1. S_n characters

The partition $\lambda \vdash n$ has the corresponding irreducible S_n character χ^λ.

For example, $\lambda = (n-i, 1^i)$ has the corresponding irreducible S_n character

$$\chi^{(n-i,1^i)}.$$

The matrix $\quad \left(\chi^\lambda(\mu) \mid \lambda, \mu \vdash n\right) \quad$ is the character table of S_n, computed for example by the Murnaghan-Nakayama rule.

2. The character $\sum_{i=0}^{n-1} \chi^{(n-i,1^i)}$

Denote $\chi_n = \sum_{i=0}^{n-1} \chi^{(n-i,1^i)}$. When $\mu \vdash n$ we study

$$\chi_n(\mu) = \sum_{i=0}^{n-1} \chi^{(n-i,1^i)}(\mu).$$

Example

μ :	(4)	$(3,1)$	$(2,2)$	$(2,1^2)$	(1^4)
$\chi_4(\mu)$	0	2	0	0	8

3. A Theorem

We find out the following phenomena:

Theorem

Let $\mu = (\mu_1, \ldots, \mu_r) \vdash n$, then

$$\sum_{i=0}^{n-1} \chi^{(n-i,1^i)}(\mu) = \begin{cases} 0 & \text{if some } \mu_j \text{ is even} \\ 2^{\ell(\mu)-1} & \text{if all } \mu_j \text{ are odd} \end{cases}$$

How to prove it?

First approach: Apply the Murnaghan-Nakayama rule. MAYBE!

Second approach: Prove a more general identity involving Lie superalgebras.

4. Partitions in the k-strip

$s_k(\lambda) =$ the number of the k-SSYT.

If $\ell(\lambda) > k$ then $s_k(\lambda) = 0$.

If $\ell(\lambda) \leq k$, $s_k(\lambda)$ is given by a hook formula, involving the "content" numbers and the "hook" numbers of λ, see for example Macdonald's book.

This leads to the k-strip $H(k, 0; n) = \{\lambda \vdash n \mid \ell(\lambda) \leq k\}$.

This partitions parametrize the Schur-Weyl Duality.

5. Partitions in the (k, ℓ)-hook

$s_{k,\ell}(\lambda) =$ the number of the $(k, \ell) - SSYT$.

$H(k, \ell; n) = \{\lambda \vdash n \mid \lambda_{k+1} \leq \ell\}$.

These partitions parametrize the "super" Schur-Weyl Duality.

There is a formula for $s_{k,\ell}(\lambda)$ for most $\lambda \in H(k, \ell; n)$.

Example

$k = \ell = 1$, then

$$s_{1,1}(\lambda) = \begin{cases} 0 & \text{if } \lambda \neq (r, 1^{n-r}) \text{ for some } r \\ 2 & \text{if } \lambda = (r, 1^{n-r}) \end{cases}$$

6. The character $\sum_{\lambda \in H(k,\ell;n)} s_{k,\ell}(\lambda) \cdot \chi^\lambda$

Construct the S_n character

$$\sum_{\lambda \in H(k,\ell;n)} s_{k,\ell}(\lambda) \cdot \chi^\lambda.$$

When $\ell = 0$, this character arises in the Schur-Weyl theory, from the action of S_n on $V^{\otimes n}$ where $\dim V = k$.

For general k, ℓ this character arises in the super Schur-Weyl theory, from the super (i.e $\pm$) action of S_n on $(V_0 \oplus V_1)^{\otimes n}$, where $\dim V_0 = k$ and $\dim V_1 = \ell$.

7. The main result

Theorem

Let $\mu = (\mu_1, \ldots, \mu_r) \vdash n$ *where* $\mu_r > 0$. *Then*

$$\sum_{\lambda \in H(k,\ell;n)} s_{k,\ell}(\lambda) \cdot \chi^{\lambda}(\mu) = \prod_{j=1}^{r} (k + (-1)^{\mu_j+1}\ell).$$

8. Special cases

The case $k = \ell = 1$. Let $\mu = (\mu_1, \ldots, \mu_r) \vdash n$ where $\mu_r > 0$. In that case the theorem is

$$\sum_{\lambda \in H(1,1;n)} 2 \cdot \chi^\lambda(\mu) = \prod_{j=1}^{r} (1 + (-1)^{\mu_j + 1}).$$

Equivalently

$$\sum_{\lambda \in H(1,1;n)} \chi^\lambda(\mu) = \begin{cases} 0 & \text{if some } \mu_j \text{ is even} \\ 2^{\ell(\mu)-1} & \text{if all } \mu_j \text{ are odd} \end{cases}$$

The case $\ell = 0$. In that case the theorem is

$$\sum_{\lambda \in H(k,0;n)} s_k(\lambda) \cdot \chi^\lambda(\mu) = k^{\ell(\mu)}.$$

This formula is known, and can be proved via the Schur-Weyl duality.

9. The case $\ell = 0$

$\dim V = k$. Let $\sigma \in S_n$ act on $V^{\otimes n}$; $\bar{v} = v_{i_1} \otimes \cdots \otimes v_{i_n} \in V^{\otimes n}$ a basis element. Compute the matrix M_σ, then its trace $tr(M_\sigma)$:

$$\sigma = \cdots \ (r, r+1, \ldots, s) \ \cdots$$

$$\bar{v} = \cdots \ (v_{i_r}, v_{i_{r+1}}, \ldots, v_{i_s}) \ \cdots$$

$$\sigma\bar{v} = \cdots \ (v_{r+1}, \ldots v_{i_s}, v_{i_r}) \ \cdots$$

To get a contribution $\neq 0$ to $tr(M_\sigma)$ we must have $v_{i_r} = \cdots = v_{i_s}$, and there are k possible values here. Q.E.D