Divisors on graphs, Connected flags, and Syzygies

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Notation:

- G is a simple graph on $[n]$
- $S = K[x_1, \ldots, x_n]$
- I_G is a canonical binomial ideal associated to G which encodes the linear equivalences of divisors on G.

Question

Describe the algebraic invariants (a minimal free resolution) of I_G in combinatorial terms of graph.

History (complete graphs)

Postnikov-Shapiro 2004

$\beta_{k-1}(R/I_G) = (k-1)!\, S(n,k)$ where $S(n,k)$ denotes the Stirling number of the second kind (i.e. the number of ways to partition a set of n elements into k nonempty subsets).

Manjunath-Sturmfels 2012

The barycentric subdivision of the $(n-1)$-simplex supports a minimal free resolution for the toppling ideal I_G.

Question: What can we say about the algebraic invariants of a general graph?

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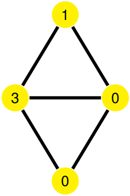
- *Div*(G): free abelian group generated by $V(G)$

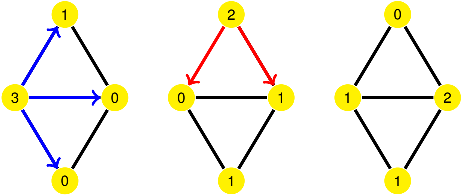
$$D = \sum_{v \in V(G)} a_v(v),$$

$D(v) := a_v \in \mathbb{Z}$.



Chip-firing game:

- initial configuration: assign an integer number of dollars to each vertex, D
- move: consists of a vertex v either borrowing one dollar from each of its neighbors or giving one dollar to each of its neighbors.
- $D \sim D'$: there is a sequence of moves taking D to D' in the chip-firing game.



- $S = K[x_i : i \in V(G)]$
- $I_G := \langle \mathbf{x}^{D_1} - \mathbf{x}^{D_2} : D_1 \sim D_2 \text{ and } D_1, D_2 \geq 0 \rangle$
- $M_G := \mathrm{in}_{\mathrm{revlex}}(I_G)$ with respect to $x_1 > \cdots > x_n$.

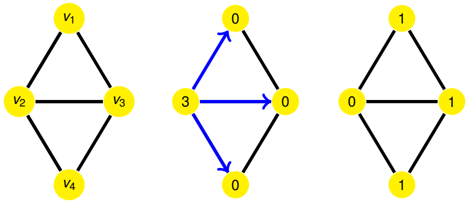


Figure : $x_2^3 - x_1 x_3 x_4$

binomial associated to an 2-acyclic orientation

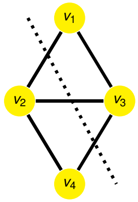


Figure : $x_1 x_3^2 - x_2^2 x_4$

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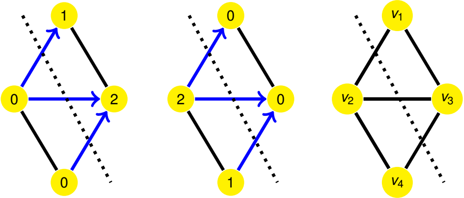
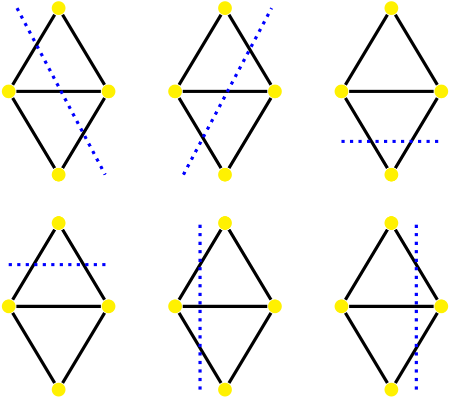
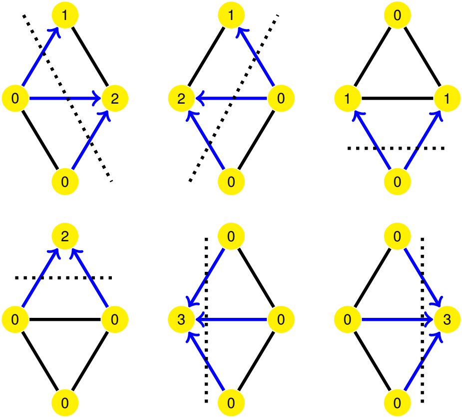


Figure : $x_1 x_3^2 - x_2^2 x_4$

Connected 2-partitions



2-acyclic orientations



$$M_G = (x_1x_2^2, x_1x_3^2, x_2x_3, x_1^2, x_2^3, x_3^3) \ .$$

Given a finitely generated R-module M and a set $z_1, \ldots, z_t$ of generators,

- a syzygy of M is an element $(a_1, \ldots, a_t) \in R^t$ for which $z_1 a_1 + \cdots + z_t a_t = 0$.
- module of syzygies of M: The set of all syzygies which is a submodule of R^t (the kernel of the map $\varepsilon : R^t \to M$ that takes the standard basis elements of R^t to the given set of generators).

- $M_G = (x^2, xy, y^2)$
- $x(y^2) - y(xy) = 0$ and $y(x^2) - x(xy) = 0$
- $0 \to R^2 \to R^3 \to M_G$

Minimal free resolution of M

- R is a polynomial ring (commutative, Noetherian local ring),
- M is a finitely generated R-module.

By choosing a minimal generating set for M, and then a minimal generating set for the first syzygy, and so on, one obtains a free resolution

$$\cdots \to R^{\beta_n} \to \cdots \to R^{\beta_1} \to R^{\beta_0} \to M \to 0$$

The syzygies are uniquely determined up to isomorphism (independent of the choice of generators at each stage).
β_i: the Betti numbers of M.

Known results:

- Coria, Rossinb, Salvy 2000: a minimal Gröbner basis for I_G in terms of 2-connected partitions of G.
- Postnikov and Shapiro 2004: the Scarf complex is a minimal free resolution for M_G in case of complete graphs.
- Perkinson, Perlman and Wilmes 2011: top Betti numbers in terms of maximal reduced divisors of G.
- Manjunath and Sturmfels 2012: the Scarf complex is a minimal free resolution for M_G and I_G (complete graphs).

Main Theorem

Theorem

There is a one-to-one correspondence between:

(1) $(k-2)^{\mathrm{th}}$ syzygies of I_G and M_G (its distinguished initial ideal)

(2) k-connected flags of G with unique source

(3) k-acyclic orientations of G with unique source

(4) maximal q-reduced divisors on the partition graphs

(5) k-dimensional bounded regions of the graphical arrangement.

Main Theorem

Theorem

The $(k-2)^{\text{th}}$ Betti number of I_G and M_G is given by

(5) the number of k-dimensional bounded regions of the graphical arrangement.

Proof

The ideals M_G and I_G are the specific specializations of some known ideals attached to the graphical arrangement. In particular

$$\beta_{ij}(I_G) = \beta_{ij}(M_G) = \beta_{ij}(O_G) = \beta_{ij}(J_G)\ .$$

Definition

- Corresponding to each edge ij of G with $i < j$

$$H_{ij} := \{v \in \mathbb{R}^n : \; h_{ij}(v) = 0 \;\; \textit{for} \;\; h_{ij}(v) := v_i - v_j\}.$$

- The graphical hyperplane arrangement of G is

$$\mathcal{A}_G := \{H_{ij} : \; ij \in E(G) \;\; \text{and} \;\; i < j\}.$$

- $\mathcal{H}_G$: The restriction of $\mathcal{A}_G$ to

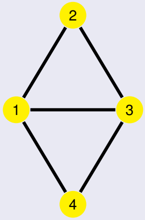
$$H_q := \{v \in \mathbb{R}^n : \; v_n = 0 \;\; \text{and} \;\; v_1 + \cdots + v_{n-1} = 1\}\ .$$

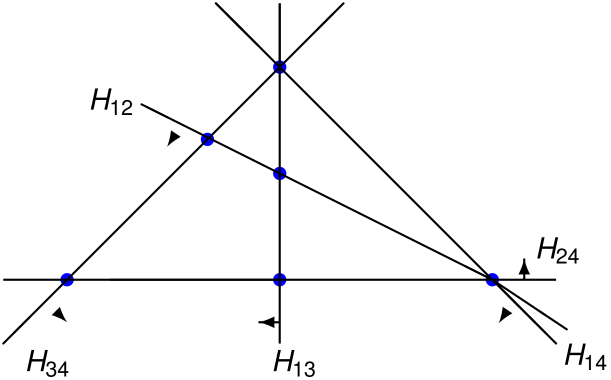
Example

$\mathcal{H}_G$ is the restriction of

$$\mathcal{A}_G := \{H_{12}, H_{24}, H_{34}, H_{14}, H_{13}\}$$

to $H_q = \{v \in \mathbb{R}^4 : \; v_4 = 0 \;\text{ and }\; v_1 + v_2 + v_3 = 1\}$.



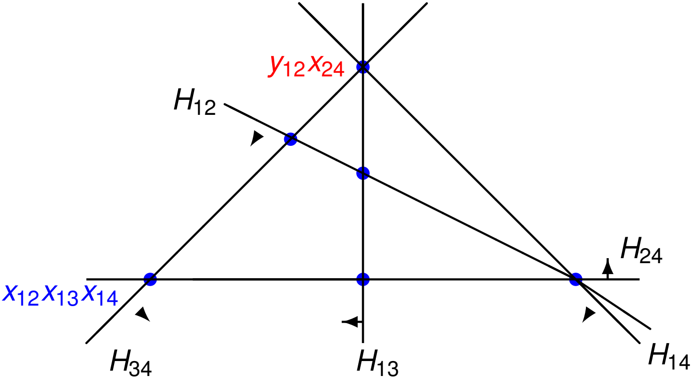

H_{12}
H_{24}
H_{34}
H_{13}
H_{14}

- $S = K[x_{ij}, y_{ij} : \; ij \in E(G)]$
- O_G: generated by the monomials

$$m(v) := \prod_{v_i > v_j} x_{ij} \prod_{v_i < v_j} y_{ij} \;\text{ for }\; v \in \mathbb{R}^n.$$

- J_G: generated by the binomials

$$b(v) := \prod_{v_i > v_j} x_{ij} \prod_{v_i < v_j} y_{ij} - \prod_{v_i > v_j} y_{ij} \prod_{v_i < v_j} x_{ij} \;\text{ for }\; v \in \mathbb{R}^n.$$


$y_{12}x_{24}$
H_{12}
H_{24}
$x_{12}x_{13}x_{14}$
H_{34}
H_{13}
H_{14}

Theorem (Novik-Postnikov-Sturmfels 2002)

The bounded complex $\mathcal{B}_G$ minimally resolves S/O_G and S/J_G. In particular the number of *k*-dimensional regions of $\mathcal{H}_G$ is $\beta_{k-2}(S/J_G) = \beta_{k-2}(S/O_G)$.

Theorem (Green-Zaslavsky 1983)

The *k*-dimensional regions of $\mathcal{H}_G$ are in one-to-one correspondence with the k-acyclic orientations of G.

- From the point of view of Gröbner theory using Schreyer's algorithm we give an explicit description of a minimal Gröbner basis for each higher syzygy module which is also a minimal generating set.

- The minimal free resolution of I_G is supported on certain cellular decomposition of the "Picard torus" of G. This new point of view allows us to generalize many concepts and results of this paper to the more general case of oriented and regular matroids.

- We apply the results mentioned in the section of Hyperplane arrangement.

Related works:

- Madhusudan Manjunath, Frank-Olaf Schreyer, John Wilmes (Nov 2012): Analogous results obtained simultaneously and independently using Gröbner degeneration.
- Horia Mania (Nov 2012): The first Betti number of I_G.
- Anton Dochtermann and Raman Sanyal (Dec 2012): Monomial ideal M_G.

Thank You!