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NORMALLY REGULAR DIGRAPHS, ASSOCIATION SCHEMES AND RELATED COMBINATORIAL STRUCTURES

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Abstract. This paper reports on the characteristics of and mutual relationships between various combinatorial structures that give rise to certain imprimitive non-symmetric three-class association schemes. The relation graphs of an imprimitive symmetric 2-class association scheme are isomorphic to $m \circ K_r$ (the union of m copies of the complete graph on r vertices) and its complement $\overline{m \circ K_r}$ (the complete m-partite strongly regular graph) for some positive integers m and r. The set of relation graphs of a non-symmetric three-class fission scheme of such a 2-class association scheme contains a pair of opposite orientations of either $m \circ K_r$ or $\overline{m \circ K_r}$, depending on m and r. For suitable parameters m and r, these graphs arise from various combinatorial objects, such as doubly regular tournaments, normally regular digraphs, skew Hadamard matrices, Cayley graphs of dicyclic groups and certain group rings. The construction and the characteristics of these objects are investigated combinatorially and algebraically, and their mutual relationships are discussed.

Key words and phrases. Cayley graphs, regular team tournaments, skew symmetric Hadamard matrices, S-rings, normally regular digraphs, association schemes.

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Contents

1. Introduction

A non-symmetric three-class association scheme consists of three nontrivial associate relations, two of which are non-symmetric relations. If A is the adjacency matrix of one of the two non-symmetric relations, then A^{t} is the adjacency matrix of the directed graph associated with the other (opposite) non-symmetric relation. The undirected graph associated with the symmetric relation has adjacency matrix $J - I - A - A^{\mathsf{t}}$ (see Section 2 for all definitions). As they are the relation graphs of an association scheme, it is required that these graphs are regular, in particular that $AJ = JA = kJ$ for some number k, and that

(1) $AA^{\mathsf{t}} = A^{\mathsf{t}}A$ is a linear combination of I, A, A^{t} and $J - I - A - A^{\mathsf{t}}$, and
(2) A^2 is a linear combination of I, A, A^{t} and $J - I - A - A^{\mathsf{t}}$.

There are many combinatorial structures that possess either of these characteristics. This paper concerns the construction of such structures and studies their relationships. Among others, it discusses

- normally regular digraphs, i.e., regular directed graphs satisfying condition 1, and

- doubly regular team tournaments, i.e., regular orientations of a complete multipartite graph satisfying condition 2.

The paper also surveys the basic properties of other related objects, including Cayley graphs and skew Hadamard matrices. It gives valuable descriptions of their mutual relationships and interpretations of certain characteristics of an object in terms of other structures. The organisation of the paper is as follows.

In Section 2, we briefly recall basic terminology and notation for graphs including matrix representations and automorphisms of graphs.

In Section 3, we describe our main objects, doubly regular tournaments, doubly regular orientations of complete multipartite strongly regular graphs, normally regular digraphs and association schemes. To keep the presentation reasonably self-contained we recall known examples and their constructions, and we compare the characteristics of these objects. In particular, the doubly regular tournaments obtained from other doubly regular tournaments T via the coclique extension (denoted by $\mathcal{C}_r(T)$) and a special way of 'doubling' and 'augmenting' with an additional pair of vertices (denoted by $\mathcal{D}(T)$), are equivalent to normally regular digraphs with certain parameter conditions. These objects are also related to a certain class of doubly regular team tournaments and imprimitive symmetrisable 3-class association schemes.

In Section 4, we focus on the characterisation of doubly regular (m, r)-team tournaments which may be viewed as the orientations of complete multipartite strongly regular graphs $\overline{m \circ K_r}$. We then treat the $(m, 2)$-team tournaments as a special class in connection with symmetrisable 3-class association schemes.

In Section 5, the relationships between the first relation graphs of imprimitive non-symmetric 3-class association schemes and special types of doubly regular tournaments $\mathcal{C}_r(T)$ and $\mathcal{D}(T)$ are discussed. In particular, it is shown that every imprimitive non-symmetric 3-class association scheme for which the first relation graph is an orientation of $\overline{m \circ K_r}$ for $r = 2$ is characterized by the relation graph being isomorphic to either $\mathcal{C}_2(T)$ or $\mathcal{D}(T)$ for a suitable doubly regular tournament T. In Section 5.2 we give a short survey of first relation graphs of imprimitive non-symmetric 3-class association schemes with $r \geq 3$, other than $\mathcal{C}_r(T)$.

In Section 6, we study the groups acting transitively on a graph $\mathcal{D}(T)$ for some doubly regular tournament T. In particular, we study the automorphism group of a vertex transitive doubly regular tournament $\mathcal{D}(T)$. We observe that for any tournament T the graph $\mathcal{D}(T)$ has a unique involutory automorphism. We also show that the Sylow 2-subgroups of the automorphism group of $\mathcal{D}(T)$ are not cyclic but are generalised quaternion groups.

In Section 7, many normally regular digraphs arise from Cayley graphs, group rings and difference sets. We investigate the conditions on the connection set $S \subset G$ under which the Cayley graph $\mathsf{Cay}(G, S)$ is a normally regular digraph. We then reformulate the condition in terms of group rings. We also show that if the Cayley graph of G with the generating set $D \setminus \{1\}$ is isomorphic to $\mathcal{D}(T)$ for some doubly regular tournament T, then D gives rise to a certain relative difference set.

In Section 8, we recall Noburo Ito's conjecture [Ito97] concerning Hadamard groups and matrices, and consider S-rings over dicyclic groups to provide a reinterpretation of the conjecture. Ito's conjecture states that every dicyclic group of order $8t$, for some t, is a Hadamard group. We consider possible values of t for which the dicyclic group of order $8t$ is a skew Hadamard group. Earlier in [Ito94], Ito proved that the Hadamard matrices corresponding to Paley tournaments P_q can be constructed from skew Hadamard groups. We state and prove the result in terms of $\mathcal{D}(P_q)$: that is, we show that $SL(2,q)$ acts as a group of automorphisms of $\mathcal{D}(P_q)$ and that this group contains a dicyclic subgroup acting regularly on the vertex set. We then give examples, found by computer search using GAP and GRAPE, showing that other groups besides the dicyclic groups may appear as regular subgroups of the automorphism groups of the graphs $\mathcal{D}(P_q)$ for some q.

In Section 9, we give some concluding remarks. In particular, we emphasise the quite unusual 20-year history of this project, discussing the reasons for its style and goals of presentation, and briefly mentioning ongoing progress in all areas considered since the project began.

2. Preliminaries

A directed graph Γ consists of a finite set $V(\Gamma)$ of vertices and a set $E(\Gamma) \subseteq \{(x,y) \mid x,y \in V(\Gamma)\}$ of arcs (or directed edges). If $(x,y) \in E(\Gamma)$, then we say that x and y are adjacent, x dominates y, and that y is an out-neighbour of x. (We also denote this by $x \to y$.) The set of out-neighbours of x is the set $N^+(x) = \{y \in V(\Gamma) \mid (x,y) \in E(\Gamma)\}$. The out-valency of x is the number $|N^+(x)|$ of out-neighbours of x. In-neighbours and in-valency are defined similarly. We say that Γ is regular of valency k if every vertex in Γ has in-valency and out-valency k.

A (simple) graph can be viewed as a directed graph Γ for which $(x,y) \in E(\Gamma)$ if and only if $(y,x) \in E(\Gamma)$. An oriented graph is a directed graph Γ where at most one of the arcs (x,y) and (y,x) is in $E(\Gamma)$. In the next sections, by a directed graph we usually refer to an oriented graph.

The adjacency matrix of a directed graph Γ with $V(\Gamma) = \{x_1,\dots,x_n\}$ is an $n \times n$ matrix A where its (i,j)-entry $(A)_{ij} = a_{ij}$ is defined by

$$a_{ij} = \begin{cases} 1, & \text{if } (x_i,x_j) \in E(\Gamma), \\ 0, & \text{otherwise.} \end{cases}$$

The matrix $J = J_n$ is the $n \times n$ matrix with 1 in every entry, and $I = I_n$ denotes the $n \times n$ identity matrix. Thus $J - I$ is the adjacency matrix of the complete graph on n vertices.

In addition to the ordinary matrix multiplication we will use two other matrix products. Let A and B be $n \times n$ matrices. Then the Schur-Hadamard product $A \circ B$ is the $n \times n$ matrix obtained by the entry-wise multiplication: $(A \circ B)_{ij} = a_{ij}b_{ij}$. For an $n \times n$ matrix A and an $m \times m$ matrix B we define the Kronecker product of A and B to be the $nm \times nm$ block matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & & \vdots \\ a_{n1}B & \dots & a_{nn}B \end{bmatrix}.$$

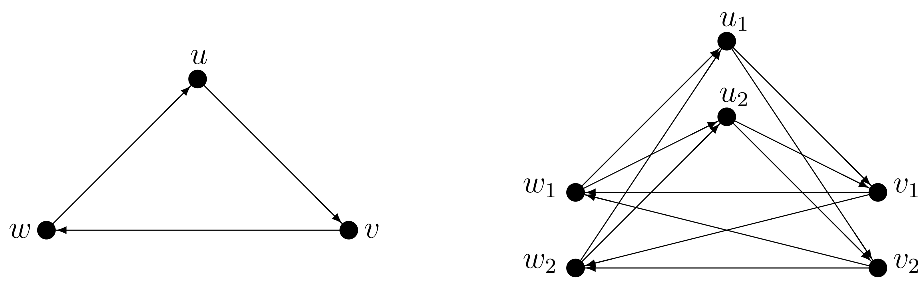

FIGURE 1. Cycle of length 3 and its 2-coclique extension.

The following lemma is an immediate consequence of the definition.

Lemma 2.1. *Let A and B be $n \times n$ matrices and let C and D be $m \times m$ matrices. Then*

$$(A \otimes C)(B \otimes D) = (AB) \otimes (CD).$$

A coclique extension of a given directed graph Γ may be conveniently defined by a Kronecker product of matrices as follows.

Definition 2.2. *Let Γ be a directed graph with adjacency matrix A. The directed graph with adjacency matrix $A \otimes J_s$ is called a coclique extension of Γ, and denoted by $\mathcal{C}_s(\Gamma)$.*

Thus $\mathcal{C}_s(\Gamma)$ is the graph obtained from Γ by replacing each vertex v with s new vertices, say $v_1, \dots, v_s$. The vertices $v_1, \dots, v_s$ form a coclique, i.e., they are pairwise non-adjacent. If w is replaced with $w_1, \dots, w_s$ then $v_i \to w_j$ is an edge if and only if $v \to w$.

Example 2.3. Let us start with the matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. Then clearly we have

$$A \otimes J_2 = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

In Figure 1 we depict two graphs, for which the matrices A and $A \otimes J_2$ serve as adjacency matrices, respectively.

Given a permutation g of a set X, we use the notation x^g for the image of $x \in X$ under g. An automorphism of a directed graph Γ is a permutation of $V(\Gamma)$ such that $(x, y) \in E(\Gamma)$ if and only if $(x^g, y^g) \in E(\Gamma)$. The set $\mathsf{Aut}(\Gamma)$ of all automorphisms of Γ forms a group called the automorphism group of Γ. Any subgroup of this group is called a group of automorphisms. A group G of automorphisms is said to be semiregular if for any two vertices x and y, there is at most one $g \in G$ such that $x^g = y$, and G is said to be transitive if for any two vertices x and y, there is at least one $g \in G$ such that $x^g = y$. If G is both semiregular and transitive then we say that G is regular.

Given a group G and a set $S \subset G$, we define the Cayley graph of G with connection set S to be the (directed) graph $\mathsf{Cay}(G, S)$ with vertex set G and arc set $\{(x, y) \mid x^{-1}y \in S\}$. We always assume that $1 \notin S$, where 1 denotes the identity of the group, so that the Cayley graph has no loops. If $S = S^{(-1)}$, where $S^{(-1)}$ denotes the set $\{s^{-1} \mid s \in S\}$, then $\mathsf{Cay}(G, S)$ is an undirected graph. If $S \cap S^{(-1)} = \emptyset$ then $\mathsf{Cay}(G, S)$ is an oriented graph. It is known that a graph Γ is a Cayley graph of a group G if and only if G is a regular group of automorphisms of Γ.

We will, in particular, consider Cayley graphs of dicyclic groups. The dicyclic group of order $4n$ is the group

$$\langle x, y \mid x^n = y^2, y^4 = 1, xyx = y\rangle.$$

The dicyclic group of order 8 is the quaternion group. A dicyclic group of order a power of 2 is also called a generalised quaternion group.

3. Main combinatorial structures

In this section, the structure and construction of various classes of doubly regular tournaments, doubly regular (m, r)-team tournaments and normally regular digraphs will be described. These are the combinatorial structures that are relevant to the three-class association schemes we are interested in. Some of the mutual relationships between the above objects will also be described.

3.1. **Doubly regular tournaments.**

A tournament is a directed graph T with the property that for any two distinct vertices x and y, either (x, y) or (y, x), but not both, belongs to $E(T)$. In terms of the adjacency matrix A, a tournament is a directed graph with the property that $A + A^{\mathrm{t}} + I = J$. If every vertex in a tournament T with n vertices has out-valency k then every vertex has in-valency $n - k - 1$. Therefore, $k = n - k - 1$, i.e., $n = 2k + 1$, so such a tournament is a regular directed graph of valency k.

Definition 3.1 ([ReiB72]). *A tournament T is called doubly regular if it is regular and for every vertex x in T the out-neighbours of x span a regular tournament.*

Example 3.2. Let us consider the Cayley graph $\Gamma = \mathsf{Cay}(\mathbb{Z}_7, X)$ over the cyclic group $\mathbb{Z}_7$, defined by the connection set $X = \{1, 2, 4\}$. The graph Γ is depicted in Figure 2. By inspection, Γ is a tournament in which the out-neighbours of the vertex 0 induce a directed triangle. The same applies to every vertex, so Γ is an example of a doubly regular tournament.

Lemma 3.3. *Suppose that T is a tournament with $|V(T)| = 4\lambda + 3$, and that its adjacency matrix A satisfies*

(a) *either $AJ = (2\lambda + 1)J$ or $JA = (2\lambda + 1)J$, and*
(b) *either $A^2 = \lambda A + (\lambda + 1)A^{\mathrm{t}}$, $AA^{\mathrm{t}} = \lambda J + (\lambda + 1)I$ or $A^{\mathrm{t}}A = \lambda J + (\lambda + 1)I$.*

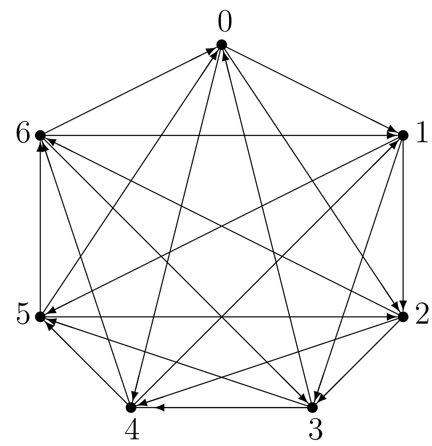

FIGURE 2. Cayley graph on 7 vertices.

Then all of these equations are satisfied and T is a doubly regular tournament. Conversely, if T is a doubly regular tournament then $|V(T)| = 4\lambda+3$ and the adjacency matrix satisfies all of the equations in (a) *and* (b)*.*

Proof. If we multiply the equation $A + A^{\mathrm{t}} = J - I$ by J we get $JA + JA^{\mathrm{t}} = (n-1)J = (4\lambda+2)J$. Thus $JA = (2\lambda+1)J$ if and only if $JA^{\mathrm{t}} = (2\lambda+1)J$. The transpose of the second equation is $AJ = (2\lambda+1)J$. Using the equations $AJ = JA = (2\lambda+1)J$ and $A + A^{\mathrm{t}} = J - I$, it can be shown that the equations involving A^2, AA^{t} and $A^{\mathrm{t}}A$ are equivalent.

Suppose that T is a doubly regular tournament with n vertices. Then it is regular of valency $k = \frac{n-1}{2}$. The subgraph spanned by the out-neighbours of a vertex is regular of valency $\lambda = \frac{k-1}{2} = \frac{n-3}{4}$. Thus $n = 4\lambda + 3$. Every vertex has out-valency $2\lambda + 1$, so $JA = (2\lambda+1)J$. Let x and y be vertices with $(x, y) \in E(T)$. Then y has out-valency λ in $N^+(x)$; i.e., x and y have exactly λ common out-neighbours. From this (and the fact that every vertex has out-valency $2\lambda + 1$) it follows that $AA^{\mathrm{t}} = \lambda J + (\lambda + 1)I$.

Conversely, if $AJ = (2\lambda+1)J$ then every vertex has out-valency $2\lambda+1$. Thus T is regular. If $AA^{\mathrm{t}} = \lambda J + (\lambda+1)I$ then for every vertex x the out-neighbours of x span a subtournament in which every vertex has out-valency λ. Thus this subtournament is regular and so T is doubly regular. □

Corollary 3.4. *Let x and y be vertices in a doubly regular tournament T with $4\lambda + 3$ vertices and suppose that $(x, y) \in E(T)$. Then the number of directed paths of length 2 from x to y is λ, the number of directed paths of length 2 from y to x is $\lambda+1$, the number of common out-neighbours of x and y is λ, and the number of common in-neighbours of x and y is λ.*

Below we discuss a couple of classical infinite series.

Paley [Pal33] constructed an infinite family of Hadamard matrices that corresponds to the following family of doubly regular tournaments. These tournaments are called

Paley tournaments (cf. Theorems 3.5, 3.7 and 3.12 for the relationship with Hadamard matrices).

Theorem 3.5 ([Pal33]). *Let $q \equiv 3 \bmod 4$ be a prime power. Let $\mathbb{F}_q$ denote the field of q elements and let Q be the set of nonzero squares in $\mathbb{F}_q$. Then the graph P_q with $V(P_q) = \mathbb{F}_q$ and $E(P_q) = \{(x, y) \mid y - x \in Q\}$ is a doubly regular tournament.*

Thus the Paley tournament P_q is the Cayley graph $\mathsf{Cay}(\mathbb{F}_q^+, Q)$ of the additive group $\mathbb{F}_q^+$.

Lemma 3.6. *The Cayley graph $\mathsf{Cay}(\mathbb{F}_q^+, -Q)$ is isomorphic to P_q.*

Proof. Multiplication by -1 defines an isomorphism between $\mathsf{Cay}(\mathbb{F}_q^+, Q)$ and $\mathsf{Cay}(\mathbb{F}_q^+, -Q)$. □

Theorem 3.7 (See e.g. [Pas92]). *Suppose that A is the adjacency matrix of a doubly regular tournament of order n. Then the matrix*

$$B = \left[\begin{array}{c|ccc|ccc} 0 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ \hline 0 & & & & & & \\ \vdots & & A & & & A+I & \\ 0 & & & & & & \\ \hline 1 & & & & & & \\ \vdots & & A & & & A^{\mathrm{t}} & \\ 1 & & & & & & \end{array}\right]$$

is the adjacency matrix of a doubly regular tournament of order $2n + 1$.

Proof. It is clear that B is the adjacency matrix of a regular tournament of valency n. An easy computation also shows that B satisfies the equation $BB^{\mathrm{t}} = (k + 1)I_{2n+1} + kJ_{2n+1}$ where $k = (n - 1)/2$. □

Another family of doubly regular tournaments was found by Szekeres [Sze69].

Theorem 3.8. *Let $q \equiv 5 \bmod 8$ be a prime power. Let $\mathbb{F}_q$ be the field of q elements and let Q be the unique subgroup of $\mathbb{F}_q^*$ of index 4. Let $Q, -Q, R, -R$ be the cosets of Q. Then the graph Sz_q with vertex set*

$$\{v\} \;\cup\; \{x_i \mid x \in \mathbb{F}_q, i = 1, 2\}$$

and arc set

$$\begin{aligned} &\{(v, x_1), (x_1, x_2), (x_2, v) \mid\; x \in \mathbb{F}_q\} \;\cup\; \{(x_1, y_1) \mid\; y - x \in Q \cup R\} \\ &\qquad \cup\; \{(x_2, y_2) \mid\; y - x \in -Q \cup -R\} \;\cup\; \{(x_i, y_{3-i}) \mid\; y - x \in Q \cup -R,\; i = 1, 2\} \end{aligned}$$

is a doubly regular tournament of order $2q + 1$.

Two more new constructions of doubly regular tournaments as Cayley graphs were presented by Ding and Yuan [DinY06] and by Ding, Wang and Xiang [DinWX07]. We refer to [Muz10], where Muzychuk constructed exponentially many doubly regular tournaments as Cayley graphs over an elementary abelian group of order q^3.

Doubly regular tournaments are also obtained from a class of Hadamard matrices.

Definition 3.9. *An $n \times n$ matrix H with entries ± 1 is called a Hadamard matrix of order n if*

$$HH^{\mathrm{t}} = nI.$$

The condition $HH^{\mathrm{t}} = nI$ implies that any two distinct rows of H are orthogonal. Multiplication of any row or any column by -1 preserves this condition. Permutation of the rows and permutation of the columns also preserve the condition.

Definition 3.10. *Let H_1 and H_2 be Hadamard matrices of order n. Then we say that H_1 and H_2 are equivalent if H_2 can be obtained from H_1 by permuting rows, permuting columns and multiplying some rows and/or columns by -1.*

It is known that if a Hadamard matrix of order $n > 2$ exists then $n \equiv 0 \bmod 4$. The well-known Hadamard matrix conjecture states that there exists a Hadamard matrix of order n for every n divisible by 4. We are interested in the following class of Hadamard matrices.

Definition 3.11. *A Hadamard matrix H is called skew if*

$$H + H^{\mathrm{t}} = -2I.$$

This implies that H is skew if $H = (h_{ij})$ satisfies $h_{ij} = -h_{ji}$ for all $i \neq j$, and $h_{ii} = -1$ for all i.

A new skew Hadamard matrix can be obtained from an old one by simultaneously multiplying the ith row and the ith column by -1 for any i. By repeating this procedure we can transform any skew Hadamard matrix to a skew Hadamard matrix of the following form

$$H = \begin{bmatrix} -1 & 1 & \cdots & 1 \\ -1 & & & \\ \vdots & & B & \\ -1 & & & \end{bmatrix} \tag{3.1}$$

where B is an $(n-1) \times (n-1)$ matrix with entries ± 1. Reid and Brown [ReiB72] proved the following equivalence between skew Hadamard matrices and doubly regular tournaments.

Theorem 3.12. *Let H and B be matrices satisfying Equation* (3.1)*. Then H is a skew Hadamard matrix of order n if and only if $A = \frac{1}{2}(B + J)$ is the adjacency matrix of a doubly regular tournament with $n - 1$ vertices.*

The proof follows by matrix computations and the properties of the adjacency matrix of a doubly regular tournament in Lemma 3.3.

If A_1 and A_2 are adjacency matrices of isomorphic doubly regular tournaments, then A_2 can be obtained from A_1 by applying the same permutation to rows and columns. Thus the Hadamard matrices constructed from A_1 and A_2 are equivalent. But it is also possible that non-isomorphic doubly regular tournaments correspond to equivalent skew Hadamard matrices.

The investigation of skew Hadamard matrices and their diverse links with other combinatorial structures is definitely a topic of independent interest. Papers such as [IonK03] and [NozS12] provide a small sample of related results.

3.2. **Doubly regular orientations of $\overline{m \circ K_r}$.** Let $m \circ K_r$ denote the disjoint union of m copies of the complete graph on r vertices, and let $\overline{m \circ K_r}$ denote its complement, i.e., the complete multipartite graph with m independent sets of size r. Let Γ be an orientation of $\overline{m \circ K_r}$, i.e., every edge $\{a, b\}$ in $\overline{m \circ K_r}$ is replaced with one of the arcs (a, b) or (b, a). Then we say that Γ is an (m, r)-team tournament.

Remark. The suggested name emphasises that there are m teams of size r, and that each member plays against all the members of the other teams, but against none of his own team. The authors feel that this terminology is consistent with the traditional use of the word "tournament" in graph theory, while future possible alternatives to this suggestion are welcome.

Definition 3.13. *An (m, r)-team tournament Γ with adjacency matrix A is said to be doubly regular if there exist integers k, α, β, γ such that*

i) *Γ is regular with valency k,*
ii) *$A^2 = \alpha A + \beta A^{\mathrm{t}} + \gamma(J - I - A - A^{\mathrm{t}})$.*

Note that the equation $A^2 = \alpha A + \beta A^{\mathrm{t}} + \gamma(J - I - A - A^{\mathrm{t}})$ means that the number of directed paths of length 2 from a vertex x to a vertex y is α if $(x, y) \in E(\Gamma)$, β if $(y, x) \in E(\Gamma)$, and γ if $\{x, y\}$ is an edge of $m \circ K_r$.

Proposition 3.14. *Let T be a doubly regular tournament of order $m = 4\lambda + 3$ and let $r \in \mathbb{N}$. Then the coclique extension $\mathcal{C}_r(T)$ of T is a doubly regular (m, r)-team tournament.*

Proof. Let A be the adjacency matrix of T. Then, by Lemma 2.1, the adjacency matrix $A \otimes J_r$ of $\mathcal{C}_r(T)$ satisfies

$$(A \otimes J_r)^2 = A^2 \otimes J_r^2 = (\lambda A + (\lambda + 1)A^{\mathrm{t}}) \otimes rJ_r = r\lambda(A \otimes J_r) + r(\lambda + 1)(A \otimes J_r)^{\mathrm{t}}.$$

□

We also have another construction of doubly regular team tournaments from doubly regular tournaments.

Definition 3.15. *Let T be a tournament with adjacency matrix A. Then the graph with adjacency matrix*

$$\begin{bmatrix} 0 & 1 \cdots 1 & 0 & 0 \cdots 0 \\ 0 & & 1 & \\ \vdots & A & \vdots & A^{\mathrm{t}} \\ 0 & & 1 & \\ 0 & 0 \cdots 0 & 0 & 1 \cdots 1 \\ 1 & & 0 & \\ \vdots & A^{\mathrm{t}} & \vdots & A \\ 1 & & 0 & \end{bmatrix}$$

is denoted by $\mathcal{D}(T)$.

In other words, if $T = (V, E)$ is a tournament with $V = \{x_1, \ldots, x_n\}$ then $\mathcal{D}(T)$ is the graph with vertex set $\{v_0, v_1, \ldots, v_n, w_0, w_1, \ldots, w_n\}$ and arc set

$$\{(v_0, v_i), (v_i, w_0), (w_0, w_i), (w_i, v_0) \mid i = 1, \ldots, n\} \\ \cup\ \{(v_i, v_j), (v_j, w_i), (w_i, w_j), (w_j, v_i) \mid (x_i, x_j) \in E\}.$$

Proposition 3.16. *For any doubly regular tournament* T *of order* $m = 2k + 1 = 4\lambda + 3$, *the graph* $\mathcal{D}(T)$ *is a doubly regular* $(m + 1,\ 2)$*-team tournament.*

Proof. Let D be the adjacency matrix of $\mathcal{D}(T)$. Then an easy computation shows that

$$D^2 = kD + kD^{\mathrm{t}} + m(J - I - D - D^{\mathrm{t}}). \tag{3.2}$$

□

3.3. **Normally regular digraphs.** The notion of a normally regular digraph was suggested by Jørgensen in [Jør94b] as a possible generalisation of the notion of strongly regular graphs. For the reader's convenience we provide here a short digest of some of the most important facts regarding this notion, in particular some facts about orientations of $\overline{m \circ K_r}$ that are normally regular digraphs.

Definition 3.17. *A directed graph* Γ *with* v *vertices and adjacency matrix* A *is called a normally regular digraph with parameters* (v, k, λ, μ) *if*

(i) $AA^{\mathrm{t}} = kI + \lambda(A + A^{\mathrm{t}}) + \mu(J - I - A - A^{\mathrm{t}})$, *and*
(ii) $A + A^{\mathrm{t}}$ *is a* $\{0, 1\}$*-matrix.*

The condition (i) says that, for any pair x, y of vertices, the number of common out-neighbours of x and y is

$$\begin{cases} k, & \text{if } x = y, \\ \lambda, & \text{if } x \text{ and } y \text{ are adjacent, i.e., if either } (x, y) \text{ or } (y, x) \text{ is in } \Gamma, \\ \mu, & \text{if } x \text{ and } y \text{ are nonadjacent.} \end{cases}$$

In particular, every vertex has out-valency k, i.e., $AJ = kJ$. The second condition says that, for any pair x, y of vertices, at most one of the arcs (x, y) and (y, x) is present in the directed graph.

Example 3.18. We present two small normally regular digraphs constructed as Cayley graphs over cyclic groups.

(1) The Cayley graph $\mathsf{Cay}(\mathbb{Z}_{13}, \{1, 3, 9\})$ is a normally regular digraph with parameters $(13, 3, 0, 1)$.
(2) The Cayley graph $\mathsf{Cay}(\mathbb{Z}_{19}, \{1, 4, 6, 7, 9, 11\})$ is a normally regular digraph with parameters $(19, 6, 1, 3)$. This is the smallest graph from the infinite series constructed in Theorem 3.21.

One may consider general normally regular digraphs, satisfying condition (i) but not necessarily (ii). However in this paper it is natural to require condition (ii), as we investigate relations to other structures satisfying (ii).

It is convenient to introduce two new parameters

$$\eta = k - \mu + (\mu - \lambda)^2 \quad \text{and} \quad \rho = k + \mu - \lambda.$$

The matrix equation in the definition of normally regular digraphs can be written as follows

$$(A + (\mu - \lambda)I)(A + (\mu - \lambda)I)^{\mathrm{t}} = (k - \mu + (\mu - \lambda)^2)I + \mu J.$$

Thus the matrix $B = (A + (\mu - \lambda)I)$ satisfies the following equations

$$BB^{\mathrm{t}} = \eta I + \mu J$$

and (as $AJ = kJ$)

$$BJ = \rho J.$$

Proposition 3.19 ([Jør94b]). *If A is the adjacency matrix of a normally regular digraph then A is normal, i.e.,*

$$A^{\mathrm{t}} A = AA^{\mathrm{t}}.$$

Proof. It is sufficient to prove that B is normal. Suppose first that B is singular. Then one of the eigenvalues of $\eta I + \mu J$ is zero: $\eta = 0$ or $\eta + \mu v = 0$. Since $\mu, v \geq 0$ this is possible only when $\eta = k - \mu + (\mu - \lambda)^2 = 0$. This implies that $k = \mu = \lambda$. But if $k > 0$ then the k out-neighbours of a vertex span a λ-regular graph and so $k \geq 2\lambda + 1$. Thus B is nonsingular.

From $BJ = \rho J$ we get $\rho^{-1} J = B^{-1} J$ and

$$B^{\mathrm{t}} = B^{-1}(BB^{\mathrm{t}}) = B^{-1}(\eta I + \mu J) = \eta B^{-1} + \mu \rho^{-1} J. \tag{3.3}$$

Using the fact that J is symmetric, we deduce from this that

$$\rho J = (BJ)^{\mathrm{t}} = JB^{\mathrm{t}} = \eta JB^{-1} + \mu \rho^{-1} J^2 = \eta JB^{-1} + \mu \rho^{-1} v J.$$

This implies that

$$JB^{-1} = \frac{\rho - \mu \rho^{-1} v}{\eta} J,$$

and so

$$vJ = J^2 = (JB^{-1})(BJ) = \frac{\rho - \mu \rho^{-1} v}{\eta} \rho v J.$$

Thus

$$\frac{\rho - \mu \rho^{-1} v}{\eta} = \rho^{-1}, \tag{3.4}$$

and $JB^{-1} = \rho^{-1} J$ or $\rho J = JB$. Now Equation (3.3) implies that

$$B^{\mathrm{t}} B = \eta I + \mu \rho^{-1} JB = \eta I + \mu J = BB^{\mathrm{t}}.$$

Thus we also have $A^{\mathrm{t}} A = AA^{\mathrm{t}}$. □

Corollary 3.20. *A normally regular digraph is a regular graph of valency k, i.e.,*

$$AJ = JA = kJ$$

and the number of common in-neighbours of vertices x and y is equal to the number of common out-neighbours of x and y.

Since the graph is regular we can now count pairs of vertices (y, z) with $x \to y \leftarrow z$, $z \neq x$, where x is a fixed vertex, in two ways. We get

$$k(k-1) = 2k\lambda + (n-1-2k)\mu, \tag{3.5}$$

which is equivalent to Equation (3.4).

The preprint [Jør94b] and its update [Jør14] contain many examples of normally regular digraphs, as well as numerical restrictions on their parameters. We present one such result below, referring to the proof of Theorem 31 in [Jør14].

Theorem 3.21 ([Jør94b])**.** *Let k be a multiple of 3, such that $k+1$ is a prime power. Then there exists a normally regular digraph with parameters $(\frac{k^2+3k+3}{3}, k, 1, 3)$, which appears as a Cayley graph over the cyclic group of order $\frac{k^2+3k+3}{3}$.*

For any normally regular digraph we have $0 \leq \mu \leq k$. We next give a structural characterisation of normally regular digraphs with equality in one of these inequalities.

Theorem 3.22 ([Jør94b])**.** *A directed graph Γ is a normally regular digraph with $\mu = k$ if and only if Γ is isomorphic to $\mathcal{C}_s(T)$ for some doubly regular tournament T and natural number s.*

Proof. It is easy to verify that if T is a doubly regular tournament on $4t+3$ vertices then $\mathcal{C}_s(T)$ is a normally regular digraph with $(v, k, \lambda, \mu) = (s(4t+3), s(2t+1), st, s(2t+1))$.

Suppose that Γ is a normally regular digraph with $\mu = k$. Suppose that x and y are non-adjacent vertices in Γ. Then, as $\mu = k$, $N^+(x) = N^+(y)$. If z is another vertex not adjacent to x then $N^+(z) = N^+(x) = N^+(y)$. Since $\lambda \leq \frac{k-1}{2}$, y and z are non-adjacent. Non-adjacency is thus a transitive relation on the vertex-set, which is therefore partitioned into classes, say $V_1, \ldots, V_r$, such that any two vertices are adjacent if and only if they belong to distinct classes.

Suppose that $x \in V_i$ dominates $y \in V_j$, where $1 \leq i, j \leq r$. Then every vertex $x' \in V_i$ dominates every vertex $y' \in V_j$. For if $x \neq x'$ then x and x' are non-adjacent and have the same out-neighbours, so that $x' \to y$, and similarly $x' \to y'$. We also have $2k = |N^+(x)| + |N^-(x)| = v - |V_i|$. Thus Γ is isomorphic to $\mathcal{C}_s(T)$ for some regular tournament T and $s = v - 2k$.

If $v \to w$ is an arc in T then it follows that the number of common out-neighbours of v and w in T is $\frac{\lambda}{s}$. It follows from Lemma 3.3 that T is a doubly regular tournament. $\square$

Theorem 3.23 ([Jør94b])**.** *A connected graph Γ is a normally regular digraph with $\mu = 0$ if and only if Γ is isomorphic to T or $\mathcal{D}(T)$ for some doubly regular tournament T, or Γ is a directed cycle.*

Proof. It is easy to verify that if Γ is a doubly regular tournament T or a directed cycle then Γ is a normally regular digraph with $\mu = 0$. (Since a doubly regular tournament has no pairs of non-adjacent vertices, μ is arbitrary but we may choose $\mu = 0$.)

Suppose that Γ is isomorphic to $\mathcal{D}(T)$ for some doubly regular tournament T with adjacency matrix A as in Definition 3.15. If the order of T is $s = 2\ell + 1$ then, by Lemma 3.3, the adjacency matrix D of $\mathcal{D}(T)$ satisfies

$$DD^{\mathrm{t}} = \begin{bmatrix} s & \ell \ \ldots \ \ell & 0 & \ell \ \ldots \ \ell \\ \ell & & \ell & \\ \vdots & \ell J + (\ell+1)I & \vdots & \ell(A + A^{\mathrm{t}}) \\ \ell & & \ell & \\ 0 & \ell \ \ldots \ \ell & s & \ell \ \ldots \ \ell \\ \ell & & \ell & \\ \vdots & \ell(A + A^{\mathrm{t}}) & \vdots & \ell J + (\ell+1)I \\ \ell & & \ell & \end{bmatrix} = sI + \ell(D + D^{\mathrm{t}}). \tag{3.6}$$

Thus Γ is a normally regular digraph with parameters $(v, k, \lambda, \mu) = (2s + 2, s, \ell, 0)$.

Let Γ be a connected normally regular digraph with $\mu = 0$. Suppose that Γ is not a directed cycle. Then $k \geq 2$. If Γ is a tournament then it follows from Lemma 3.3 that T is a doubly regular tournament. So we may assume that Γ is not a tournament. Let x be a vertex in Γ and let y be a vertex not adjacent to x.

Since $\mu = 0$, it follows from Equation (3.5) that $\lambda = \frac{k-1}{2} \geq 1$. Thus the subgraphs spanned by $N^+(x)$ and $N^-(x)$ are regular tournaments of valency λ. Since $\mu = 0$, y has no out-neighbour in $N^+(x)$. But since Γ is connected, we may assume that y has an out-neighbour in $N^-(x)$. Let $U = N^+(y) \cap N^-(x)$ and suppose that $U \neq N^-(x)$. Since $N^-(x)$ is a regular tournament, there exist vertices $u \in U$ and $u' \in N^-(x) \setminus U$ such that $u' \to u$. But y and u' have a common out-neighbour and so they are adjacent. So we have $y \to u'$, a contradiction. Thus $N^+(y) = N^-(x)$.

Suppose there is another vertex y' non-adjacent to x such that y' has an out-neighbour in $N^-(x)$. Then as above $N^+(y') = N^-(x)$. But then y and y' have k common out-neighbours, a contradiction to $k > \lambda > \mu = 0$. Thus, as $N^-(x)$ is regular of valency λ, any vertex in $u \in N^-(x)$ has $k - 1 - \lambda = \lambda$ in-neighbours in $N^+(x)$. It also has λ out-neighbours in $N^+(x)$, as x and u have λ common out-neighbours.

Now there is a vertex $w \in N^+(x)$ such that w and y have a common out-neighbour in $N^-(x)$. Thus $w \to y$. As above, it follows that $N^-(y) = N^+(x)$ and that any vertex in $N^+(x)$ has λ out-neighbours and λ in-neighbours in $N^-(x)$.

Since Γ is connected, it has vertex set $\{x, y\} \cup N^+(x) \cup N^-(x)$. For every vertex there is a unique non-adjacent vertex. Thus Γ is a $(\frac{v}{2}, 2)$-team tournament. In fact, we have now proved that it is a doubly regular $(\frac{v}{2}, 2)$-team tournament with $\alpha = \beta = \lambda$ and $\gamma = k$. (Since x is an arbitrary vertex we need only consider paths of length 2 starting at x.) The statement now follows from Theorem 4.6, see Section 4. $\square$

3.4. **Association schemes and their relation graphs.**

Definition 3.24. *Let X be finite set and let $\{R_0, R_1, \ldots, R_d\}$ be a partition of $X \times X$. Then we say that $\mathcal{X} = (X, \{R_0, R_1, \ldots, R_d\})$ is a d-class association scheme if the following conditions are satisfied:*

- $R_0 = \{(x, x) \mid x \in X\}$,
- *for each $i \in \{0, \ldots, d\}$ there exists $i' \in \{0, \ldots, d\}$ such that*
$$R_{i'} = \{(x, y) \mid (y, x) \in R_i\},$$
- *for each triple (i, j, h), $i, j, h \in \{0, \ldots, d\}$, there exists a number p_{ij}^h such that for all $x, y \in X$ with $(x, y) \in R_h$ there are exactly p_{ij}^h elements $z \in X$ such that $(x, z) \in R_i$ and $(z, y) \in R_j$.*

The binary relations $R_1, \ldots, R_d$ can be identified with (arc sets of) graphs on X. A relation R_i with $i' = i$ is an undirected graph and a relation R_i with $i' \neq i$ is a directed (oriented) graph. If the graphs $R_1, \ldots, R_d$ are all connected then we say that $\mathcal{X}$ is primitive; otherwise it is imprimitive.

If $i = i'$ for all i then $\mathcal{X}$ is said to be symmetric; otherwise it is non-symmetric. For a symmetric 2-class association scheme the graph R_1 is a strongly regular graph and R_2 is its complement. Conversely, if Γ is a strongly regular graph then Γ and $\overline{\Gamma}$ (i.e., $E(\Gamma)$ and $E(\overline{\Gamma})$) form the relations of a symmetric 2-class association scheme. It is known that for an imprimitive symmetric 2-class association scheme, either R_1 or R_2 is isomorphic to $m \circ K_r$ for some m, r.

For a non-symmetric 2-class association scheme the graph R_1 is a doubly regular tournament and conversely (see Lemma 3.3) for every doubly regular tournament R_1 there is a non-symmetric 2-class association scheme $(X, \{R_0, R_1, R_2\})$. Every non-symmetric 2-class association scheme is primitive.

Let $X = \{x_1, \ldots, x_n\}$ be a finite set and let $R_0, \ldots, R_d$ be graphs (possibly with loops) with vertex set X. Let A_i be the adjacency matrix of R_i, for $i = 0, \ldots, d$. Let $\mathcal{A}$ be the vector space spanned by $A_0, \ldots, A_d$. Then $\{R_0, \ldots, R_d\}$ is a partition of $X \times X$ if and only if $A_0 + \cdots + A_d = J$ and $A_i \circ A_j = 0$ for all $i \neq j$. The other conditions in Definition 3.24 are equivalent to

- $A_0 = I$,
- $A_i^{\mathrm{t}} = A_{i'}$,
- $\mathcal{A}$ is closed under matrix multiplication and $A_i A_j = \sum_h p_{ij}^h A_h$.

Furthermore, if these conditions are satisfied then $\mathcal{A}$ is closed under the Schur-Hadamard product. The algebra $\mathcal{A}$ with two operations of multiplication, namely the usual matrix product and the Schur-Hadamard product, is called the adjacency algebra of the association scheme. The association scheme is symmetric if and only if $\mathcal{A}$ consists of symmetric matrices.

An association scheme $\mathcal{X}$ is called commutative if $p_{ij}^h = p_{ji}^h$ for all i, j, h. Thus $\mathcal{X}$ is commutative if and only if $\mathcal{A}$ is commutative. The adjacency algebra of a commutative association scheme is usually called the Bose–Mesner algebra of this scheme. Every symmetric association scheme is commutative, for if A and B belong to an algebra of symmetric matrices then $AB = A^{\mathrm{t}}B^{\mathrm{t}} = (BA)^{\mathrm{t}} = BA$. Higman [Hig75] proved that every d-class association scheme with $d \le 4$ is commutative.

For a commutative non-symmetric association scheme $\mathcal{X} = (X, \{R_0, R_1, \ldots, R_d\})$ it is known (see [BanI84, Remark in Section 2.2]) that a symmetric association scheme $\tilde{\mathcal{X}}$, called the symmetrisation of $\mathcal{X}$, can be constructed by replacing each pair $R_i, R_{i'}$ ($i \ne i'$) with $R_i \cup R_{i'}$. The corresponding Bose–Mesner algebra is

$$\tilde{\mathcal{A}} = \{A \in \mathcal{A} \mid A = A^{\mathrm{t}}\}.$$

Its basis consists of the matrices A_i where $i = i'$, and $A_i + A_{i'}$ where $i \ne i'$.

Let $\mathcal{X}$ be a non-symmetric 3-class association scheme. Then $\mathcal{X}$ consists of a pair of non-symmetric relations R_1, R_2 and one symmetric relation R_3 (in addition to R_0). By the theorem of Higman mentioned above, $\mathcal{X}$ is commutative. Thus the symmetrisation of $\mathcal{X}$ exists and is a 2-class association scheme with relations $R_1 \cup R_2$ and R_3.

A non-symmetric association scheme $\mathcal{X}$ is imprimitive if and only if the symmetrisation $\tilde{\mathcal{X}}$ is imprimitive. Now suppose that $\mathcal{X}$ is imprimitive. Then R_1 is an orientation of either $m \circ K_r$ or $\overline{m \circ K_r}$.

Proposition 3.25. *R_1 is an orientation of $m \circ K_r$ if and only if R_1 consists of m disjoint doubly regular tournaments of order r.*

Proof. Suppose first that R_1 is an orientation of $m \circ K_r$. Then R_1 consists of m disjoint tournaments of order r, and R_1 and each of the tournaments are regular of valency p^0_{12}. Let T be one of the components of R_1. Let $(x, y) \in T$. Then there are p^1_{11} paths of length 2 from x to y. Thus, according to Lemma 3.3, T is a doubly regular tournament.

Conversely, if R_1 is a disjoint union of m doubly regular tournaments of order r (not necessarily isomorphic) then $(V(R_1), \{R_0, R_1, R_2, R_3\})$ is an imprimitive 3-class association scheme with $R_2 = R_1^{\mathrm{t}}$ and $R_3 = \overline{R_1 \cup R_2}$. □

We now consider the case where R_1 is an orientation of $\overline{m \circ K_r}$. First we give two types of examples.

Proposition 3.26. *Let R_1 be one of the graphs $\mathcal{C}_r(T)$ where $r \ge 2$, or $\mathcal{D}(T)$ where T is a doubly regular tournament. Let $R_2 = R_1^{\mathrm{t}}$ and $R_3 = \overline{R_1 \cup R_2}$. Then*

$$(V(R_1), \{R_0, R_1, R_2, R_3\})$$

is an imprimitive 3-class association scheme.

Proof. Let T be a doubly regular tournament on v vertices and with adjacency matrix A. Then $\mathcal{C}_r(T)$ has adjacency matrix $A \otimes J_r$. The relations R_2, R_3 and R_0 in this case have adjacency matrices $A^{\mathrm{t}} \otimes J_r$, $I_v \otimes (J_r - I_r)$ and $I_v \otimes I_r$. It follows from Lemma 2.1 and Lemma 3.3 that the set of linear combinations of these matrices is closed under multiplication.

Now suppose that D is the adjacency matrix of $\mathcal{D}(T)$ for some doubly regular tournament T. From Equations (3.2) and (3.6) we know that $D^2 = \ell D + \ell D^{\mathrm{t}} = (D^{\mathrm{t}})^2$ and $DD^{\mathrm{t}} = sI + \ell(D + D^{\mathrm{t}})$ for some $s = 2\ell + 1$. By Proposition 3.19, $D^{\mathrm{t}}D = DD^{\mathrm{t}}$. We also know that $JD = DJ = JD^{\mathrm{t}} = D^{\mathrm{t}}J = sJ$. It follows that any product of two matrices in $B = \{I, D, D^{\mathrm{t}}, J - I - D - D^{\mathrm{t}}\}$ is a linear combination of matrices in B. Thus the relations with adjacency matrices in B form an imprimitive association scheme with 3 classes. □

In Section 5 we will continue our analysis of links between orientations of $\overline{m \circ K_r}$ and association schemes based on a characterisation of doubly regular (m,r)-team tournaments which will be achieved in the next section. We will also mention some constructions of doubly regular team tournaments with $r > 2$ in Section 5.2.

4. Characterisation of doubly regular (m,r)-team tournaments

4.1. **Three types of graphs.** Let Γ be a doubly regular (m,r)-team tournament and let α, β, γ be as in Definition 3.13. Let $V(\Gamma) = V_1 \cup \cdots \cup V_m$ be the partition of the vertex set into m independent sets of size r.

In this section we prove that Γ is of one of three types:

Type I. $\mathcal{C}_r(T)$ for some doubly regular tournament T.

Type II. Every vertex in V_i dominates exactly half of the vertices in each V_j, for $j \neq i$. This type includes $\mathcal{D}(T)$ for a doubly regular tournament T.

Type III. Every vertex in V_i dominates either all vertices of V_j, exactly half of the vertices in each V_j or none of the vertices of V_j, for $j \neq i$, but Γ is not one of the above types. (No examples of this structure are known.)

More details about the structure of the last two types are given below.

For $x \in V_i$, let $d_j(x) = |N^+(x) \cap V_j|$ be the number of out-neighbours of x in V_j.

Lemma 4.1. *Let $x \in V_i$ and $y \in V_j$ and suppose that $(x, y) \in E(\Gamma)$. Then*

$$d_j(x) - d_i(y) = \beta - \alpha.$$

Proof. The vertex x has $k - d_j(x)$ out-neighbours outside V_j. Exactly α of these out-neighbours are in-neighbours of y and $k - d_j(x) - \alpha$ vertices are common out-neighbours of x and y. Similarly, y has $k - d_i(y)$ out-neighbours outside V_i. Then β of these vertices are in-neighbours of x and $k - d_i(y) - \beta$ vertices are common out-neighbours of x and y. Thus $k - d_j(x) - \alpha = k - d_i(y) - \beta$. This proves the lemma. □

Lemma 4.2. *For each pair (i, j), $i \neq j$, either*

1) *there exists a constant c_{ij} such that $d_j(x) = c_{ij}$ for every $x \in V_i$ or else*
2) *V_i is partitioned into two nonempty sets $V_i = V_i' \cup V_i''$ so that all arcs are directed from V_i' to V_j and from V_j to V_i''.*

Proof. Suppose that such a constant c_{ij} does not exist. Let $x', x'' \in V_i$ be such that $d_j(x') \neq d_j(x'')$. Let $y \in V_j$. If $(x', y), (x'', y) \in E(\Gamma)$ then

$$d_j(x') - d_i(y) = \beta - \alpha = d_j(x'') - d_i(y),$$

i.e., $d_j(x') = d_j(x'')$, a contradiction. If $(y, x'), (y, x'') \in E(\Gamma)$ then

$$d_i(y) - d_j(x') = \beta - \alpha = d_i(y) - d_j(x''),$$

a contradiction. Thus V_j is partitioned into two sets $V_j = V_j' \cup V_j''$ (one of the sets may be empty) so that V_j' is the set of out-neighbours of x' in V_j and V_j'' is the set of out-neighbours of x'' in V_j.

For every vertex $x \in V_i$ with an out-neighbour in V_j', $d_j(x) = d_j(x') \neq d_j(x'')$. Thus x has no out-neighbour in V_j'' and so the set of out-neighbours of x in V_j is exactly V_j'. Similarly, if x has an out-neighbour in V_j'' then the set of out-neighbours of x in V_j is V_j''. If x has no out-neighbour in V_j then x has a common in-neighbour with either x' or x'', say with x''. Then $d_j(x'') = d_j(x) = 0$ and so the set of out-neighbours of x in V_j is $V_j'' = \emptyset$. Thus we also get a partition $V_i = V_i' \cup V_i''$ so that the arcs between V_i and V_j are directed from V_i' to V_j', from V_j' to V_i'', from V_i'' to V_j'' and from V_j'' to V_i'. If either V_j' or V_j'' is empty then the second case of the lemma holds. So suppose that V_j' and V_j'' are both nonempty. Let $y' \in V_j'$ and $y'' \in V_j''$. From

$$|V_j''| - |V_i'| = d_j(x'') - d_i(y'') = \beta - \alpha = d_i(y') - d_j(x'') = |V_i''| - |V_j''|,$$

we get by adding $|V_i'| + |V_i''| = r = |V_j'| + |V_j''|$ that $d_j(x'') = |V_j''| = |V_j'| = d_j(x')$, a contradiction. □

Theorem 4.3. *Let Γ be a doubly regular (m,r)-team tournament. Then Γ satisfies one of the following:*

Type I. $\beta - \alpha = r$ *and Γ is isomorphic to $\mathcal{C}_r(T)$, for some doubly regular tournament T.*

Type II. $\beta - \alpha = 0$, *r is even and $d_i(x) = \frac{r}{2}$ for all $x \notin V_i$.*

Type III. $\beta - \alpha = \frac{r}{2}$ *and for every pair $\{i,j\}$ either V_i is partitioned into two sets V_i' and V_i'' of size $\frac{r}{2}$ so that all arcs between V_i and V_j are directed from V_i' to V_j and from V_j to V_i'', or similarly with i and j interchanged.*

Proof. Suppose that for some given pair (i,j), case (2) of Lemma 4.2 is satisfied. Let $x' \in V_i', x'' \in V_i'', y \in V_j$. By Lemma 4.1,

$$|V_j| - |V_i''| = d_j(x') - d_i(y) = \beta - \alpha = d_i(y) - d_j(x'') = |V_i''|.$$

Thus $|V_i''| = \frac{1}{2}|V_j| = \frac{r}{2}$, $|V_i'| = \frac{r}{2}$ and $\beta - \alpha = \frac{r}{2}$. A similar argument applies if the pair (j,i) satisfies case (2).

Suppose that there exist constants c_{ij} and c_{ji} such that $d_j(x) = c_{ij}$ for every $x \in V_i$ and $d_i(x) = c_{ji}$ for every $x \in V_j$. Then there are two possibilities.

a) One of c_{ij}, c_{ji} is 0, suppose that $c_{ji} = 0$ and $c_{ij} = r$. Then $\beta - \alpha = c_{ij} - c_{ji} = r$.
b) c_{ij} and c_{ji} are both positive, so that there exist arcs from V_i to V_j and arcs from V_j to V_i. Then $c_{ij} - c_{ji} = \beta - \alpha = c_{ji} - c_{ij}$, i.e., $c_{ij} = c_{ji} = \frac{r}{2}$ and $\beta - \alpha = 0$.

Since the value of $\beta - \alpha$ is independent of the choice of $\{i,j\}$, we see that if case (2) of Lemma 4.2 is satisfied for at least one pair (i,j) then Γ is of Type III, and if b) is satisfied for some pair (i,j) then Γ is of Type II.

Suppose now that a) holds for every pair $\{i,j\}$. Then clearly Γ is isomorphic to $\mathcal{C}_r(T)$ for some regular tournament T. Suppose that $(x,y) \in E(T)$ and that there are λ paths of length 2 from x to y in T. Then there are exactly $r\lambda$ paths of length 2 between vertices in Γ in the cocliques corresponding to x and y. Thus $\lambda = \frac{\alpha}{r}$ is constant and so T is doubly regular. □

For doubly regular (m,r)-team tournaments of Type II and Type III it is possible to compute the parameters α, β, γ from m and r and to give some restrictions on m and r.

Theorem 4.4. *Let Γ be a doubly regular (m,r)-team tournament of Type II. Then*

1) $\alpha = \beta = \frac{(m-2)r}{4}$,
2) $\gamma = \frac{(m-1)r^2}{4(r-1)}$, *and*
3) $r-1$ *divides* $m-1$.

Proof. 1) Let $x \in V_1$. The number of directed paths of length 2 from x to a vertex $y \notin V_1$ is $(m-1)\frac{r}{2} \cdot (m-2)\frac{r}{2}$. The number of such vertices y is $(m-1)r$, so

$$\alpha = \beta = \frac{(m-1)\frac{r}{2}(m-2)\frac{r}{2}}{(m-1)r} = \frac{(m-2)r}{4}.$$

2) The number of directed paths of length 2 from x to a vertex $y \in V_1$ is $(m-1)\frac{r}{2} \cdot \frac{r}{2}$. The number of such vertices y is $r-1$. Thus

$$\gamma = \frac{(m-1)\frac{r}{2}\frac{r}{2}}{r-1} = \frac{(m-1)r^2}{4(r-1)}.$$

3) Since γ is an integer and r and $r-1$ are relatively prime, $r-1$ divides $m-1$. $\square$

Theorem 4.5. *Let Γ be a doubly regular (m,r)-team tournament of Type III. Then*

1) $\alpha = \frac{(m-1)r}{4} - \frac{3r}{8}$,
2) $\beta = \frac{(m-1)r}{4} + \frac{r}{8}$,
3) $\gamma = \frac{(m-1)r^2}{8(r-1)}$,
4) *for every i and for every $x \in V_i$, the sets $\{j \mid d_j(x) = 0\}$, $\{j \mid d_j(x) = \frac{r}{2}\}$ and $\{j \mid d_j(x) = r\}$ have cardinality $\frac{m-1}{4}$, $\frac{m-1}{2}$ and $\frac{m-1}{4}$, respectively.*
5) 8 *divides* r, *and*
6) $4(r-1)$ *divides* $m-1$.

Proof. For each pair $\{i,j\}$, either (i,j) or (j,i) satisfies case (2) of Lemma 4.2. Thus in the subgraph spanned by $V_i \cup V_j$ there are $\frac{r^3}{4}$ directed paths of length 2 joining two nonadjacent vertices. The total number of directed paths of length 2 joining two nonadjacent vertices in Γ is $\binom{m}{2}\frac{r^3}{4}$. The number of ordered pairs of nonadjacent vertices is $mr(r-1)$. Thus $\gamma = \frac{(m-1)r^2}{8(r-1)}$ as $\binom{m}{2}\frac{r^3}{4} = mr(r-1)\gamma$. Since the number of directed paths of length 2 starting at a vertex x is independent of x, Γ satisfies property (4).

The number of arcs in Γ is mrk and the number of directed paths of length 2 is mrk^2. Thus

$$mrk^2 - \binom{m}{2}\frac{r^3}{4} = mrk(\alpha + \beta).$$

Since $k = \frac{(m-1)r}{2}$,

$$\frac{(m-1)^2r^2}{4} - \frac{(m-1)r^2}{8} = \frac{(m-1)r}{2}(\alpha + \beta),$$

i.e.,

$$\frac{(m-1)r}{2} - \frac{r}{4} = \alpha + \beta.$$

Since $\beta - \alpha = \frac{r}{2}$, we get the required solutions for α and β.

Since β and $\frac{m-1}{4}$ are integers, $\frac{r}{8}$ is also an integer.

Since γ is an integer and $r-1$ and r are relatively prime, $r-1$ divides $m-1$. Since 4 divides $m-1$ and $r-1$ is odd, $4(r-1)$ divides $m-1$. □

Remark. It follows from 5) and 6) in Theorem 4.5 that the first feasible case is $(m, r) = (29, 8)$. Suppose that a doubly regular $(29, 8)$-team tournament of Type III exists. Consider a fixed set V_i. For exactly half of the other sets V_j, say $V_{j_1}, \ldots, V_{j_{14}}$, the set V_i is partitioned into two sets, say $V_i'(j)$ and $V_i''(j)$, as in Lemma 4.2. It follows from the existence of the parameter γ that $V_i'(j_1), \ldots, V_i'(j_{14})$ are the blocks of a 2-design with point set V_i and with parameters $(v, b, r, k, \lambda) = (8, 14, 7, 4, 3)$. There are 4 non-isomorphic designs with these parameters; one of them is a Hadamard 3-design, see [BethJL85].

In order to construct a doubly regular $(29, 8)$-team tournament of Type III, we therefore need to put together 29 such designs (one for each V_i), in such a way that the parameters α and β also exist. It is not clear whether this is possible.

4.2. **Doubly regular $(m, 2)$-team tournaments.** We are mainly interested in the case $r = 2$. In this case Song [So95] proved that either $m \equiv 0 \bmod 4$ or $m \equiv 3 \bmod 4$.

Theorem 4.6. *Let Γ be a doubly regular $(m, 2)$-team tournament. Then Γ is isomorphic to either $\mathcal{C}_2(T)$ or $\mathcal{D}(T)$ for some doubly regular tournament T.*

Proof. Suppose that Γ is not isomorphic to $\mathcal{C}_2(T)$ for any doubly regular tournament T. Then, by Theorem 4.5, Γ is of Type II. By Theorem 4.3, every vertex in V_i has exactly one out-neighbour in V_j, $i \neq j$. Thus the subgraph spanned by $V_i \cup V_j$ is a directed 4-cycle.

Let v_0 be a vertex in Γ and $v_1, \ldots v_k$ be the out-neighbours of v_0. For $i = 0, \ldots, k$ let w_i be the unique vertex in Γ not adjacent to v_i. Then $V(\Gamma) = \{v_0, \ldots, v_k, w_0, \ldots, w_k\}$ and these vertices are distinct.

Suppose that $(v_i, v_j) \in E(\Gamma)$. Since the subgraph spanned by $\{v_i, v_j, w_i, w_j\}$ is a 4-cycle, we also have (v_j, w_i), $(w_i, w_j), (w_j, v_i) \in E(\Gamma)$. Thus Γ is isomorphic to $\mathcal{D}(T)$ where T is the tournament spanned by $N^+(v_0)$. Since there are α paths of length 2 from v_0 to v_i, v_i has in-valency α in T. Thus T is regular of valency α.

Suppose that $(v_i, v_j) \in E(T)$. Let λ denote the number of paths of length 2 in T from v_i to v_j. Then v_i and v_j have $\alpha - 1 - \lambda$ common out-neighbours in T. Thus v_j has $\alpha - (\alpha - 1 - \lambda) = \lambda + 1$ out-neighbours in T which are in-neighbours of v_i, i.e., there are $\lambda + 1$ paths of length 2 in T from v_j to v_i.

Any path of length 2 from v_i to v_j contained in Γ but not in T has the form $v_i w_\ell v_j$. But $v_i w_\ell v_j$ is a directed path if and only if $v_j v_\ell v_i$ is a directed path from v_j to v_i. Thus Γ contains $\lambda + (\lambda + 1)$ paths of length 2 from v_i to v_j. Since this number is α, $\lambda = \frac{\alpha-1}{2}$ is constant, and so T is doubly regular. □

Remarks. 1. In the definition of $\mathcal{D}(T)$ (Definition 3.15) the vertex v_0 plays a special role. However, the above proof shows that if T is a doubly regular tournament then any vertex in $\mathcal{D}(T)$ can be chosen to play this role. Thus for any vertex x, the subgraph spanned by $N^+(x)$ is a doubly regular tournament. This tournament need not be isomorphic to $N^+(v_0)$, but Hadamard matrices constructed from them as in Theorem 3.12 are equivalent. The relationship between the tournaments $N^+(v_0)$ and $N^+(x)$ was investigated in [Jør94].

2. Due to the existence of infinitely many doubly regular tournaments, we obtain infinitely many examples of doubly regular (m, r)-team tournaments.

5. Equivalence of main structures

5.1. **Simple corollaries.** We will begin with a simple corollary of some results which were obtained in previous sections.

Corollary 5.1. *Every doubly regular (m, r)-team tournament with $r = 2$ is a relation of an imprimitive 3-class association scheme.*

Proof. This follows from Theorem 4.6 and Proposition 3.26. □

Proposition 5.2. *A doubly regular (m, r)-team tournament of Type III is not a relation of a 3-class association scheme.*

Proof. Suppose that R_1 is a doubly regular (m, r)-team tournament of Type III and that R_0, R_1, R_2, R_3 are the relations of a 3-class association scheme, where $R_2 = R_1^{\mathrm{t}}$. Then R_3 is an undirected graph with components $V_1, \ldots, V_m$, each of which spans a complete graph. Let $(x, y) \in R_1$. Then there exist $i \neq j$ such that $x \in V_i$ and $y \in V_j$. Then in R_1, x has $p^1_{13} + p^1_{10}$ out-neighbours in V_j. But by Theorem 4.5, this number is not a constant. □

Proposition 5.3. *Let $(X, \{R_0, R_1, R_2, R_3\})$ be a non-symmetric imprimitive 3-class association scheme such that R_1 and R_2 are directed graphs and R_3 is a disconnected undirected graph. Then R_1 is a doubly regular (m, r)-team tournament of Type I or Type II for some $m, r \in \mathbb{N}$.*

Proof. Let A_0, A_1, A_2, A_3 be the adjacency matrices of R_0, R_1, R_2, R_3. Then $A_1^2 = p^0_{11}A_0 + p^1_{11}A_1 + p^2_{11}A_2 + p^3_{11}A_3 = p^1_{11}A_1 + p^2_{11}A_1^{\mathrm{t}} + p^3_{11}(J - I - A_1 - A_1^{\mathrm{t}})$, as $p^0_{11} = 0$. Thus R_1 is a doubly regular team tournament. By Proposition 5.2, it is of Type I or Type II. □

Therefore, the imprimitive non-symmetric 3-class association schemes for which R_1 is an orientation of $\overline{m \circ K_r}$ for $r = 2$ are characterized by R_1 being isomorphic to either $\mathcal{C}_2(T)$ or $\mathcal{D}(T)$ for some doubly regular tournament T. For $r \geq 3$, we have the following theorem of Goldbach and Claasen [GolC96]. For completeness we give an alternative proof of this statement here.

Theorem 5.4. *Let $(X, \{R_0, R_1, R_2, R_3\})$ be a non-symmetric imprimitive 3-class association scheme such that R_1 is an orientation of $\overline{m \circ K_r}$. Then either R_1 is isomorphic to $\mathcal{C}_r(T)$ for some doubly regular tournament T, or else*

- *$r - 1$ divides $m - 1$ and*
- *r and m are both even.*

Proof. If R_1 is not isomorphic to $\mathcal{C}_r(T)$ then by Proposition 5.3, R_1 is a doubly regular (m,r)-team tournament of Type II. It follows from Theorem 4.3 and Theorem 4.4 that r is even and that $r-1$ divides $m-1$. In Corollary 5.8 below, it will be proved that m is even. □

We have proved above that a non-symmetric relation of an imprimitive 3-class association scheme is a doubly regular team tournament of Type I or Type II. The converse is also true. The following theorem is the corresponding formal claim.

Theorem 5.5. *Let R_1 be a doubly regular (m,r)-team tournament of Type I or Type II. Let $R_2 = R_1^t$ and $R_3 = \overline{R_1 \cup R_2}$. Then $(V(R_1), \{R_0, R_1, R_2, R_3\})$ is an imprimitive 3-class association scheme.*

We know from Proposition 3.26 that Theorem 5.5 is true for doubly regular (m,r)-team tournaments of Type I. In order to prove the theorem for doubly regular (m,r)-team tournaments of Type II we need the following lemma.

Lemma 5.6. *Let A be the adjacency matrix of a doubly regular (m,r)-team tournament Γ of Type II. Then A is normal.*

Proof. We must prove that $AA^{\mathrm{t}} = A^{\mathrm{t}}A$, i.e., that for every pair x,y of vertices, the number of common out-neighbours is equal to the number of common in-neighbours. Since Γ is regular, this is true for $x=y$. Suppose that x and y are distinct. By Theorem 4.3, x and y both have exactly $\frac{r}{2}$ out-neighbours in each V_i, where $x,y \notin V_i$. Thus for each V_i the number of common out-neighbours of x and y in V_i is equal to the number of common in-neighbours of x and y (these are the vertices in V_i that are out-neighbours of neither x nor y.) □

It follows from the classical spectral theorem for normal matrices that A has an orthogonal diagonalisation. Let $\lambda_0, \ldots, \lambda_s$ be the distinct eigenvalues. Let $E_0, \ldots, E_s$ be the orthogonal projections onto the corresponding eigenspaces. Then

$$A = \sum_{i=0}^{s} \lambda_i E_i,$$

and $I = \sum_{i=0}^{s} E_i$. Then

$$A^2 = \sum_{i=0}^{s} \lambda_i^2 E_i,$$

and since orthogonal projections are self-adjoint,

$$A^{\mathrm{t}} = \sum_{i=0}^{s} \overline{\lambda_i} E_i.$$

We may assume that $\lambda_0 = k = \frac{(m-1)r}{2}$. Then $J = mrE_0$.

Lemma 5.7. *Let Γ be a doubly regular (m,r)-team tournament of Type II. Then the adjacency matrix A of Γ has exactly four distinct eigenvalues.*

Proof. If we use the above expressions for A, A^2 and A^{t} in the equation

$$A^2 = \alpha A + \beta A^{\mathrm{t}} + \gamma(J - I - A - A^{\mathrm{t}})$$

and multiply by E_j, $j \neq 0$, then we get

$$\lambda_j^2 = \alpha\lambda_j + \beta\overline{\lambda_j} + \gamma(-1 - \lambda_j - \overline{\lambda_j}).$$

Let $\lambda_j = a + bi$, where a and b are real. Since $\beta = \alpha$,

$$a^2 - b^2 + 2abi = 2a\alpha + \gamma(-1 - 2a).$$

Since the imaginary part of this is $2ab = 0$, either $b = 0$ (and λ_j is real) or $a = 0$ and $b^2 = \gamma$.

The adjacency matrix of $\overline{m \circ K_r}$ is

$$A + A^{\mathrm{t}} = \sum_{j=0}^{s} (\lambda_j + \overline{\lambda_j}) E_j.$$

Thus $\lambda_j + \overline{\lambda_j} = 2a$ is one of the three eigenvalues $2k = (m-1)r, 0$ or $-r$ of $\overline{m \circ K_r}$. Thus the eigenvalues of Γ are $k, -\frac{r}{2}, \pm i\sqrt{\gamma}$. □

Corollary 5.8. *Let Γ be a doubly regular (m,r)-team tournament of Type II. Then m is even.*

Proof. Since the multiplicity of 0 as an eigenvalue of $\overline{m \circ K_r}$ is $m(r-1)$, the eigenvalues $\pm i\sqrt{\gamma}$ of Γ both have multiplicity $\frac{m(r-1)}{2}$, and so m must be even. □

Proof of Theorem 5.5. Let A be the adjacency matrix of R_1. By the above lemma, A has four eigenvalues $\lambda_0, \ldots, \lambda_3$ with corresponding orthogonal projections $E_0, \ldots, E_3$. Let $\mathcal{A}$ denote the set of matrices that are linear combinations of $E_0, \ldots, E_3$. Then $\mathcal{A}$ is a four-dimensional algebra closed under matrix multiplication.

The set $B = \{I, A, A^{\mathrm{t}}, J - I - A - A^{\mathrm{t}}\}$ consists of four linearly independent matrices, and by the above remarks they are contained in $\mathcal{A}$. Thus B is a basis of $\mathcal{A}$ and any product of matrices from B is a linear combination of elements of B. The matrices in B are the adjacency matrices of relations R_0, R_1, R_2, R_3 of an imprimitive 3-class association scheme. □

5.2. **Type II doubly regular (m,r)-team tournaments with $r > 2$.** We have seen above that, whenever r and m are positive even integers such that $r-1$ divides $m-1$, it is possible that a doubly regular (m,r)-team tournament of Type II or equivalently an imprimitive non-symmetric 3-class association scheme exists. For many such pairs (m,r) existence is still open. However, there are also some existence results, in particular when $r = m$ and $m \equiv 0 \bmod 4$. The constructions are related to a certain type of Hadamard matrices.

Definition 5.9. *A Hadamard matrix H of order m^2 is said to be a Bush-type Hadamard matrix if it is a block matrix with $m \times m$ blocks H_{ij} of size $m \times m$ such that $H_{ii} = J_m$ and $H_{ij}J_m = J_mH_{ij} = 0$ for all $i \neq j$.*

We say that a Bush-type Hadamard matrix is skew if $H_{ij} = -(H_{ji})^{\mathrm{t}}$ for all $i \neq j$.

[GolC98] proved that these Hadamard matrices are equivalent to the association schemes considered here, see also [Jør09].

Theorem 5.10. *A doubly regular (m, r)-team tournament of Type II with $r = m$ is equivalent to a skew Bush-type Hadamard matrix of order m^2.*

Bush-type Hadamard matrices of order m^2 and thus doubly regular (m, m)-team tournaments where constructed by Ionin and Kharaghani for many values of m where $m \equiv 0 \bmod 4$.

Theorem 5.11 ([IonK03])**.** *If there exists a Hadamard matrix of order m then there exists a skew Bush-type Hadamard matrix of order m^2.*

Davis and Polhill found a family of highly symmetric doubly regular team tournaments.

Theorem 5.12 ([DavP10])**.** *For every $s \geq 2$ there exists a Cayley graph over $(\mathbb{Z}_4)^s$ which is a doubly regular (m, r)-team tournament of Type II with $m = r = 2^s$ and with automorphism group of rank 4, i.e., the automorphism group acts transitively on vertices, on directed edges, and on ordered pairs of nonadjacent vertices.*

Example 5.13. The Cayley graph $\Gamma = \mathsf{Cay}(\mathbb{Z}_4 \times \mathbb{Z}_4, X)$ with connection set

$$X = \{(0, 1), (1, 0), (1, 1), (1, 2), (3, 1), (2, 3)\}$$

is a doubly regular $(4, 4)$-team tournament with $\alpha = \beta = 2$ and $\gamma = 4$. The four independent sets of 4 vertices are the cosets of the subgroup $\{(0, 0), (0, 2), (2, 0), (2, 2)\}$. Γ has automorphism group of rank 4.

For $m \equiv 2 \bmod 4$ not much is known, but in [Jør09] it was shown that there are at least four doubly regular $(6, 6)$-team tournaments.

Also for $2 < r < m$, most feasible cases are open, but in [Jør10] 40 non-isomorphic doubly regular $(16, 4)$-team tournaments were constructed as Cayley graphs.

In the remaining sections of this paper we will focus on theory related to the case $r = 2$.

6. Vertex transitive doubly regular team tournaments

In [Jør94] it was proved that $\mathcal{D}(T)$ is vertex transitive in the two cases where T is the Paley tournament P_q (cf. Theorem 3.7), and where T is the Szekeres tournament Sz_q. We now consider groups acting transitively on a graph $\mathcal{D}(T)$ for some doubly regular tournament T. It is a well known fact that a tournament does not have any automorphism of order 2. A modification of the proof of this shows that $\mathcal{D}(T)$ has only one automorphism of order 2.

Lemma 6.1. *Let Γ be the graph $\mathcal{D}(T)$ for some tournament T. Then the automorphism group of Γ contains a unique involution ϕ. This involution ϕ maps a vertex x to the unique vertex in Γ nonadjacent to x.*

Proof. It is easy to verify that ϕ is an automorphism. Let ψ be an involutory automorphism of Γ. If for some vertex x, $(x, \psi(x)) \in E(\Gamma)$ then $(\psi(x), \psi^2(x)) = (\psi(x), x) \in E(\Gamma)$, a contradiction. Similarly, it is impossible that $(\psi(x), x) \in E(\Gamma)$. If $\psi(x) = x$ then for any vertex $y \in N^+(x)$ we have $\psi(y) = y$, as $N^+(x)$ is a tournament. Then it follows from the structure of Γ that $\psi(y) = y$ for every vertex $y \in V(\Gamma)$. Thus for every vertex x, $\psi(x)$ is the unique vertex not adjacent to x, and so $\psi = \phi$. □

Lemma 6.2. *Let G be a group of automorphisms of $\Gamma = \mathcal{D}(T)$ for some doubly regular tournament T, and suppose that G has order a power of 2. Then the action of G on $V(\Gamma)$ is semiregular.*

Proof. Let $x \in V(\Gamma)$. By Lemma 6.1, G_x has odd order, but the order of G_x is a power of 2. Thus $|G_x| = 1$. □

We wish to prove that the Sylow 2-subgroups of the automorphism group of a vertex transitive graph $\mathcal{D}(T)$ are generalised quaternion groups. Ito [Ito94] showed that the Sylow 2–subgroups of a Hadamard group cannot be cyclic (see Section 8 below). We prove a similar theorem for the automorphism group of $\mathcal{D}(T)$, but we do not require that the graph is a Cayley graph. First we need a few lemmas.

Lemma 6.3. *Let Γ be a directed graph with vertex set $\{x_0, \dots, x_{4m-1}\}$ such that*

- *the permutation $g = (x_0, \dots, x_{4m-1})$ is an automorphism;*
- *the pairs of non-adjacent vertices are the pairs $\{x_i, x_{i+2m}\}$, for $i = 0, \dots, 2m-1$; and*
- *if $x_0 \to x_\ell$ then $x_\ell \to x_{2m}$.*

Then the number of common out-neighbours of x_0 and x_1 in Γ is even.

Proof. If $x_0 \to x_\ell$ then, since g^{2m} is an automorphism, $x_0 \to x_\ell \to x_{2m} \to x_{2m+\ell} \to x_0$, and by applying $g^{-\ell}$ to this we get $x_0 \to x_{2m-\ell} \to x_{2m} \to x_{4m-\ell} \to x_0$.

The proof is by induction on the number s of out-neighbours of x_0 in $\{x_{2m+1}, \dots, x_{4m-1}\}$. If $s = 0$ then $N^+(x_0) = \{x_1, \dots, x_{2m-1}\}$, and so $N^+(x_1) = \{x_2, \dots, x_{2m}\}$. Thus x_0 and x_1 have $2m-2$ common out-neighbours.

For the induction step, suppose that for some ℓ such that $1 \le \ell \le m$ the arcs in the set $B = \{x_i \to x_{i+\ell}, x_i \to x_{i+2m-\ell} \mid i = 0, \dots 4m-1\}$ are reversed.

Suppose first that $1 < \ell < m$. If $x_0 \to x_{\ell-1}$ and thus $x_1 \to x_\ell$, $x_0 \to x_{2m-\ell+1}$, $x_0 \leftarrow x_{-\ell+1}$ and $x_1 \leftarrow x_{2m+\ell}$, then by reversal of arcs in B the vertices x_ℓ and $x_{2m-\ell+1}$ are removed from the set of common out-neighbours of x_0 and x_1. If $x_0 \leftarrow x_{\ell-1}$ then $x_{-\ell+1}$ and $x_{2m+\ell}$ will be new common out-neighbours of x_0 and x_1 after reversal of arcs in B. If $x_0 \to x_{\ell+1}$ and thus $x_1 \to x_{2m-\ell}$, $x_0 \leftarrow x_{2m+\ell+1}$ and $x_1 \leftarrow x_{-\ell}$, then reversal will remove $x_{\ell+1}$ and $x_{2m-\ell}$ from the list of common out-neighbours, but if $x_0 \leftarrow x_{\ell+1}$ then $x_{2m+\ell+1}$ and $x_{-\ell}$ will be new common out-neighbours. Thus the number of common out-neighbours of x_0 and x_1 will increase by $\pm 2 \pm 2$.

The case $\ell = 1$ is essentially the same as the above except that arcs $x_0 \to x_{\ell-1}$ and $x_0 \to x_{2m-\ell+1}$ cannot exist. The number of common out-neighbours will increase by ± 2.

In the case $\ell = m$, $x_0 \to x_{\ell-1}$ exists if and only if $x_0 \to x_{\ell+1}$ exists. The number of common out-neighbours will increase by ± 2.

In each case the number of common out-neighbours of x_0 and x_1 will remain even. □

Lemma 6.4. *Let* Γ *be a directed graph with vertex set* $\{x_0, \ldots, x_{4m-1}, v_0, \ldots, v_{4m-1}\}$ *such that*

- *the permutation* $g = (x_0, \ldots, x_{4m-1})(v_0, \ldots, v_{4m-1})$ *is an automorphism,*
- *for each* $i = 0, \ldots, 2m-1$*, either* $x_0 \to v_i$ *or* $x_0 \leftarrow v_i$*, and*
- *if* $x_0 \to v_l$ *then* $v_{l+2m} \to x_0$*.*

Then the number of common out-neighbours of x_0 *and* x_1 *in* $D = \{v_0, \ldots, v_{4m-1}\}$ *is odd.*

Proof. The proof is by induction on the number s of out-neighbours of x_0 in $\{v_{2m}, \ldots, v_{4m-1}\}$.

If $s = 0$ then x_0 is adjacent to $v_0, \ldots, v_{2m-1}$, and since g is an automorphism, x_1 is adjacent to $v_1, \ldots, v_{2m}$. Thus the number of common out-neighbours of x_0 and x_1 in D is $2m-1$.

For the induction step, suppose that for some l, the arcs $x_i \to v_{i+l}$ and thus $v_{i+l+2m} \to x_i$, $i = 0, \ldots, 4m-1$, are reversed. This reversal affects only common out-neighbours in $\{v_l, v_{l+1}, v_{l+2m}, v_{l+2m+1}\}$. Thus the number of common out-neighbours of x_0 and x_1 in D increases by $2, 0$ or -2, if x_0 was adjacent to none of x_{l-1} or x_{l+1}, to exactly one of them, or to both of them, respectively.

Thus x_0 and x_1 always have an odd number of common out-neighbours in D. □

Lemma 6.5. *Let* Γ *be isomorphic to* $\mathcal{D}(T)$ *for some doubly regular tournament* T*. Let* g *be a semiregular automorphism of* Γ*. Then the number of orbits of* g *is even.*

Proof. Suppose that the number of orbits is odd. Then 8 divides the order of g. Let the order of g be $4m$. Then g^{2m} is the unique automorphism of order 2. Let $x_0 \in V(\Gamma)$ and let C be the orbit containing x_0. Let $x_i = {x_0}^{g^i}$. Then the subgraph of Γ spanned by C satisfies the conditions in Lemma 6.3. Thus the number of common out-neighbours of x_0 and x_1 in C is even.

Let $D = \{v_0, \ldots, v_{4m-1}\}$ be any other orbit of g with $v_i = {v_0}^{g^i}$. The subgraph spanned by C and D satisfies the conditions of Lemma 6.4. Thus the number of common out-neighbours of x_0 and x_1 in D is odd. Since there is an even number of orbits different from C, x_0 and x_1 have an even number of common out-neighbours outside C and thus an even number of common out-neighbours in Γ. However, the number of common out-neighbours of two adjacent vertices in $\mathcal{D}(T)$ is odd, a contradiction. □

Theorem 6.6. *Suppose that for some doubly regular tournament* T*, the graph* $\Gamma = \mathcal{D}(T)$ *is vertex transitive and has automorphism group* G*. Then a Sylow* 2*-subgroup* S *of* G *is a generalised quaternion group of order* 2^n*, where* 2^n *is the highest power of* 2 *that divides the order of* Γ*.*

Proof. Since Γ is vertex transitive the order of G is divisible by the order of Γ and thus by 2^n. It follows that the order of S is at least 2^n. Since the action of S on $V(\Gamma)$ is semiregular, the order of S is not divisible by 2^{n+1}. Thus $|S| = 2^n$.

Since G and thus S has a unique involution and the order of S is a power of 2, S is either a cyclic group or a generalised quaternion group, see [Gor68, Theorem 5.4.10(ii)] or [Hup67, Satz III.8.2(b)]. Since S is semiregular, the number of orbits under the action of S is $|\Gamma|/2^n$, which is odd. Thus by Lemma 6.5, S cannot be cyclic. □

Corollary 6.7. *Suppose that for some doubly regular tournament T, the graph $\Gamma = \mathcal{D}(T)$ is vertex transitive and has order 2^n. Then Γ is a Cayley graph for the generalised quaternion group of order 2^n.*

Proof. By Theorem 6.6, a Sylow 2-subgroup S of the automorphism group of Γ is a generalised quaternion group of order 2^n. By Lemma 6.2, S is semiregular and thus regular. □

In the next section we will investigate the possibilities for a graph Γ satisfying the hypotheses of Corollary 6.7.

7. Cayley graphs and group rings

Many normally regular digraphs arise from Cayley graphs, group rings and difference sets. We consider first under what conditions on the connection set $S \subset G$ the Cayley graph $\mathsf{Cay}(G, S)$ becomes a normally regular digraph.

Proposition 7.1. *Let G be a group with identity 1 and let $S \subset G$. Let $S^{(-1)}$ denote the set $\{s^{-1} \mid s \in S\}$. Let Γ be $\mathsf{Cay}(G, S)$. Then Γ is a normally regular digraph with parameters $(|G|, |S|, \lambda, \mu)$ if and only if $S \cap S^{(-1)} = \emptyset$ and for any $g \in G$ the number of pairs $(s, t) \in S \times S$ such that $st^{-1} = g$ is*

$$\begin{cases} |S|, & \text{if } g = 1, \\ \lambda, & \text{if } g \in S \cup S^{(-1)}, \\ \mu, & \text{otherwise.} \end{cases}$$

Proof. The condition $S \cap S^{(-1)} = \emptyset$ ensures that the graph does not have undirected edges or loops. Let x and y be distinct vertices in Γ. There exists a unique $g \in G$ such that $xg = y$. Then $x \to y$ if $g \in S$ and $y \to x$ if $g \in S^{(-1)}$. Let z be a vertex in Γ. Then $x \to z$ if $z = xs$ for some $s \in S$, and $y \to z$ if $z = yt$ for some $t \in S$. Thus z is a common out-neighbour of x and y if and only if $xs = yt = xgt$. This is equivalent to $g = st^{-1}$, and so the number of common out-neighbours of x and y is equal to the number of pairs $(s, t) \in S \times S$ such that $g = st^{-1}$. □

For a (multiplicative) group G, the group ring $\mathbb{Z}G$ is the set of formal sums $\sum_{g \in G} c_g g$, where $c_g \in \mathbb{Z}$. Then $\mathbb{Z}G$ is a ring with sum

$$\left(\sum_{g \in G} c_g g\right) + \left(\sum_{g \in G} d_g g\right) = \sum_{g \in G} (c_g + d_g) g$$

and product

$$\left(\sum_{g \in G} c_g g\right) \cdot \left(\sum_{g \in G} d_g g\right) = \sum_{g \in G} \left(\sum_{h \in G} c_h d_{h^{-1}g}\right) g.$$

For a set $S \subseteq G$ we define $\underline{S} = \sum_{g \in S} g \in \mathbb{Z}G$. We write $\underline{\{g\}}$ as g.

The conditions in Proposition 7.1 can be reformulated in terms of the group ring.

Corollary 7.2. $\mathsf{Cay}(G,S)$ *is a normally regular digraph with parameters* (v,k,λ,μ) *where* $v = |G|$ *and* $k = |S|$ *if and only if*

- $S \cap S^{(-1)} = \emptyset$, *and*
- $\underline{S} \cdot \underline{S^{(-1)}} = k1 + \lambda(\underline{S} + \underline{S^{(-1)}}) + \mu(\underline{G} - \underline{S} - \underline{S^{(-1)}} - 1)$ *in the group ring* $\mathbb{Z}G$.

For the particular case $\mu = 0$, we get the following corollary.

Corollary 7.3. $\mathsf{Cay}(G,S)$ *is isomorphic to* $\mathcal{D}(T)$ *for some doubly regular tournament* T *if and only if*

- $|G| = 4\lambda + 4$ *and* $|S| = 2\lambda + 1$, *for some* λ,
- $S \cap S^{(-1)} = \emptyset$, *and*
- $\underline{S} \cdot \underline{S^{(-1)}} = (2\lambda + 1)1 + \lambda(\underline{S} + \underline{S^{(-1)}})$.

The condition $S \cap S^{(-1)} = \emptyset$ implies that $1 \notin S$. We will next consider a property of sets satisfied by $S \cup \{1\}$.

Definition 7.4. *Let* G *be a group of order* mn. *Let* N *be a subgroup of* G *of index* m. *A subset* $D \subset G$ *is said to be a relative* (m,n,k,λ) *difference set with forbidden subgroup* N *if* $|D| = k$ *and for any* $g \in G$ *the number of pairs* $(s,t) \in D \times D$ *such that* $g = st^{-1}$ *is exactly*

$$\begin{cases} k, & \text{if } g = 1, \\ 0, & \text{if } g \in N,\ g \neq 1, \\ \lambda, & \text{if } g \notin N. \end{cases}$$

Example 7.5. Let $G = \mathbb{Z}_2^3$ be the elementary abelian group of order 8 and let $N = \{(0,0,0),(1,1,1)\}$. Then $D = \{(0,0,0),(1,0,0),(0,1,0),(0,0,1)\}$ is a relative $(4,2,4,2)$ difference set. However, $S = D \setminus \{(0,0,0)\}$ does not satisfy the condition $S \cap S^{(-1)} = \emptyset$. In Example 7.9 we present another relative difference set with the same parameters satisfying all the conditions in Corollary 7.3.

It is usually assumed that N is a normal subgroup. Since N is a forbidden subgroup, D has at most one element in each (right) coset of N. We consider the case where $m = k$; that is, D has exactly one element in each coset of N.

The conditions in the definition of relative difference sets can be stated in terms of the group ring.

Lemma 7.6. *Let* $N < G$ *be as in the above definition. Then a subset* $D \subset G$ *with* $|D| = k$ *is a relative difference set if and only if*

$$\underline{D} \cdot \underline{D^{(-1)}} = k1 + \lambda(\underline{G} - \underline{N}).$$

In the next lemma we see that from one relative difference set several new relative difference sets can be constructed by shifting D.

Lemma 7.7. *If $D \subset G$ is a relative difference set then for any $g \in G$, $Dg = \{dg \mid d \in D\}$ is a relative difference set with the same forbidden subgroup and the same parameters.*

Proof.

$$\underline{Dg} \cdot \underline{(Dg)^{(-1)}} = \underline{Dg} \cdot \underline{g^{-1}D^{(-1)}} = \underline{D} \cdot \underline{D^{(-1)}}.$$

□

In Lemma 6.1 we have observed that for any tournament T the graph $\mathcal{D}(T)$ has a unique automorphism of order 2. This automorphism is forbidden as a "difference" of elements in the connection set if $\mathcal{D}(T)$ is a Cayley graph.

Proposition 7.8. *Suppose that $\Gamma = \mathsf{Cay}(G, S)$ is isomorphic to $\mathcal{D}(T)$ for some doubly regular tournament T and $|G| = 4\lambda + 4$. Then $S \cup \{1\}$ is a relative $(2n, 2, 2n, n)$ difference set in G with forbidden subgroup $N = \langle\phi\rangle$, where ϕ is the unique involutory automorphism of Γ and $n = \lambda + 1$.*

Proof. From Corollary 7.3 we get

$$\underline{(S \cup \{1\})} \cdot \underline{(S \cup \{1\})^{-1}} = \underline{S} \cdot \underline{S^{(-1)}} + \underline{S} + \underline{S^{(-1)}} + 1 =$$

$$(2\lambda + 2)1 + (\lambda + 1)(\underline{S} + \underline{S^{(-1)}}) = 2n \cdot 1 + n\underline{(S \cup S^{(-1)})}.$$

By Lemma 6.1, $S \cup S^{(-1)} = G \setminus N$. Thus we have

$$\underline{(S \cup \{1\})} \cdot \underline{(S \cup \{1\})^{-1}} = 2n \cdot 1 + n(\underline{G} - \underline{N}).$$

□

Example 7.9. The smallest, almost trivial, example of a doubly regular tournament is, clearly, a directed triangle Δ. It is clear that $\mathcal{D}(\Delta)$ has 8 vertices. We are in a position to represent $\mathcal{D}(\Delta)$ as a Cayley graph. For this purpose, let us consider the well-known quaternion group $Q = \{1, -1, i, j, k, -i, -j, -k\}$ with the usual multiplication table. Let $S = \{i, j, k\}$. Then the Cayley graph $\Gamma = \mathsf{Cay}(Q, S)$ satisfies all the requirements of Proposition 7.8. Indeed,

$$\underline{\{1, i, j, k\}} \cdot \underline{\{1, -i, -j, -k\}} = 4 \cdot \underline{\{1\}} + 2 \cdot \underline{\{i, j, k, -i, -j, -k\}}.$$

This implies that the set $\{1, i, j, k\}$ is a relative $(4, 2, 4, 2)$ difference set in Q with forbidden subgroup $N = \langle -1 \rangle = \{-1, 1\}$. Here -1 is the unique involutory automorphism of Γ (provided that we regard Q as a regular group, acting on the vertices of Γ).

Suppose that $D \subset G$ is a relative $(2n, 2, 2n, n)$ difference set. Then, by shifting if necessary, we may assume that $1 \in D$.

Proposition 7.10. *If $D \subset G$ is a relative $(2n, 2, 2n, n)$ difference set with forbidden subgroup $N = \langle\phi\rangle$ such that $D \cap D^{(-1)} = \{1\}$, then $\mathsf{Cay}(G, D \setminus \{1\})$ is isomorphic to $\mathcal{D}(T)$ for some doubly regular tournament T.*

Proof. Let $S = D \setminus \{1\}$. Since $D \cap D^{(-1)} = \{1\}$, we get $S \cap S^{(-1)} = \emptyset$, $\underline{D} + \underline{D^{(-1)}} = \underline{G} - \phi + 1 = \underline{G} - \underline{N} + 2 \cdot 1$ and $S \cup S^{(-1)} = G \setminus N$. Thus $\underline{S} \cdot \underline{S^{(-1)}} = (\underline{D} - 1) \cdot (\underline{D^{(-1)}} - 1) = \underline{D} \cdot \underline{D^{(-1)}} - \underline{D} - \underline{D^{(-1)}} + 1 = 2n \cdot 1 + n(\underline{G} - \underline{N}) - (\underline{G} - \underline{N} + 2) + 1 = (n-1)(\underline{G} - \underline{N}) + (2n-1) \cdot 1 = (n-1)(\underline{S} + \underline{S^{(-1)}}) + (2n-1) \cdot 1$. If we let $\lambda = n - 1$ then we get all the conditions in Corollary 7.3. □

8. Ito's Conjecture and S-rings over dicyclic groups

In [Ito94], N. Ito proposed the construction of Hadamard matrices from relative difference sets, and he introduced the following definition.

Definition 8.1. *A group of order $4n$ is called a Hadamard group if it contains a relative $(2n, 2, 2n, n)$ difference set, relative to some normal subgroup of order 2.*

Ito [Ito94] proved that the existence of a Hadamard group of order $4n$ implies the existence of a Hadamard matrix of order $2n$.

Theorem 8.2. *Let G be a group of order $4n$ and let $N = \langle u \rangle$ be a normal subgroup of order 2. Let $D \subset G$ be a relative $(2n, 2, 2n, n)$ difference set. Let $a_1, \dots, a_{2n}$ be representatives of the cosets of N. Let $H = (h_{ij})$ be the $2n \times 2n$ matrix with*

$$h_{ij} = \begin{cases} -1, & \text{if } a_j \in a_i D, \\ 1, & \text{if } u a_j \in a_i D. \end{cases}$$

Then H is a Hadamard matrix.

Proof. First, we must prove that h_{ij} is well defined for every i, j, i.e., that $|\{a_j, ua_j\} \cap a_i D| = 1$. Suppose that $\{a_j, ua_j\} \subset a_i D$. Then there exist $d_1, d_2 \in D$ such that $a_j = a_i d_1$ and $ua_j = a_i d_2$, and so $d_1 d_2^{-1} = (a_i^{-1} a_j)(a_i^{-1} u a_j)^{-1} = a_i^{-1} a_j a_j^{-1} u^{-1} a_i = a_i^{-1} u a_i = u$ as N is a normal subgroup. But this is a contradiction, as N is the forbidden subgroup. Thus $|\{a_j, ua_j\} \cap a_i D| \le 1$. Since $a_i D$ has cardinality $2n$ and there are $2n$ sets $\{a_j, ua_j\}$, $|\{a_j, ua_j\} \cap a_i D| = 1$ for every i, j.

Since H has entries ± 1, we need only to prove that $HH^{\mathrm{t}} = 2nI$. For $1 \le i, \ell \le n$, $i \ne \ell$, the cardinality of the set $a_i D \cap a_\ell D$ is

$$\begin{aligned} & |\{g \in G : g = a_i d_1 \text{ and } g = a_\ell d_2, \text{for some } d_1, d_2 \in D\}| \\ = \; & |\{(d_1, d_2) \in D \times D : a_i d_1 = a_\ell d_2\}| \\ = \; & |\{(d_1, d_2) \in D \times D : d_1 d_2^{-1} = a_i^{-1} a_\ell\}| \\ = \; & n, \end{aligned}$$

as D is a relative $(2n, 2, 2n, n)$ difference set and $a_i^{-1} a_\ell \notin N$. The number of columns j such that $h_{ij} = h_{\ell j} = -1$ is $|\{j \mid a_j \in a_i D \cap a_\ell D\}|$, and the number of columns j where $h_{ij} = h_{\ell j} = 1$ is $|\{j \mid ua_j \in a_i D \cap a_\ell D\}|$. Since $G = \{a_1, \dots, a_{2n}, ua_1, \dots, ua_{2n}\}$, the number of values of j where h_{ij} and $h_{\ell j}$ are equal is $|a_i D \cap a_\ell D| = n$. Also h_{ij} and $h_{\ell j}$ are different for the remaining n values of j. Thus the dot product of rows i and ℓ is 0. □

Example 8.3. Consider the relative difference set from Example 7.5 and let $a_1 = (0, 0, 0)$, $a_2 = (1, 0, 0)$, $a_3 = (0, 1, 0)$ and $a_4 = (0, 0, 1)$ be representatives of the cosets of N. Then the Hadamard matrix constructed in Theorem 8.2 is

$$\begin{bmatrix} -1 & -1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix}.$$

If we use the relative difference set from Example 7.9 and let $a_1 = 1$, $a_2 = i$, $a_3 = j$ and $a_4 = k$ be representatives of the cosets of N then we get the following skew Hadamard matrix

$$\begin{bmatrix} -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix}.$$

Ito [Ito94] also proved that the Sylow 2-subgroups of a Hadamard group are not cyclic or dihedral. In [Ito97] he conjectured that every dicyclic group of order $8t$, for some t, is a Hadamard group. By Theorem 8.2 this would imply the Hadamard matrix conjecture. Theorem 8.7 below shows that the conjecture is true for infinitely many values of t. Schmidt [Sch99] proved that the conjecture is true for $t \leq 46$. Ito [Ito94] also investigated conditions for the constructed Hadamard matrix to be skew.

Theorem 8.4. *Let G be a Hadamard group of order $4n$ and let $N = \langle u \rangle$ be a normal subgroup of order 2. Let $D \subset G$ be a relative $(2n, 2, 2n, n)$ difference set and let $H = (h_{ij})$ be the Hadamard matrix constructed from D in Theorem 8.2. Then H is a skew Hadamard matrix if and only if $D \cap D^{(-1)} = \{1\}$.*

Proof. A diagonal entry h_{ii} is -1 if and only if $1 \in D$.

Suppose that $d \neq 1$ satisfies that $d, d^{-1} \in D$. There exist indices $i \neq j$ and $u' \in N$ such that $a_i \in N$ and $u'a_j = d$. Then $a_j = (u'a_i)a_i d$, as $a_i^2 = 1$, and $a_i = (a_i u')a_j d^{-1}$. The value of h_{ij} depends on $u'a_i$, and the value of h_{ji} depends on $a_i u' = u'a_i$. Thus $h_{ij} = h_{ji}$.

Suppose next that $D \cap D^{(-1)} = \{1\}$. Then for $g \in G \setminus N$, $g \in D$ if and only if $g^{-1} \notin D$. For $i \neq j$, $h_{ij} = -1 \Leftrightarrow a_i^{-1} a_j \in D \Leftrightarrow a_j^{-1} a_i \notin D \Leftrightarrow h_{ji} = 1$. □

Ito [Ito94] called a group with the property of Theorem 8.4 a skew Hadamard group. From Corollary 7.3, Proposition 7.10 and Theorem 8.4 we get the following corollary.

Corollary 8.5. *A group is a skew Hadamard group if and only if it is a regular subgroup of the automorphism group of $\mathcal{D}(T)$ for some doubly regular tournament T.*

In view of Ito's conjecture we consider the following problem.

Problem 8.6. *For which values of t is the dicyclic group of order $8t$ a skew Hadamard group?*

In what follows, we consider this problem in detail. In particular, we prove that if $4t - 1$ is a prime power then the dicyclic group of order $8t$ is a skew Hadamard group. A computer search has shown that there are no skew Hadamard groups of order 72, see also [ItoO96].

Ito [Ito94] showed that the Hadamard matrices corresponding to Paley tournaments can be constructed from skew Hadamard groups. We state this result in terms of $\mathcal{D}(P_q)$.

Theorem 8.7. *Let $q \equiv 3$ mod 4 be a prime power and let P_q be the Paley tournament of order q. Let $\Gamma = \mathcal{D}(P_q)$. Then $SL(2,q)$ acts as a group of automorphisms of Γ and this group contains a dicyclic subgroup acting regularly on $V(\Gamma)$.*

The proof below uses the following basic property of the special linear group $SL(2,q)$ of 2×2 matrices over $\mathbb{F}_q$ with determinant 1.

Lemma 8.8. *$SL(2,q)$ has order $q(q-1)(q+1)$ and contains a subgroup isomorphic to the dicyclic group of order $2(q+1)$.*

Proof. Since $SL(2,q)$ is isomorphic to the special unitary group $SU(2,q^2)$ (see [Hup67, Hilfssatz II.8.8]), it suffices to find a subgroup of $SU(2,q^2)$ isomorphic to the dicyclic group. Let $\lambda \in \mathbb{F}_{q^2}^*$ be an element of order $q+1$. Let

$$x = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^q \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then $x, y \in SU(2,q^2)$, $x^{\frac{q+1}{2}} = y^2, y^4 = 1, xyx = y$ and so x and y generate a dicyclic subgroup of $SU(2,q^2)$ of order $2(q+1)$. $\square$

Proof of Theorem 8.7. Let $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$ be the multiplicative group of the finite field of order q. Let Q be the set of nonzero squares in $\mathbb{F}_q$ and let $N = \mathbb{F}_q^* \setminus Q$. Then Q is a subgroup of $\mathbb{F}_q^*$ of index 2, and $N = -Q$ as $-1 \notin Q$. Let $V = {\mathbb{F}_q}^2$ be the 2-dimensional vector space of row vectors over $\mathbb{F}_q$. We define a relation $\sim$ on $V \setminus \{0\}$ by

$$v \sim w \ \text{ if } \ v = \alpha w \ \text{ for some } \alpha \in Q.$$

This relation is an equivalence relation and we have a partition of $V \setminus \{0\}$ into equivalence classes. This relation partitions each 1-dimensional subspace into two classes. The class containing $v = (x,y)$ will be denoted by $[v] = [x,y]$. We denote by $X = (V \setminus \{0\})/\!\sim$ the set of all equivalence classes. The number of classes is $|X| = \frac{q^2-1}{(q-1)/2} = 2(q+1)$.

If $v = \alpha w$ for $v, w \in V \setminus \{0\}$ and $\alpha \in Q$ then $vA = \alpha wA$ for any matrix $A \in GL(2,q)$. Thus the natural action of $GL(2,q)$ on $V \setminus \{0\}$ by right multiplication preserves the relation $\sim$.

For any $[\alpha,\beta] \in X$, the matrix $\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix}$, or if $\alpha = 0$ the matrix $\begin{pmatrix} \alpha & \beta \\ -\beta^{-1} & 0 \end{pmatrix}$, has determinant 1 and maps $[1,0]$ to $[\alpha,\beta]$. It follows that the special linear group

$$G = SL(2,q) = \{g \in GL(2,q) \mid \det(g) = 1\}$$

acts transitively on X by right multiplication. If some matrix $g \in G$ fixes every $[v] \in X$ then $g = \lambda I$ for some $\lambda \in Q$. Since $\det(g) = \lambda^2 = 1$ and $\lambda \in Q$, $\lambda = 1$. Thus G acts faithfully on X. The stabilizer of $[1,0]$ is

$$G_{[1,0]} = \left\{ \begin{pmatrix} \alpha & 0 \\ \gamma & \alpha^{-1} \end{pmatrix} \mid \alpha \in Q, \gamma \in \mathbb{F}_q \right\}.$$

The group G has an action on $X \times X$ defined by $(x,y)^g = (xg, yg)$ for all $x, y \in X$ and $g \in G$. Let E be the orbit in this action containing $([1,0],[0,1])$ and let Γ be the graph with vertex set X and arc set E. We need to prove that Γ is isomorphic to $\mathcal{D}(P_q)$.

The set of out-neighbours of $[1,0]$ is the orbit containing $[0,1]$ under the action of $G_{[1,0]}$ on X. The orbit consists of elements of the form $[\gamma, \alpha^{-1}] = [\beta, 1]$ (where $\beta = \alpha\gamma$).

Since the matrix $\begin{pmatrix} \beta & -1 \\ 1-\beta^2 & \beta \end{pmatrix} \in G$ maps $([1,0],[\beta,1])$ to $([\beta,-1],[1,0])$ the in-neighbours of $[1,0]$ are the elements $[\beta,-1]$, $\beta \in \mathbb{F}_q$.

Thus $[-1,0]$ is the unique vertex not adjacent to $[1,0]$; the automorphism $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $\in G$ shows that $[-1,0]$ has in-neighbours $[\beta,1]$ and out-neighbours $[\beta,-1]$.

Any automorphism $g \in G$ mapping $[1,0]$ to $[v] \in X$ also maps $[-1,0]$ to the unique vertex $[-v]$ not adjacent to $[v]$. Since G acts transitively on arcs of Γ (by definition of $E = E(\Gamma)$), on $E^{\mathrm{t}} = \{(x,y) \mid (y,x) \in E\}$ and on pairs of nonadjacent vertices, Γ is a doubly regular $(q+1,2)$-team tournament. Since there are q paths of length 2 from $[1,0]$ to $[-1,0]$, Γ is isomorphic to $\mathcal{D}(N^+([1,0]))$ by Theorem 4.6.

Let $[\alpha,1],[\beta,1] \in N^+([1,0])$. An element $g \in G$ maps $([1,0],[0,1])$ to $([\alpha,1],[\beta,1])$ if and only if $g = \begin{pmatrix} \delta\alpha & \delta \\ \epsilon\beta & \epsilon \end{pmatrix}$, where $\delta, \epsilon \in Q$. Since $\det(g) = (\alpha - \beta)\delta\epsilon$, such δ, ϵ exist if and only $\alpha - \beta \in Q$. Thus $([\alpha,1],[\beta,1]) \in E(\Gamma)$ if and only $\alpha - \beta \in Q$. By Lemma 3.6, $N^+([1,0])$ is isomorphic to the Paley tournament P_q and so Γ is isomorphic to $\mathcal{D}(P_q)$.

It follows from Lemma 8.8 that there is a subgroup $H < G = SL(2,q)$ isomorphic to the dicyclic group of order $2(q+1)$, and that the stabilizer $G_{[1,0]}$ in G of $[1,0]$ has order $\frac{q(q-1)(q+1)}{2(q+1)} = \frac{1}{2}q(q-1)$, as $|X| = 2(q+1)$. Thus the order of the stabilizer $H_{[1,0]}$ in H of $[1,0]$ divides $\frac{1}{2}q(q-1)$. It also divides $|H| = 2(q+1)$. Since $\frac{1}{2}q(q-1)$ and $2(q+1)$ are relatively prime, $H_{[1,0]} = 1$. Since $|H| = |X|$, H acts regularly on X. □

Example 8.9. Other groups than the dicyclic groups may appear as regular subgroups of the automorphism groups of graphs $\mathcal{D}(T)$. We show some examples found with the use the computer algebra system GAP [GAP] and its share package GRAPE [Soi93] together with nauty [McKay90].

(1) By Theorem 8.7, the automorphism group of $\mathcal{D}(P_q)$ has a regular group isomorphic to the dicyclic group of order $2q+2$. However, for some values of q there are additional regular subgroups.

The automorphism group of $\mathcal{D}(P_{11})$ has two conjugacy classes of regular subgroups. One class consists of dicyclic groups of order 24. The other class consists of groups isomorphic to $\langle x, y, z \mid x^2 = y^2, y^4 = z^3 = 1, xyx = y, zx = yz, zy = xyz \rangle \simeq SL(2,3)$.

The automorphism group of $\mathcal{D}(P_{23})$ has three conjugacy classes of regular subgroups. One class consists of dicyclic groups of order 48. The other two classes consist of groups isomorphic to $\langle x, y, z, w \mid x^2 = y^2 = w^2, y^4 = z^3 = 1, xyx = y, zx = yz, zy = xyz, wx = yw, wy = xw, wz = xz^2w\rangle$.

The automorphism group of $\mathcal{D}(P_{59})$ also has three conjugacy classes of regular subgroups. One class consists of dicyclic groups of order 120. The other two classes consist of groups isomorphic to $SL(2, 5)$.

(2) Let EP_q be the doubly regular tournament of order $2q + 1$ obtained by applying Theorem 3.7 to the Paley tournament P_q for a prime power $q \equiv 3 \bmod 4$, and let G be the automorphism group of $\mathcal{D}(\mathsf{EP}_q)$. Then it was proved in [Jør94] that G acts transitively on the vertices. Computer experiments show that for small values of q (at least up to $q = 47$) G has a regular subgroup isomorphic to the dicyclic group.

For $q = 11$ there is one additional regular subgroup, isomorphic to the group $\langle x, y, z, w \mid x^2 = y^2 = w^2, y^4 = z^3 = 1, xyx = y, zx = yz, zy = xyz, wx = yw, wy = xw, wz = xz^2w\rangle$ of order 48 appearing in part (1).

(3) There is a tournament T of order 23 such that $\mathcal{D}(T)$ is a Cayley graph for the group $G = \langle x, y, z \mid x^6 = y^2, x^3 = z^2, y^4 = 1, xyx = y, zx = x^5z, zy = x^3yz\rangle$. This is the full group of automorphisms of $\mathcal{D}(T)$, i.e., T has a trivial automorphism group.

There is, in fact, a natural explanation for the exceptional groups of order 24, 48 and 120 appearing in Example 8.9(1). They are the well-known binary polyhedral groups (see [CoxM80, § 6.5], for instance), which are double coverings of the finite rotation groups of the tetrahedron (isomorphic to the alternating group A_4), of the cube and octahedron (isomorphic to the symmetric group S_4) and of the icosahedron and dodecahedron (isomorphic to A_5). These rotation groups are isomorphic to subgroups of the projective special linear group $PSL(2, q) = SL(2, q)/\{\pm I\}$ for certain prime powers q (see [Hup67, Hauptsatz II.8.27] for details), and the corresponding binary groups are their inverse images in $SL(2, q)$. As subgroups of $SL(2, q)$, these binary groups act semiregularly on the $2(q + 1)$ vertices of $\mathcal{D}(P_q)$ since they have trivial intersections with the vertex-stabilisers in $SL(2, q)$. When $q = 11, 23$ or 59 respectively they have order $2(q+1)$ and therefore act regularly. Moreover, it follows from Dickson's description of the subgroups of $PSL(2, q)$ [Hup67, Hauptsatz II.8.27] that these are the only examples of non-dicyclic regular subgroups of $SL(2, q)$. In a similar way, the dicyclic regular subgroups in Theorem 8.7 and Example 8.9(1) are the double covers in $SL(2, q)$ of dihedral rotation groups of order $q+1$ in $PSL(2, q)$; these arise for all prime powers $q \equiv 3 \bmod 4$.

9. Concluding remarks

In this section we wish to discuss a number of issues which do not belong properly to the main part of the text. However we hope that they will clarify some historical and presentational aspects, and will help better to define the genre of the paper, putting it in context within the wider scope of Algebraic Graph Theory.

9.1 This project belongs to the area of Algebraic Graph Theory (which we abbreviate to AGT). This name was coined about 40 years ago by Norman Biggs; initially there was also a tendency to use the alternative name Algebraic Combinatorics (as suggested in [BanI84]), though nowadays this latter name is used in a much wider context.

Loosely speaking, AGT deals with diverse techniques which allow one to investigate all possible aspects of symmetry of graphs, distinguishing group-theoretic, spectral, number-theoretic and purely combinatorial features of the concept of symmetry. In our eyes, the recent book [BroH12] vividly, though concisely, reflects almost all the significant facets of AGT.

9.2 This project began in 1994, when the author MK visited SYS at Ames. We started with a 12-page handwritten draft, prepared by SYS after discussion of a few examples of class 3 association schemes and their links with S-rings. A year later this was transformed into a first printed text with a brief outline of further steps which needed to be completed.

Another source, at that time independent, was a one-page handwritten text prepared by GJ, who realised that some of the isolated examples provided by MK and SYS could be generalised to an infinite series with the aid of the groups $SL(2,q)$. This happened in 1997 at Beer Sheva, where a few days later the draft was extended to a 4-page dense text, in which the concept of doubly regular orientation was considered.

In 2000 LJ, visiting Beer Sheva, drew MK's attention to his preprint [Jør94b] with its agenda around the newly introduced concept of a normally regular digraph, various constructions of such graphs and their links with Cayley graphs, difference sets and doubly regular tournaments. LJ's familiarity with these topics, as well as a few new examples discovered by him with the aid of a computer, stimulated the authors to combine their efforts. In 2001, during a visit of MK to Aalborg, a first draft of the existing text (at that time 17 pages) was prepared and distributed between the authors. Nevertheless it took another two years before MK and LJ added six more pages to the text, when they met at Newark, DE.

In 2005, at Southampton, the current structure of the text was formed, and in 2008 it reached its existing state. Unfortunately, due to a lack of efficient communication and, probably, adequate leadership, it took us other six years to realise that the project was, in principle, finished and should be published as soon as possible. Here we remind the reader of the saying "better late than never".

9.3 Needless to say, during this 20-year period the entire mathematical community was not dreaming, patiently awaiting our contribution. This is why nowadays the available literature contains many intersections with our approach and results, some of them briefly discussed below. However we believe that it still makes sense to publish our text as it is. It seems that the traditions of the journal SLC do not contradict this vision.

9.4 The style of this paper is a mixture of several relevant genres: research report, survey of ideas and examples, short review of the literature and also elements of an essay. An essential feature of the chosen approach is that it is in a sense genetic. Step by step we have introduced necessary background, new definitions, helpful examples, and proofs of results, while the discussion of some links with the existing literature has intentionally been postponed to the very end of this final section or even neglected.

To give just one example, we think that the spectral theorem for normal matrices is a classical result, but it would be difficult to attribute it fairly to one specific reference. Fortunately, in the era of Wikipedia a single click on the internet allows one to reach a wide range of relevant information.

9.5 Below is a brief summary of the material presented here, intentionally expressed in a slightly different wording in comparison with the preceding main part of the text.

We have investigated combinatorial structures obtained by relaxing the conditions on a non-symmetric three-class association scheme and in particular the conditions for an imprimitive three-class association scheme. In many cases there are relations with certain types of Hadamard matrices.

For doubly regular team tournaments we have seen that all known examples are relations of an imprimitive three-class association scheme, and that examples that are not association schemes can exist only for one particular family of parameters (Type III). Similarly in [Jør14] it is proved that a group divisible normally regular digraph (i.e. an orientation of a complete multipartite graph) has either four or five distinct eigenvalues, and if it has four eigenvalues then it is a relation of a three-class association scheme. It is not known whether there exist any group divisible normally regular digraphs with five eigenvalues.

In the primitive case the situation is quite different. Only one infinite family and a few exceptional examples of primitive non-symmetric three-class association schemes are known (see [Jør10]). However, several families of normally regular digraphs are known, see [Jør94b] or [Jør14]. We have not investigated general digraphs satisfying the equation of Definition 3.13.

9.6 Association schemes form one of the background concepts in this paper. A classical book [BanI84] still serves as an introduction to this part of algebraic combinatorics. The main line of the presentation in [BanI84] deals with P- and Q-polynomial association schemes, and also with related distance regular graphs. Schemes with few classes are related to another possible view of this area. In particular, symmetric schemes with two classes are well known in the equivalent form of pairs of complementary strongly regular graphs, while non-symmetric schemes with two classes are associated in the same way with doubly regular tournaments.

Surprisingly, the next case of schemes with three classes has received less attention in the literature. Significant sources for the study of symmetric 3-class schemes are the thesis [Cha94] and [vDam99].

Our paper is strictly related to the case of non-symmetric association schemes with three classes. The most significant text here is the unpublished report [GolC93]. Only small extracts and extensions of it have been published, see [GolC96a], [GolC96] and [GolC98]. An interesting precursor of this approach (in purely permutation group theoretic language) is [Iwa73]. Both texts by SYS, [So95] and [So96], were prepared without any previous knowledge of or access to [GolC93]. Our acquaintance with this source goes back to a time when most of the current text had already been written. This is one of the main reasons for the chosen genetic style of presentation.

The paper [Ma07] discusses some families of association schemes on 4^k vertices in relation to Galois rings of characteristic 4. Non-symmetric commutative schemes with few classes are also investigated in [ChiaK06]. We also mention the paper [Jør09] by LJ.

9.7 Relative difference sets form a significant source of inspiration, which influenced research in related areas over a long period. This concept was introduced by J.E.M. Elliot and A.T. Butson [EllB66] and was the subject of investigations in design theory, which are of their own independent interest. A generalisation of this concept together with a number of resulting combinatorial objects was suggested in [AraJP90]. In [DavP10] various kinds of difference sets are exploited for the construction of non-symmetric association schemes with three classes. We also mention a few other applications of relative difference sets: extremal graphs of girth 5 in [Jør05], antipodal distance regular Cayley graphs in [CheL05], and nonlinear functions for cryptographic purposes in [Hor10].

9.8 Noboru Ito, who in a sequence of papers coined the concept of a Hadamard group, started to use such groups in design theory and initiated a hope that their use might provide "a golden way" towards a proof of the Hadamard conjecture. Ito's ideas have been successfully developed in several directions. Here we mention just [AraCP01], where Hadamard groups are used in conjunction with relative difference sets, and also [KimN02], devoted to Hadamard matrices of dihedral group type.

In a more sophisticated way, group-theoretic constructions are used in [IonK03], supporting diverse links between designs and doubly regular tournaments (digraphs).

We also mention a recent paper [NozS12], where doubly regular tournaments and normal matrices are considered together with the concept of an almost regular tournament.

9.9 In this paper, we have studied groups of automorphisms of various combinatorial structures, which act transitively, and in many cases regularly, on the points. The full automorphism groups of these structures are also of interest: for instance, one can show that the automorphism group of the digraph $\mathcal{D}(P_q)$ is $\Sigma L(2,q)$, an extension of the group $SL(2,q)$ in Theorem 8.7 by the Galois group of the field $\mathbb{F}_q$ (a cyclic group of order e where $q = p^e$ for some prime p). However, not wishing to add to the already unusual gestation period of this paper (see Section 9.2), we have left a closer investigation of these groups to a later date.

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