Dual Bayer-Billera relations and Kazhdan-Lusztig polynomials

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0-1 sequences

\{0, 1\}^n \text{ set of binary sequences of length } i \text{ and } \{0, 1\}^* \overset{\text{def}}{=} \bigcup_{n=0}^{\infty} \{0, 1\}^n

\{0, 1\}^* \text{ has a monoid structure given by juxtaposition.}
\( \alpha : \{0, 1\}^n \rightarrow \mathbb{C} \) satisfies the Bayer-Billera (or, generalized Dehn-Sommerville) relations if

for all \( T \in \{0, 1\}^n \), \( T = P \cdot 0^j \cdot S \), \( 0^j \) maximal sequence of consecutive 0's in \( T \),

\[
\sum_{i=0}^{j-1} (-1)^i \alpha(P \cdot 0^i 10^{j-i-1} \cdot S) = (1 + (-1)^{j-1})\alpha(T)
\]

The vector space of all functions \( \alpha : \{0, 1\}^n \rightarrow \mathbb{C} \) satisfying the Bayer-Billera relations has dimension \( F_n \) (where \( F_0 = F_1 = 1 \), \( F_n = F_{n-1} + F_{n-2} \) for \( n \geq 2 \)).
Eulerian posets

$P$ a finite graded Eulerian poset of rank $n + 1$, with $\hat{0}$, $\hat{1}$, and rank function $\rho$.
For any chain $C : \hat{0} < x_1 < \cdots < x_k < \hat{1}$ let

$$\text{supp}(C) = d_1 \cdots d_n \in \{0, 1\}^n$$

given by $d_i = 1$ if and only if $\rho(x_j) = i$ for some $j$.
The flag f-vector of $P$ is the function $f_P : \{0, 1\}^n \to \mathbb{N}$ given by

$$f_P(S) = |\{C : \text{supp}(C) = S\}| \quad \forall S \in \{0, 1\}^n$$

Well-known that the function $f_P$ satisfies Bayer-Billera relations.
Special cases if poset $P$ is the face lattice of a convex polytope or a Bruhat interval.
Recent generalization by Ehrenborg-Goresky-Readdy for a Whitney stratification of a closed subset of a smooth manifold.
Coxeter groups

The Bayer-Billera relations also arise in another way in the theory of Coxeter groups. Recall that a Coxeter system is a pair \((W, S)\), where \(S\) is a finite set and \(W\) is a group defined by:

\[
W \overset{\text{def}}{=} < S | (st)^{m(s,t)} = e, \forall s, t \in S >
\]

where, for all \(s, t \in S:\)

\[
m(s, s) = 1 \quad (\Rightarrow s^2 = e \quad \forall s)
\]

\[
m(s, t) = m(t, s)
\]

\[
m(s, t) \in \{2, 3, \ldots, \infty\}, \text{ if } s \neq t
\]

Given \(u \in W\) one lets

\[
\ell(u) \overset{\text{def}}{=} \min\{r : u = s_1 \cdots s_r \text{ for some } s_1, \ldots, s_r \in S\}
\]

(length of \(u\)).
We let $$T \overset{\text{def}}{=} \{usu^{-1} : u \in W, s \in S\}$$ (set of reflections of $W$).

The *Bruhat graph* of $(W, S)$ is the directed graph $B(W, S)$ with

- $W$ as vertex
- directed edges $x \xrightarrow{t} xt$ where $t \in T$ and $\ell(x) < \ell(xt)$.

*Bruhat order* on $W$ is the partial order which is the transitive closure of the Bruhat graph.
The symmetric group $S_n$ is a Coxeter group with respect to the generating set

$$S \overset{\text{def}}{=} \{(1, 2), (2, 3), \ldots, (n-1, n)\}$$

The reflections are

$$T = \{(a, b) : 1 \leq a < b \leq n\}$$

and the length function is

$$\ell(u) = \text{inv}(u)$$

for all $u \in S_n$ (number of inversions of $u$).

If $u = u_1 \cdots u_n \in S_n$ and $t = (a, b) \in T$, $(a \not\leftrightarrow b)$ then

$$u \xrightarrow{t} ut$$

if and only if

$$u_a < u_b$$

($ut$ is obtained from $u$ by switching $u_a$ and $u_b$.)
A reflection ordering is a total ordering $<_T$ on $T$ satisfying additional properties. In $S_n$ the lex order is a reflection ordering. Given a path $\Delta = (a_0 \xrightarrow{t_1} a_1 \xrightarrow{t_2} \cdots \xrightarrow{t_r} a_r)$ in $B(W, S)$ from $a_0$ to $a_r$, we define its length to be

$$l(\Delta) \overset{\text{def}}{=} r,$$

and its descent string, with respect to $<_T$, to be

$$D(\Delta) \overset{\text{def}}{=} d_1 \cdots d_{r-1} \in \{0, 1\}^{r-1}$$

where $d_i = 1$ if and only if $t_i >_T t_{i+1}$.
An example

If $<_T$ is the lex order on $S_n$ and $\Delta$ is the path

\[
123456 \xrightarrow{(2,4)} 143256 \xrightarrow{(1,2)} 413256 \xrightarrow{(2,5)} 453216
\]

\[
\xrightarrow{(5,6)} 453261 \xrightarrow{(2,5)} 463251
\]

then $l(\Delta) = 5$ and $D(\Delta) = 1001$
Given \( u, v \in W \), and \( n \in \mathbb{N} \), we denote by \( B_n(u, v) \) the set of all directed paths in \( B(W, S) \) from \( u \) to \( v \) of length \( n \), and we let
\[
B(u, v) \overset{\text{def}}{=} \bigcup_{n \geq 0} B_n(u, v).
\]
For \( u, v \in W \), \( n \in \mathbb{P} \) and \( S \in \{0, 1\}^{n-1} \) we let
\[
c_{S,n}(u, v) \overset{\text{def}}{=} |\{\Delta \in B_n(u, v) : D(\Delta) \leq S\}|.
\]
\( c_{S,n}(u, v) \) does not depend on \( <_T \) and

**Theorem (Brenti '98)**

Let \( (W, S) \) be a Coxeter system, \( u, v \in W \), and \( n > 0 \). The function \( S \mapsto c_{S,n}(u, v) \) satisfies the Bayer-Billera relations.
It is often the case in combinatorics that whenever a function \( \alpha : \{0, 1\}^n \to \mathbb{C} \) satisfies the Bayer-Billera relations then the function \( \beta : \{0, 1\}^n \to \mathbb{C} \) defined by

\[
\beta(S) = \sum_{T \prec S} (-1)^{|S| - |T|} \alpha(T)
\]  

is also of combinatorial interest. In fact, very often the function \( \beta \) is the one that is \textit{most} interesting.

This was one of the motivations that led us to ask the following natural question.

\textit{If a function \( \alpha : \{0, 1\}^n \to \mathbb{C} \) satisfies the Bayer-Billera relations, then what relations are satisfied by the function \( \beta : \{0, 1\}^n \to \mathbb{C} \) defined by (1) ?}
Dual Bayer-Billera relations

It is well known that the function \( \beta \) satisfies the relations

\[
\beta(S) = \beta(\overline{S})
\]

for all \( S \in \{0, 1\}^n \), where \( \overline{S} \) is the complement string of \( S \). However, these relations are not enough since they are \( 2^{n-1} \), while at least \( 2^n - F_n \) are needed.

**Theorem (Brenti-C, '13)**

Let \( \alpha, \beta : \{0, 1\}^n \to \mathbb{C} \) be such that (1) holds. Then the following are equivalent:

1. \( \alpha \) satisfies the Bayer-Billera relations;
2. \( \beta(S) = \beta(\overline{S}) \) for all \( S \in \{0, 1\}^n \) and, for all \( i \in [2, n - 1] \), \( S \in \{0, 1\}^{i-1} \) and \( T \in \{0, 1\}^{n-i} \)

\[
\beta(S \cdot 0 \cdot T) + \beta(S \cdot 1 \cdot T) = \beta(S \cdot 0 \cdot \overline{T}) + \beta(S \cdot 1 \cdot \overline{T})
\]
Call the relations (2) in the previous theorem the dual Bayer-Billera relations.

Call the submonoid of $\{0, 1\}^*$ generated by 0 and 01 the monoid of sparse strings.

It is easy to see that there are $F_n$ sparse strings in $\{0, 1\}^n$.

If $\beta : \{0, 1\}^n \to \mathbb{C}$ satisfies the dual Bayer-Billera relations then it is uniquely determined by its values on the sparse strings, the other values being linear combinations of these.

We look for such a linear expansion explicitly.
Let $a_1, \ldots, a_k \in \mathbb{P}$

Define $\mathcal{F}(a_1, \ldots, a_k)$ to be the set of all integer sequences $(i_1, \ldots, i_k)$ such that:

i) $0 \leq i_j \leq a_j$ for all $j$;

ii) if $i_{j-1} = a_{j-1}$ then $i_j = 0$;

iii) if $i_j = 0$ and $a_j \equiv 1 \pmod{2}$ then $i_{j-1} = a_{j-1}$;

for all $j = 1, \ldots, k$.

**Example**

If $(a_1, a_2, a_3) = (2, 1, 3)$ then

\[ \mathcal{F}(2, 1, 3) = \{(2, 0, 1), (2, 0, 2), (2, 0, 3), (0, 1, 0), (1, 1, 0)\} \]
A sparse expansion

We let $E_{0,a} = 0^a$ and

$$E_{i,a} = 0^{i-1}10^{a-i} \forall i = 1, \ldots, a.$$ 

Note that the definition of $F(a_1, \ldots, a_k)$ implies that

$$E_{i_1,a_1} \cdots E_{i_k,a_k}$$

does not have two consecutive 1’s.

$E \in \{0, 1\}^n$; its exponent composition $(a_1, a_2, \ldots)$ is given by

$$E = \begin{cases} 
1^{a_1} 0^{a_2} 1^{a_3} \cdots, & \text{if } E_1 = 1, \\
0^{a_1} 1^{a_2} 0^{a_3} \cdots, & \text{if } E_1 = 0.
\end{cases}$$

For example, the exponent composition of 00110 is (2, 2, 1).
A sparse expansion

Theorem (Brenti-C, '13)

Let $\beta : \{0, 1\}^n \rightarrow \mathbb{C}$ satisfy the dual Bayer-Billera relations, $T \in \{0, 1\}^n$ and $(a_1, \ldots, a_k)$ be its exponent composition. Then

$$\beta(T) = \sum_{(i_2, \ldots, i_k) \in F(a_2, \ldots, a_k)} (-1)^{\sum_{j: i_j \neq 0} i_j} \beta(0^{a_1} E_{i_2, a_2} \cdots E_{i_k, a_k}),$$
Application: new non-recursive formula for the Kazhdan-Lusztig polynomials of a Coxeter system \((W, S)\) which holds in complete generality. Recall \(S \mapsto c_{S,n}(u, v)\) satisfies the Bayer-Billera relations. In this case the function given by (1) is

\[
b_{S,n}(u, v) = |\{\Delta \in B_n(u, v) : D(\Delta) = S\}|,
\]

for all \(u, v \in W, \ n \in \mathbb{P}\), and \(S \subseteq [n - 1]\). Hence, the function \(S \mapsto b_{S,n}(u, v)\) satisfies the conclusion of Theorem 3 for all \(u, v \in W\) and \(n \in \mathbb{P}\).
A lattice path on \([n]\) is a function \(\Gamma : [n] \to \mathbb{Z}\) such that \(\Gamma(0) = 0\) and
\[
|\Gamma(i + 1) - \Gamma(i)| = 1
\]
for all \(i \in [n - 1]\).

We let
\[
N(\Gamma) \stackrel{\text{def}}{=} \eta_1 \cdots \eta_{n-1} \in \{0, 1\}^{n-1},
\]
where \(\eta_i = 1\) if and only if \(\Gamma(i) < 0\),
and \(d_+(\Gamma) \stackrel{\text{def}}{=} \text{number of up-steps of } \Gamma\).
Let $\mathcal{L}(n)$ the set of all the lattice paths $\Gamma$ on $[0, n]$ such that $\Gamma(n) < 0$.

Given $E \in \{0, 1\}^{n-1}$ we define a polynomial $\Upsilon_E(q)$ by

$$
\Upsilon_E(q) \overset{\text{def}}{=} (-1)^{m_0(E)} \sum_{\{\Gamma \in \mathcal{L}(n): N(\Gamma) = E\}} (-q)^{d_+(\Gamma)}
$$

where $m_0(E) \overset{\text{def}}{=} |\{j \in [n - 1] : E_j = 0\}|$. 
For $E \in \{0, 1\}^{n-1}$ let
\[
\partial(E) \overset{\text{def}}{=} \{i \in [n - 2] : E_i \neq E_{i+1}\}.
\]
Let $T \in \{0, 1\}^{n-1}$ and $s_1 < \cdots < s_t$ be the positions of the 1’s in $T$, $s_0 \overset{\text{def}}{=} 0$, $s_{t+1} \overset{\text{def}}{=} n$. We define $G(T)$ to be the set of all $E \in \{0, 1\}^{n-1}$ such that:

i) $|\partial(E) \cap (s_j, s_{j+1})| = 1$ for all $j \in [0, t - 1]$;

ii) $|\partial(E) \cap (s_t, s_{t+1})| \leq 1$;

iii) if $\partial(E) \cap (s_t, s_{t+1}) = \{x\}$ then $x \equiv n - 1 \pmod{2}$.

Note that $G(T)$ is empty if $T$ is not sparse.
Given such an \( E \in \mathcal{G}(T) \) we define

\[
\text{sgn}(E, T) \overset{\text{def}}{=} (-1)^{\sum_{i=1}^{t}(s_i-x_i-1)}
\]

where \( \{x_i\} \overset{\text{def}}{=} \partial(E) \cap (s_{i-1}, s_i) \) for \( i \in [t] \), and let

\[
\Omega_T(q) \overset{\text{def}}{=} \sum_{E \in \mathcal{G}(T)} \text{sgn}(E, T) \Upsilon_E(q).
\]

Recall that \( \mathcal{G}(T) = \emptyset \) (and hence \( \Omega_T(q) = 0 \)) if \( T \) is not sparse.
Given a Coxeter system \((W, S)\), \(u, v \in W\), \(u < v\), let \(P_{u,v}(q)\) be the Kazhdan-Lusztig polynomial of \(u, v\). Then we have

**Theorem (Brenti-C, '13)**

Let \((W, S)\) be a Coxeter system and \(u, v \in W\), \(u < v\). Then

\[
P_{u,v}(q) = \sum_{T \in \{0,1\}^*} q^{\frac{\ell(u,v)-1 - \ell(T)}{2}} b_T(u, v) \Omega_T(q).
\]
All combinatorial nonrecursive formulas known for the Kazhdan-Lusztig polynomials which hold in complete generality express each coefficient of the Kazhdan-Lusztig polynomial of $u, v$ as a linear combination of the numbers $b_T(u, v)$. This was one of the motivations that led us to ask the following natural question:

*Are there any other linear relations that hold among the numbers $b_S(u, v)$, besides those implied by Theorem 2 (i.e., by the dual Bayer-Billera relations)?*

*Are there any non-trivial linear relations that hold among the numbers $\{b_T(u, v) : T \in \{0, 1\}^*, T_{\text{sparse}}\}$?*
The answer to the last question is “no”.

**Theorem**

Let \( \{a_T\}_{T \in \{0,1\}^*, T_{\text{sparse}}} \subseteq \mathbb{C} \) be such that

\[
\sum_{\{T \in \{0,1\}^*, T_{\text{sparse}}\}} a_T \, b_T(u, v) = 0
\]

for all Coxeter systems \((W, S)\) and all \(u, v \in W\). Then \(a_T = 0\) for all \(T \in \{0,1\}^*, T_{\text{sparse}}\).