure of the Feynman diagrams.

Definition 4.1. A Feynman graph Γ of a QFT model is equipped with a set of *external constraints* $\{c_k^\Gamma(\vec{p}_i) = 0\}$ on the external momenta. We denote the space of the external momenta of Γ by P_Γ. The Feynman amplitudes are then generalized functions on P_Γ, as before.

Remark 4.1. Notice that this generalizes the case of local QFT, where the external constraints impose the momentum conservation. In contrast, in tensor field theories the interactions are combinatorially non-local in the sense that the boundary variables are identified according to tensorial invariance. Thus, the notion of tensorial invariance in TFT constitutes the substitute for the notion of locality in local QFT. A gluing of tensor invariant graphs always yields another tensor invariant graph.

Accordingly, we also need to generalize the notion of residue to tensorial Feynman graphs.

Definition 4.2. A residue $\mathrm{Res}(\Gamma)$ of a tensorial Feynman graph $\Gamma \in \mathcal{G}_{1\mathrm{PI}}$ of the BGR model is the graph obtained, in the non-stranded representation, by shrinking all internal edges of Γ to a point, and retaining the external constraints and structure of the graph. In the stranded representation, this corresponds to the removal of all internal faces and internal propagators of Γ.

(ii) There are vertices of valence four and vertices of valence six. By inserting a self-loop into a 6-valent vertex $V_{6;1}$, we obtain a self-loop Feynman graph with four external edges and divergence degree 1. This leads us to introduce a new 4-valent counter-term vertex $V^*_{4;1}$ that has the same structure as $V_{4;1}$ but a linear counter-term attached. This may be illustrated as in Figure 15.

(iii) Similarly, the contraction of superficially divergent subgraphs with two external edges leads us to introduce 2-valent counter-term vertices $V_{2;1}$, $V_{2;2}$, and $V_{2;3}$, with a quadratic, linear, and logarithmic counter-term attached, respectively. This is very different, for example, from the ϕ^4-model, for which we only have quadratically divergent 2-point functions. See Figure 16 for illustration. Of the 2-valent counter-term vertices, only the one with a quadratic divergence can be replaced by the original propagator, since only in this case the divergence degree of the graph is preserved.

Let us make here a few more remarks on the renormalizability of the BGR model. When one does the Taylor expansion of the two-point function (see Lemmas 11 and 12 in [3, 4] for the respective analytic details), the subleading divergence of the first graph in Figure 16 is logarithmic and renormalizes the wave function. Note that there is no linear subleading divergence. Analogously to the "usual" $\lambda\phi^4$ model (see Section 2), there is no distinction between these two counterterms (quadratic, mass and logarithmic, wave function) at the level of the graph drawing, but this is given by the external structure. Moreover, there are no subleading divergences for the rest of the graphs of Figure 16.

Remark 4.2. Note that a given two-point graph cannot have several counterterms of type 0, 1 and 2 (see again Figure 16), since one can identify to which line of Table 1 the respective tensor graph belongs (due to the numbers V_4, V_2'', and to the graph topology). If the graph belongs to line 2_1, then it has a counterterm of type 2 (which means a quadratic counterterm of mass type and a logarithmic counterterm of wave function-type, see above), if it belongs to line 2_2, then it has a counterterm of type 1, and so on.

(iv) The superficial divergence degree does not only depend on the external data of a Feynman graph, such as the number of external edges, but also some internal combinatorial and topological information, in particular, the number of 4-valent vertices and the sum over the genera of pinched jackets. Out of these, the first one is easy enough to understand, but to have a control over the second requires a bit more work. To this aim, we prove the following auxiliary result.

Lemma 4.1. *Let $\Gamma \in \mathcal{G}$, $\gamma \in \mathcal{G}_{sd}$ be Feynman graphs of the BGR model such that $\gamma \subseteq \Gamma$. For the sum over the genera of pinched jackets, we have the relation*

$$\sum_J g_{\tilde{J}}(\Gamma) = \sum_J g_{\tilde{J}}(\gamma) + \sum_J g_{\tilde{J}}(\Gamma/\gamma)\,.$$

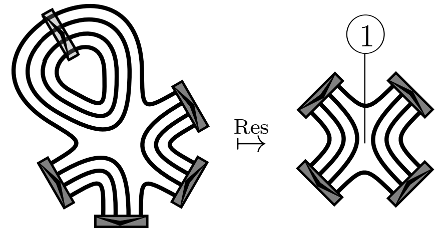

FIGURE 15. Illustration of the linear 4-valent counter-term vertex.

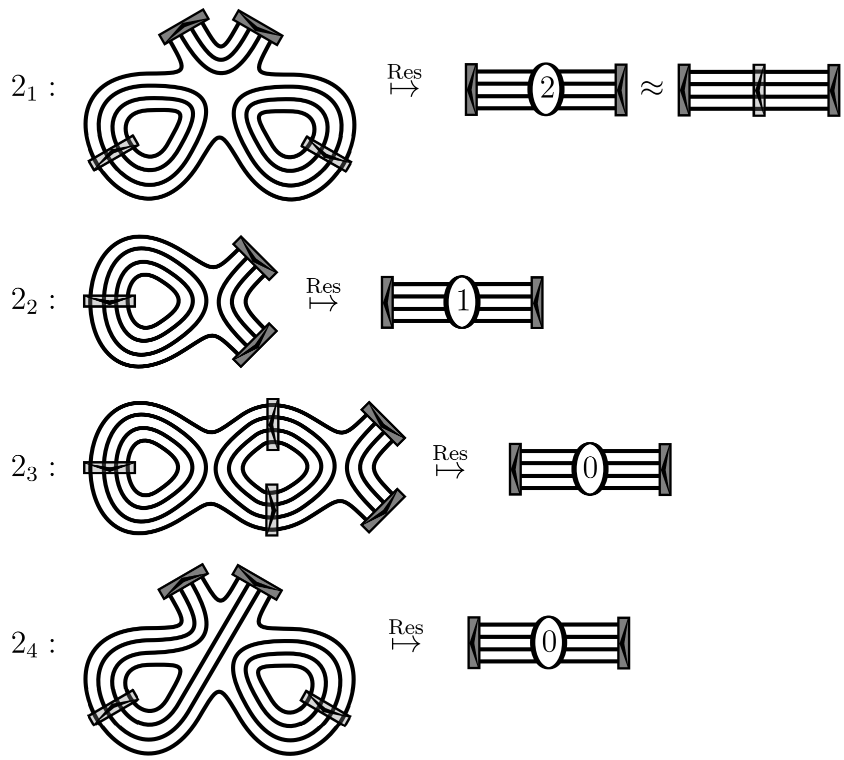

FIGURE 16. Some examples of the 2-valent counter-term as residues of 2-point functions from the classes 2_1, 2_2, 2_3, and 2_4, in the respective order. Only the quadratic counter-term vertex may be inserted into a propagator, since this conserves the divergence degree.

Proof. First note that the genus for a single pinched jacket may be expressed via the Euler characteristic formula as $g_{\tilde{J}} = -\frac{1}{2}(F_{\tilde{J}} - L_{\tilde{J}} + V_{\tilde{J}} - 2C_{\tilde{J}})$, where $F_{\tilde{J}}$, $L_{\tilde{J}}$, $V_{\tilde{J}}$, and $C_{\tilde{J}}$ are the numbers of faces, edges, vertices, and connected components of $\tilde{J}$. (For a disconnected jacket, we define the genus to be given by the sum over the genera of its connected components.) One may further write $F_{\tilde{J}} = F_{\tilde{J},ext} + F_{\tilde{J},int}$, where $F_{\tilde{J},ext}$ is the number of faces formed by pinching the external edges of J, and $F_{\tilde{J},int}$ is the number of (internal) faces of J. Now, one may easily verify that each of $F_{\tilde{J},int}$, $L_{\tilde{J}}$, and $V_{\tilde{J}}$ separately satisfies the relation

$$Q(\Gamma) - Q(\gamma) = Q(\Gamma/\gamma) - Q(\mathrm{Res}(\gamma))$$

with respect to the contraction of subgraphs $\gamma \in \mathcal{G}_{sd}$, as defined for Feynman graphs of the BGR model. (Notice, in particular, that, in general, $F_{\tilde{J},int}(\mathrm{Res}(\gamma)) \neq 0$, $L_{\tilde{J}}(\mathrm{Res}(\gamma)) \neq 0$, and $V_{\tilde{J}}(\mathrm{Res}(\gamma)) \neq 0$, since the vertices of the BGR model correspond to extended subgraphs in the coloured extension of the Feynman graphs, from which the jackets are derived.) For example, it is clear that both sides of

$$Q(\Gamma) - Q(\Gamma/\gamma) = Q(\gamma) - Q(\mathrm{Res}(\gamma))$$

represent the number of internal faces, edges, or vertices, which are removed from $\tilde{J}$ by contracting $\gamma \subseteq \Gamma$, since by definition the subgraph contraction operation replaces γ by $\mathrm{Res}(\gamma)$ inside Γ, which does not affect the numbers of internal faces, edges, or vertices that do not belong to γ. (Notice that some of the internal faces of Γ not internal to γ may still pass through γ, but are not affected by the contraction of γ.) Accordingly, we have

$$Q(\Gamma) - Q(\mathrm{Res}(\gamma)) = Q(\gamma) - Q(\mathrm{Res}(\gamma)) + Q(\Gamma/\gamma) - Q(\mathrm{Res}(\gamma))\,,$$

so $G(\Gamma) := Q(\Gamma) - Q(\mathrm{Res}(\gamma))$ gives a grading of jackets with respect to the contraction of γ, in the sense that $G(\Gamma) = G(\gamma) + G(\Gamma/\gamma)$. Moreover, if we have a set of gradings G_i, then it is easy to verify that $\sum_i (-1)^{n_i} G_i$ is also a grading for any choice of $n_i \in \{0,1\}$.

Now, note that we may write

$$\begin{aligned}
&g_{\tilde{J}}(\Gamma) + \frac{1}{2}\big(F_{\tilde{J},ext}(\Gamma) + F_{\tilde{J}}(\mathrm{Res}(\gamma)) - L_{\tilde{J}}(\mathrm{Res}(\gamma)) + V_{\tilde{J}}(\mathrm{Res}(\gamma)) - 2C_{\tilde{J}}(\mathrm{Res}(\gamma))\big)\\
&= -\frac{1}{2}\big((F_{\tilde{J},int}(\Gamma) - F_{\tilde{J},int}(\mathrm{Res}(\gamma))) - (L_{\tilde{J}}(\Gamma) - L_{\tilde{J}}(\mathrm{Res}(\gamma))) + (V_{\tilde{J}}(\Gamma) - V_{\tilde{J}}(\mathrm{Res}(\gamma)))\big)\,.
\end{aligned}$$

Thus, we find that $g_{\tilde{J}}(\Gamma) + \frac{1}{2}F_{\tilde{J},ext}(\Gamma) + X_{\tilde{J}}(\mathrm{Res}(\gamma))$ is a grading with respect to the contraction of γ, where we wrote

$$X_{\tilde{J}}(\mathrm{Res}(\gamma)) := \frac{1}{2}\big(F_{\tilde{J},int}(\mathrm{Res}(\gamma)) - L_{\tilde{J}}(\mathrm{Res}(\gamma)) + V_{\tilde{J}}(\mathrm{Res}(\gamma)) - 2C_{\tilde{J}}(\mathrm{Res}(\gamma))\big)\,.$$

Accordingly, we find that, as a sum of gradings,

$$\sum_J g_{\tilde{J}}(\Gamma) + \frac{1}{2}\sum_J F_{\tilde{J},ext}(\Gamma) + \sum_J X_{\tilde{J}}(\mathrm{Res}(\gamma))$$

is also a grading with respect to the contraction of γ. Let us write $X(\mathrm{Res}(\gamma)) := \sum_J X_{\tilde{J}}(\mathrm{Res}(\gamma))$. Moreover, $\sum_J F_{\tilde{J},ext}(\Gamma) = 2F_{\partial\Gamma}$, the number of faces of the boundary $\partial\Gamma$ of Γ. Then, due to the grading property, we have

$$\sum_J g_{\tilde{J}}(\Gamma) + F_{\partial\Gamma} = \sum_J g_{\tilde{J}}(\gamma) + F_{\partial\gamma} + \sum_J g_{\tilde{J}}(\Gamma/\gamma) + F_{\partial(\Gamma/\gamma)} + X(\mathrm{Res}(\gamma))\,.$$

We may further simplify by noting that $F_{\partial\Gamma} = F_{\partial(\Gamma/\gamma)}$, since a subgraph contraction does not affect the boundary. Hence, we may write

$$\sum_J g_{\tilde{J}}(\Gamma) = \sum_J g_{\tilde{J}}(\gamma) + \sum_J g_{\tilde{J}}(\Gamma/\gamma) + F_{\partial\gamma}(\mathrm{Res}(\gamma)) + X(\mathrm{Res}(\gamma))\,.$$

But now

$$
\begin{aligned}
&F_{\partial\gamma}(\mathrm{Res}(\gamma)) + X(\mathrm{Res}(\gamma)) \\
&\qquad = \frac{1}{2}\sum_J \big(F_{\tilde{J}}(\mathrm{Res}(\gamma)) - L_{\tilde{J}}(\mathrm{Res}(\gamma)) + V_{\tilde{J}}(\mathrm{Res}(\gamma)) - 2C_{\tilde{J}}(\mathrm{Res}(\gamma))\big) \\
&\qquad = -\sum_J g_{\tilde{J}}(\mathrm{Res}(\gamma))\,,
\end{aligned}
$$

so we finally obtain

$$
\sum_J g_{\tilde{J}}(\Gamma) = \sum_J g_{\tilde{J}}(\gamma) + \sum_J g_{\tilde{J}}(\Gamma/\gamma) - \sum_J g_{\tilde{J}}(\mathrm{Res}(\gamma))\,.
$$

One may then explicitly check that all the genera of pinched jackets vanish for different choices of $\mathrm{Res}(\gamma)$ for $\gamma \in \mathcal{G}_{sd}$, by using the Euler character formula for $g_{\tilde{J}}$. For example, for $\mathrm{Res}(\gamma) = V_{6;1}$, we have $V_{\tilde{J}}(V_{6;1}) = 6$, $L_{\tilde{J}}(V_{6;1}) = 12$, and $C_{\tilde{J}}(V_{6;1}) = 1$ for all J, so $g_{\tilde{J}} = -\frac{1}{2}(F_{\tilde{J}}(V_{6;1}) - 8)$. We must then count the number of faces for each jacket, which gives $F_{\tilde{J}}(V_{6;1}) = 8$ for all J, so $g_{\tilde{J}}(V_{6;1}) = 0$ for all J. Similar calculations may be done for the other possible residues of superficially divergent graphs, which results in the above statement. □

With the above generalizations of the usual Connes–Kreimer framework, and the power-counting for the model given in Table 1, we are then ready to define the Hopf algebra of BGR model Feynman graphs in the same fashion as above for local QFT. In particular, we define the coproduct for the 1PI Feynman graphs $\Gamma \in \mathcal{G}_{1\mathrm{PI}}$ by

$$
\Delta(\Gamma) = \Gamma \otimes \mathbf{1} + \mathbf{1} \otimes \Gamma + \sum_{\substack{\gamma \in \bigcup \mathcal{G}^{\omega}_{sd} \\ \gamma \subsetneq \Gamma}} \gamma \otimes \Gamma/\gamma\,, \tag{4.1}
$$

where the subgraph contraction is defined as in Definition 2.14. Recall that the summation runs over the set $\bigcup \mathcal{G}^{\omega}_{sd}$ of all disconnected unions of superficially divergent 1PI subgraphs. (See Figure 17 for an example.)

We have the following auxiliary result.

Lemma 4.2. *The set of superficially divergent Feynman diagrams of the BGR model is closed under the operations of subgraph contraction and insertion.*

Proof. Let us first consider the subgraph contraction on a class-by-class basis. Notice that a subgraph contraction does not affect the boundary of a graph, and therefore we do not need to consider the boundary properties. Accordingly, the contractions of subgraphs lead to the following, depending on the class to which they belong:

- 6_1: Contracting a subgraph in $6_1, 4_1, 4_3, 2_1$ gives a graph in 6_1. Other classes are not possible, since 6_1 does not contain 4-valent vertices, and the genera of its pinched jackets are zero.
- 4_1: Contracting a subgraph in $6_1, 4_1, 4_3, 2_1$ gives a graph in 4_1. Other classes are not possible for the same reasons as above.
- 4_2: Contracting a subgraph in $6_1, 4_1, 4_2, 4_3, 2_1$ gives a graph in 4_2. Contracting a subgraph in 2_2 gives a graph in 4_1. Other subgraph classes are not possible.
- 4_3: Contracting a subgraph in $6_1, 4_1, 4_3, 2_1$ gives a graph in 4_3. Other subgraph classes are not possible.

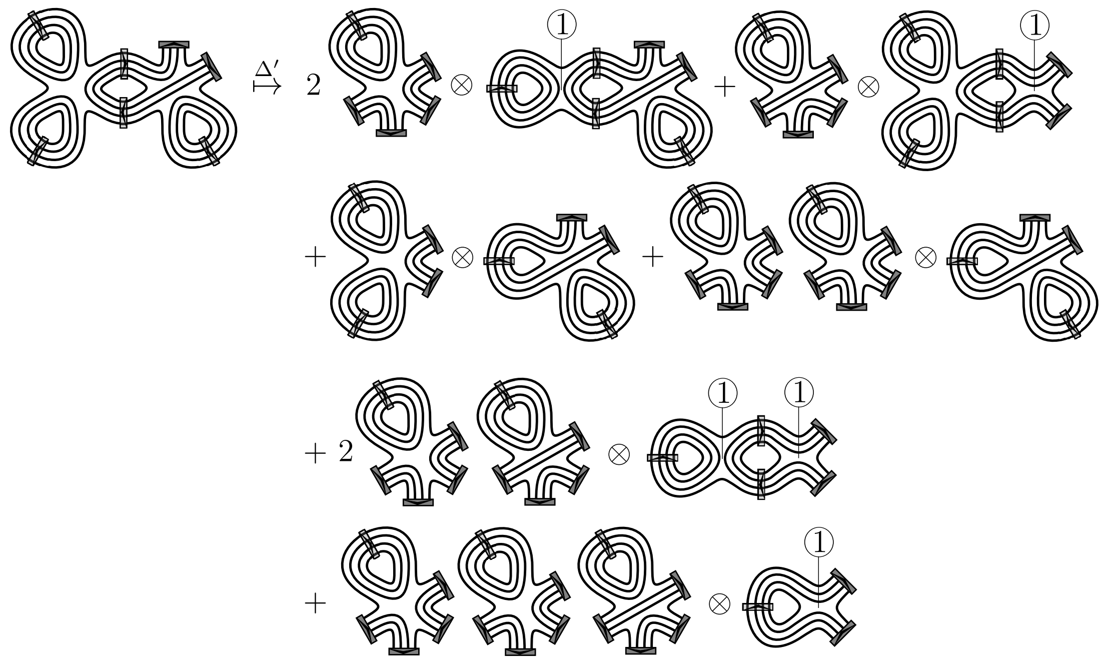

FIGURE 17. An example of the coproduct $\Delta(\Gamma) \equiv \Gamma \otimes \mathbf{1} + \mathbf{1} \otimes \Gamma + \Delta'(\Gamma)$.

row/col	6_1	4_1	4_2	4_3	2_1	2_2	2_3	2_4
6_1	6_1	6_1	0	6_1	6_1	0	0	0
4_1	4_1	4_1	0	4_1	4_1	0	0	0
4_2	4_2	4_2	4_2	4_2	4_2	4_1	0	0
4_3	4_3	4_3	0	4_3	4_3	0	0	0
2_1	2_1	2_1	0	2_1	2_1	0	0	0
2_2	2_2	2_2	2_2	2_2	2_2	2_1	0	0
2_3	2_3	2_3	2_3	2_3	2_3	2_2	2_1	0
2_4	2_4	2_4	0	2_4	2_4	0	0	2_1

TABLE 2. Various possibilities of contractions of superficially divergent Feynman graphs of the BGR model.

2_1: Contracting a subgraph in $6_1, 4_1, 4_3, 2_1$ gives a graph in 2_1. Other subgraph classes are not possible.

2_2: Contracting a subgraph in $6_1, 4_1, 4_2, 4_3, 2_1$ gives a graph in 2_2. Contracting a subgraph in 2_2 gives a graph in 2_1. Other subgraph classes are not possible.

2_3: Contracting a subgraph in $6_1, 4_1, 4_2, 4_3, 2_1$ gives a graph in 2_3. Contracting a subgraph in 2_2 gives a graph in 2_2. Contracting a subgraph in 2_3 gives a graph in 2_1. Other subgraph classes are not possible.

2_4: Contracting a subgraph in $6_1, 4_1, 4_3, 2_1$ gives a graph in 2_4. Contracting a subgraph in 2_4 gives a graph in 2_1. Other subgraph classes are not possible.

We may condense the above analysis into Table 2. We have marked a vanishing entry in the table if a graph from the column class cannot be a subgraph of a graph from the row class.

Similarly, we have to consider the insertions of superficially divergent graphs into other superficially divergent graphs, as defined in Definition 2.15. If we consider insertions into the original Feynman graphs of the model, which do not contain counter-term vertices, we must note the following: first, we may only insert graphs into a vertex, whose residue is equal to that vertex. Moreover, into bare propagators we may only insert superficially divergent graphs with two external edges from class 2_1, since these have divergence degree 2, which is needed in order to conserve the divergence degree of the graph under insertions. The other superficially divergent graphs with $N_{ext} = 2$ can only be inserted into those counter-term vertices which have the same divergence degree. With these restrictions in mind, we provide the following analysis of the insertions into the graphs of the model:

6_1: Inserting a graph from class $6_1, 4_3, 2_1$ yields a graph in class 6_1. Insertion of a graph in 4_2 is not possible, since 6_1 does not contain $V_{4;1}$ vertices.
4_1: Inserting a graph from class $6_1, 4_3, 2_1$ yields a graph in class 4_1. Insertion of a graph in 4_2 is not possible, since 4_1 does not contain $V_{4;1}$ vertices.
4_2: Inserting a graph from class $6_1, 4_2, 4_3, 2_1$ yields a graph in class 4_2.
4_3: Inserting a graph from class $6_1, 4_3, 2_1$ yields a graph in class 4_3. Insertion of a graph in 4_2 is not possible, since 4_3 does not contain $V_{4;1}$ vertices.
2_1: Inserting a graph from class $6_1, 4_3, 2_1$ yields a graph in class 2_1. Insertion of a graph in 4_2 is not possible, since 2_1 does not contain $V_{4;1}$ vertices.
2_2: Inserting a graph from class $6_1, 4_2, 4_3, 2_1$ yields a graph in class 2_2.
2_3: Inserting a graph from class $6_1, 4_2, 4_3, 2_1$ yields a graph in class 2_3.
2_4: Inserting a graph from class $6_1, 4_3, 2_1$ yields a graph in class 2_4. Insertion of a graph in 4_2 is not possible, since 2_4 does not contain $V_{4;1}$ vertices.

Accordingly, all the allowed insertions lead to superficially divergent graphs, and the claim follows. □

Given the above lemma, we may now follow along the lines of the coassociativity proof for the coproduct in [16].

Lemma 4.3. *The coproduct defined in* (4.1) *is coassociative, if the set of superficially divergent Feynman diagrams is closed under the operations of subgraph contraction and insertion.*

Proof. First, by a simple calculation, we find that

$$(\Delta \otimes \mathrm{id}_{\mathcal{H}}) \circ \Delta - (\mathrm{id}_{\mathcal{H}} \otimes \Delta) \circ \Delta = (\Delta' \otimes \mathrm{id}_{\mathcal{H}}) \circ \Delta' - (\mathrm{id}_{\mathcal{H}} \otimes \Delta') \circ \Delta'$$

for $\Delta(\Gamma) \equiv \Gamma \otimes \mathbf{1} + \mathbf{1} \otimes \Gamma + \Delta'(\Gamma)$, and therefore

$$(\Delta \otimes \mathrm{id}_{\mathcal{H}}) \circ \Delta = (\mathrm{id}_{\mathcal{H}} \otimes \Delta) \circ \Delta \quad \text{if and only if} \quad (\Delta' \otimes \mathrm{id}_{\mathcal{H}}) \circ \Delta' = (\mathrm{id}_{\mathcal{H}} \otimes \Delta') \circ \Delta' \,.$$

Moreover, it is enough to show the coassociativity property for 1PI Feynman graphs, which are the generators of $\mathcal{H}$, since Δ' is an algebra homomorphism, i.e., $\Delta(\Gamma \cup \Gamma') = \Delta(\Gamma) \cup \Delta(\Gamma')$, which is also easy to verify by a direct calculation. Now, on the one hand, we have

$$(\Delta' \otimes \mathrm{id}_{\mathcal{H}}) \circ \Delta'\Gamma = \sum_{\substack{\gamma \in \bigcup \mathcal{G}^{\omega}_{sd} \\ \gamma \subsetneq \Gamma}} \Delta'\gamma \otimes \Gamma/\gamma = \sum_{\substack{\gamma \in \bigcup \mathcal{G}^{\omega}_{sd} \\ \gamma \subsetneq \Gamma}} \sum_{\substack{\gamma' \in \bigcup \mathcal{G}^{\omega}_{sd} \\ \gamma' \subsetneq \gamma}} \gamma' \otimes \gamma/\gamma' \otimes \Gamma/\gamma \,.$$

On the other hand, we have

$$(\mathrm{id}_{\mathcal{H}} \otimes \Delta') \circ \Delta'\Gamma = \sum_{\substack{\gamma' \in \bigcup \mathcal{G}^{\omega}_{sd} \\ \gamma' \subsetneq \Gamma}} \gamma' \otimes \Delta'(\Gamma/\gamma') = \sum_{\substack{\gamma' \in \bigcup \mathcal{G}^{\omega}_{sd} \\ \gamma' \subsetneq \Gamma}} \sum_{\substack{\gamma'' \in \bigcup \mathcal{G}^{\omega}_{sd} \\ \gamma'' \subsetneq \Gamma/\gamma'}} \gamma' \otimes \gamma'' \otimes (\Gamma/\gamma')/\gamma'' .$$

For the coassociativity property, we must show that these are equal for all algebra generators $\Gamma \in \mathcal{G}_{1\mathrm{PI}}$. Notice first that in the first sum we may interchange the order of the two summations,

$$(\Delta' \otimes \mathrm{id}_{\mathcal{H}}) \circ \Delta'\Gamma = \sum_{\substack{\gamma' \in \bigcup \mathcal{G}^{\omega}_{sd} \\ \gamma' \subsetneq \Gamma}} \sum_{\substack{\gamma \in \bigcup \mathcal{G}^{\omega}_{sd} \\ \gamma' \subsetneq \gamma \subseteq \Gamma}} \gamma' \otimes \gamma/\gamma' \otimes \Gamma/\gamma .$$

It is therefore enough to show that, for a fixed γ', we have

$$\sum_{\substack{\gamma \in \bigcup \mathcal{G}^{\omega}_{sd} \\ \gamma' \subsetneq \gamma \subsetneq \Gamma}} \gamma/\gamma' \otimes \Gamma/\gamma = \sum_{\substack{\gamma'' \in \bigcup \mathcal{G}^{\omega}_{sd} \\ \gamma'' \subsetneq \Gamma/\gamma'}} \gamma'' \otimes (\Gamma/\gamma')/\gamma'' .$$

These two expressions coincide, if we can set $\gamma'' = \gamma/\gamma'$ in the summation for all $\gamma, \gamma', \gamma'' \in \bigcup \mathcal{G}^{\omega}_{sd}$ such that $\gamma' \subsetneq \gamma \subsetneq \Gamma$ and $\gamma'' \subsetneq \Gamma/\gamma'$, since then we have

$$(\Gamma/\gamma')/\gamma'' = (\Gamma/\gamma')/(\gamma/\gamma') = \Gamma/\gamma.$$

This amounts to two requirements:

(1) Closure of $\mathcal{G}^{\omega}_{sd}$ under contraction: $\gamma/\gamma' \in \mathcal{G}^{\omega}_{sd}$ for all $\gamma, \gamma' \in \mathcal{G}^{\omega}_{sd}$ such that $\gamma' \subsetneq \gamma$, so that we can find $\gamma'' = \gamma/\gamma' \subsetneq \Gamma/\gamma'$ in $\bigcup \mathcal{G}^{\omega}_{sd}$.
(2) Closure of $\mathcal{G}^{\omega}_{sd}$ under insertion: for all $\gamma', \gamma'' \in \mathcal{G}^{\omega}_{sd}$ we have $\gamma' \circ_v \gamma'' \in \mathcal{G}^{\omega}_{sd}$ for any allowed insertion, so that we can find $\gamma = \gamma' \circ_v \gamma'' \subsetneq \Gamma$ in $\bigcup \mathcal{G}^{\omega}_{sd}$.

As we have assumed these properties to hold, this finishes the proof of the lemma. □

We are now in the position to establish our main result.

Theorem 4.1. *Let $\mathcal{H}_{BGR}$ be the unital associative algebra freely generated by the 1PI Feynman graphs of the Ben Geloun–Rivasseau TGFT model. Then $(\mathcal{H}_{BGR}, u, m, \epsilon, \Delta, S)$ is a Hopf algebra, where*

- *the unit $u : \mathbb{C} \to \mathcal{H}_{BGR}$ is given by $u(1) = \mathbf{1}$, where $\mathbf{1}$ is the empty graph.*
- *the product $m : \mathcal{H}_{BGR} \otimes \mathcal{H}_{BGR} \to \mathcal{H}_{BGR}$ is given by the the disjoint union of graphs.*
- *the counit $\epsilon : \mathcal{H}_{BGR} \to \mathbb{C}$ is given by $\epsilon(\mathbf{1}) = 1$ and $\epsilon(\Gamma) = 0$ for $\Gamma \neq \mathbf{1}$.*
- *the coproduct $\Delta : \mathcal{H}_{BGR} \to \mathcal{H}_{BGR} \otimes \mathcal{H}_{BGR}$ is given by*

$$\Delta(\Gamma) = \Gamma \otimes \mathbf{1} + \mathbf{1} \otimes \Gamma + \sum_{\substack{\gamma \in \bigcup \mathcal{G}^{\omega}_{sd} \\ \gamma \subsetneq \Gamma}} \gamma \otimes \Gamma/\gamma ,$$

  which is extended to $\mathcal{H}_{BGR}$ as an algebra homomorphism.
- *the antipode $S : \mathcal{H}_{BGR} \to \mathcal{H}_{BGR}$ is given by the recursive formula*

$$S(\Gamma) = -\Gamma - \sum_{\substack{\gamma \in \mathcal{G}^{\omega}_{sd} \\ \gamma \subsetneq \Gamma}} S(\gamma)\Gamma/\gamma,$$

  with $S(\mathbf{1}) = \mathbf{1}$.

Proof. We have shown above that the coproduct is coassociative. All the other Hopf algebra properties follow easily. In particular, we note that $(\mathcal{H}_{\mathrm{BGR}}, u, m, \epsilon, \Delta)$ is a graded connected bialgebra, graded by the number of internal edges. Therefore, the antipode follows from Formula (2.14). □

The 6-point function Feynman amplitude (without two- or four-point sub-divergences) of the Ben Geloun–Rivasseau model can then by formally identified with the general formula given in Theorem 2.2,

$$\phi_R(\Gamma) = S_R^\phi(\phi(\Gamma)) \star \phi(\Gamma)\,,$$

where S_R^ϕ is given recursively by

$$S_R^\phi(\phi(\Gamma)) = -R[\phi(\Gamma)] - R\left[\sum_{\substack{\gamma\subsetneq\Gamma\\ \gamma\in\mathcal{G}_{sd}^\omega}} S_R^\phi(\phi(\gamma))\phi(\Gamma/\gamma)\right].$$

If one uses a dimensional regularization scheme, then the formula above can be identified with the renormalized Feynman amplitude of any graph (recall however that in the original paper [3, 4] the position space multi-scale analysis Taylor expansion was used to prove renormalizability).

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