Singleton free set partitions avoiding a 3-element set

Ricardo Mamede

71th Séminaire Lotharingien de Combinatoire
Joint session with
XIX Incontro Italiano di Combinatoria Algebrica

Introduction

A **partition** π of a set $S \subseteq [n]$, $n \geq 1$, is a collection of nonempty disjoint subsets $B_1, \ldots, B_t$ of S, called **blocks**, whose union is S.

A block with only one element is said to be a **singleton**.

$\pi = 13/245/6/7$ is a partition of [7] with $b(\pi) = 4$ blocks

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A partition $\sigma \vdash [n]$ is **layered** if it is of the form $[1, i]/[i+1, j]/[j+1, k]/\cdots/[\ell+i, n]$.

A partition σ is said to be a **matching** if $\#B \leq 2$, for all block B of σ. When the cardinality of each block is exactly 2 the partition is a **perfect matching**.

If $S \subseteq [m]$ with $\#S = n$, then the standardization map corresponding to S is the unique order-preserving bijection

$$st_S : S \to [n].$$

For example, if $S = \{2, 5, 7\}$ then $st(2) = 1, st(5) = 2$ and $st(7) = 3$. Thus, if $\pi = 27/5$ its standardization is $st(\pi) = 13/2$.

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A **subpartition** of a partition $\pi = B_1/B_2/\cdots/B_t$ of S is a partition π' of $S' \subseteq S$ such that each block of π' is contained in a different block of π.

For example, $27/5$ is a subpartition of $1356/27/4$ but not of $1357/26/4$.

Let $\pi \in \Pi_k$ be a given set partition called the **pattern**. A partition $\sigma \in \Pi_n$ **contains the pattern** π if there exists a subpartition σ' of σ such that $st(\sigma') = \pi$. In this case, σ' is called an occurrence of the pattern π in σ.

If σ as no occurrences of π, then we say that σ *avoids* the pattern π.

For example, $\sigma = 16/23/45$ avoids the pattern 123 but contains the pattern $13/2$ since the standardization of the subpartition $\sigma' = 16/2$ is $13/2$.

$R \subseteq \Pi_k$

$$\Pi_n(R) = \{\sigma \in \Pi_n : \sigma \text{ avoids every pattern } \pi \in R\}.$$

$$\Pi'_n(R) = \{\sigma \in \Pi'_n : \sigma \text{ avoids every pattern } \pi \in R\}$$

The set $\Pi(R)$, with $R \subseteq \Pi_3$, was studied by Sagan when $\#R = 1$ and by Goyt for $\#R \geq 2$:

- B. E. Sagan, Pattern avoidance in set partitions. Ars Combin. 94 (2010), 79-96.
- A.M. Goyt, Avoidance of partitions of a three-element set. Adv. in Appl. Math. 41 (2008), no. 1, 95–114.
- T. Mansour, Combinatorics of set partitions, CRC Press [Taylor and Francis Group], 2013.
- M. Klazar, On abab-free and abba-free set partitions, European J. Combin. 17, 1 (1996), 53–68.

Singleton free set partitions, $\#R = 1$

Let π a pattern in Π_3, namely $123, 1/23, 12/3$, $1/2/3$ and $13/2$.

$$F_I(x) = \sum_{i \in I} \frac{x^i}{i!},$$

for I a set of nonnegative integers. In particular, when $I = [0, m]$, we write

$$\exp_m(x) = \sum_{i=0}^{m} \frac{x^i}{i!}.$$

Let $a^I_{n,\ell}$ denote the number of partitions of $[n]$ with ℓ blocks with cardinalities in the set $I \subseteq \mathbb{N}$. It follows that

$$\sum_{n \geq 0} a^I_{n,\ell} \frac{x^n}{n!} = \frac{F_I(x)^\ell}{\ell!}$$

is the exponential generating function for the number of partitions of $[n]$ with ℓ blocks, each of them having sizes in the set I.

Finally, we write

$$F_\pi(x) = \sum_{n \geq 0} \#\Pi'_n(\pi) \frac{x^n}{n!}.$$

For example, with $I = \mathbb{N} \setminus \{1\}$, the exponential generating function for the number of singleton free set partitions of $[n]$ is

$$\begin{aligned} F(x) &= \sum_{n \geq 0} \#\Pi'_n \frac{x^n}{n!} = \sum_{n,\ell \geq 0} a^I_{n,\ell} \frac{x^n}{n!} \\ &= \sum_{\ell \geq 0} \frac{(e^x - 1 - x)^\ell}{\ell!} = \exp(e^x - 1 - x). \end{aligned}$$

$\pi = 12/3, 1/23$

Given positive integers $i < m$, let π_m^i be the layered pattern

$$1/2/\cdots/i-1/i(i+1)/i+2/\cdots/m$$

in Π_m, where all blocks are singletons with the exception of $B_i = \{i, i+1\}$.

Theorem

For $n \geq 2$,

$$\Pi_n'(\pi_m^i) = \{\sigma \in \Pi_n' : b(\sigma) \leq m-2\},$$
$$F_{\pi_m^i}(x) = exp_{m-2}(exp(x) - 1 - x).$$

Corollary

For $n \geq 2$,

$$\Pi_n'(12/3) = \Pi_n'(1/23) = \{12\cdots n\},$$
$$F_{1/23}(x) = F_{12/3}(x) = e^x - x.$$

$\pi = 123$

Theorem

For $n \geq 2$,

$$\Pi'_n(12\cdots m) = \{\sigma \in \Pi_n : 2 \leq \#B \leq m-1, \text{ for all block } B \in \sigma\},$$
$$F_{12\cdots m}(x) = exp(exp_{m-1}(x) - 1 - x).$$

The **double factorial** of an odd positive integer $2i-1$ is defined as the product of all positive odd integers up to $2i-1$:

$$(2i-1)!! = (2i-1)(2i-3)\cdots 5\cdot 3\cdot 1.$$

Corollary

For $n \geq 2$,

$$\Pi'_n(123) = \{\sigma \in \Pi_n : \sigma \text{ is a perfect matching}\},$$
$$\#\Pi'_n(123) = \begin{cases} (2k-1)!! & \text{if } n = 2k \\ 0 & \text{otherwise} \end{cases}.$$

$\pi = 1/2/3$

Theorem

For $n \geq 2$,

$$\Pi'_n(1/2/\cdots/m) = \{\sigma \in \Pi'_n : b(\sigma) \leq m-1\},$$
$$F_{1/2/\cdots/m}(x) = exp_{m-1}(exp(x) - 1 - x).$$

Corollary

We have

$$\Pi'_n(1/2/3) = \{\sigma \in \Pi'_n : b(\sigma) \leq 2\},$$
$$\#\Pi'_n(1/2/3) = 2^{n-1} - n, \text{ for } n \geq 3,$$

with $\#\Pi'_0(1/2/3) = \#\Pi'_2(1/2/3) = 1$ and $\#\Pi'_1(1/2/3) = 0$.

The **Eulerian number** $e(n, m)$ is the number of permutations $p_1 p_2 \cdots p_n$ of $[n]$ with exactly m descents, that is, m places in which $p_j > p_{j+1}$, for $1 \leq j \leq n-1$. Let $E(n, m)$ be the set of all permutations of $[n]$ with exactly m descents.

Theorem

There is a bijection between $\Pi'_n(1/2/3)$ and $E(n-1, 1)$, for $n \geq 1$.

Proof:

$$\psi : E(n-1,1) \longrightarrow \Pi'_n(1/2/3),\ \ n \geq 3$$

$S = \{p_1, \ldots, p_k\} \subseteq [n-1]$ such that $S \neq [k]$ and $p_1 < \cdots < p_k$.

- If $\#S \neq n-2$ set $\psi(S) = \{1, p_1+1, \ldots, p_k+1\}/B$
- If $\#S = n-2$ then $S = \{1, \ldots, \hat{i}, \ldots, n-1\}$ for some i. Set

$$\psi(S) = \begin{cases} \{1, \ldots, i\}/\{i+1, \ldots, n\}, & \text{if } i \neq 1 \\ \{1, \ldots, n\}, & \text{if } i = 1 \end{cases}. \quad \square$$

$\pi = 13/2$

Denote by F_n the n-th Fibonacci number which is defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2,$$

with the initial conditions $F_0 = 0$ and $F_1 = 1$

Theorem

For $n \geq 1$,

$$\begin{aligned}
&\Pi'_n(13/2) = \{\sigma \in \Pi'_n : \sigma \text{ is layered}\}, \\
&\#\Pi'_n(13/2) = F_{n-1}.
\end{aligned}$$

Corollary

The number of layered set partitions of $[n]$ with at least one singleton is given by $2^{n-1} - F_{n-1}$.

π	$\Pi'_n(\pi)$	$\#\Pi'_n(\pi)$
$12/3$	$12\cdots n$	1
$1/23$	$12\cdots n$	1
$1/2/3$	partitions with at most 2 blocks	$2^{n-1}-n$
$13/2$	layered partitions	F_{n-1}
123	perfect matchings	$(2k-1)!!$ if $n=2k$ 0 otherwise

Table: Singleton free partitions avoiding a 3-letter pattern

$\#R \geq 2$

R	$\Pi'_n(R)$
$\{12/3, \pi\}$	$\emptyset$ if $\pi = 123$ $\{12\cdots n\}$ if $\pi \neq 123$
$\{123,13/2\}$	$\{12/34/\cdots/(n-1)n\}$ if n even $\emptyset$ if n odd
$\{123,1/2/3\}$	$\emptyset$ if $n \neq 4$ $\{12/34, 13/24, 14/23\}$ if $n = 4$
$\{13/2,1/2/3\}$	$\{1\cdots i/(i+1)\cdots n : i \in [2, n-2]\} \cup \{12\cdots n\}$
$\{12/3, 13/2, 1/2/3\}$	$\{12\cdots n\}$
$\{12/3, 123, \pi\}$	$\emptyset$ for $\pi = 1/2/3$ or $\pi = 13/2$
$\{13/2, 123, 1/2/3\}$	$\{12/34\}$ if $n = 4$ $\emptyset$ if $n \neq 4$

Table: Singleton free partitions with more than one restriction

Even and Odd Singleton Free Set Partitions

A partition $\sigma \vdash [n]$ with $b(\sigma) = k$ has **sign**

$$sgn(\sigma) = (-1)^{n-k}.$$

A partition σ of $[n]$ is **even** if $sgn(n) = 1$, and is **odd** if $sgn(n) = -1$.

Denote by $E\Pi'_n$ (resp. $O\Pi'_n$) the set of all singleton free even (resp. odd) set partitions of $[n]$. Given $R \subset \Pi_3$, let $E\Pi'_n(R)$ (resp. $O\Pi'_n(R)$) be the set of all singleton free even (resp. odd) set partitions of $[n]$ that avoids the patterns in R.

Lemma

For $n \geq 1$, $\#E\Pi'_n(12/3) = \#E\Pi'_n(1/23)$.

Theorem

For $n \geq 1$, $E\Pi'_n(12/3) = \begin{cases} \emptyset, & \text{if } n \text{ is even} \\ \{12\cdots n\}, & \text{if } n \text{ is odd} \end{cases}.$

Theorem

For $n \geq 1$,

$$E\Pi'_n(1/2/3) = \begin{cases} \{\sigma \in \Pi'_n : b(\sigma) = 2\}, & \text{if } n \text{ is even} \\ \{12\cdots n\}, & \text{if } n \text{ is odd} \end{cases},$$

$$\#E\Pi'_n(1/2/3) = \begin{cases} 2^{n-1} - n - 1, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd} \end{cases};$$

Theorem

If n is an odd integer then $E\Pi'_n(123) = O\Pi'_n(123) = \emptyset$.
If $n = 2k \geq 1$, then

$$E\Pi'_n(123) = \Pi'_n(123) \text{ and } O\Pi'_n(123) = \emptyset, \text{ if } k \text{ is even.}$$

and

$$O\Pi'_n(123) = \Pi'_n(123) \text{ and } E\Pi'_n(123) = \emptyset, \text{ if } k \text{ is odd.}$$

Theorem

For $n \geq 1$,

$$E\Pi'_n(13/2) = \{\sigma \in \Pi'_n : \sigma \text{ is layered and } b(\sigma) \text{ has the parity of } n\},$$
$$\#E\Pi'_n(13/2) = \frac{1}{2}\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right) - \frac{1}{2}\left(\frac{\gamma^n - \delta^n}{\gamma - \delta}\right),$$

where

$$\alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2}, \quad \gamma = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad \delta = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

are the roots of the equation $x^4 + 2x^3 + x^2 - 1 = 0$.

Proof: We have $\#E\Pi'_n(13/2) = \#O\Pi'_{n-2}(13/2) + \#O\Pi'_{n-1}(13/2)$ since any partition $\sigma \in E\Pi'_n(13/2)$ is uniquely obtained from a partition in $\Pi'_{n-2}(13/2)$, with parity different from n, by adding the block $\{n-1, n\}$, or from a partition in $\Pi'_{n-1}(13/2)$, with parity different from n, by adding n to the block having the letter $n-1$. Thus,

$$\begin{aligned}\#E\Pi'_n(13/2) &= \#O\Pi'_{n-2}(13/2) + \#O\Pi'_{n-1}(13/2) \\ &= F_{n-3} + F_{n-2} - \#E\Pi'_{n-2}(13/2) - \#E\Pi'_{n-1}(13/2).\end{aligned}$$

Solving this linear recursion we find that the generating function for $\#E\Pi'_n(13/2)$ is

$$G(x) = \frac{x^2(x+1)}{(1-x-x^2)(1+x+x^2)}. \quad \square$$

P-recursion

A sequence $(a_n)_{n\geq 0}$ is said to be P-recursive (short for *polynomial recursive*) if there exist polynomials $p_0(x), p_1(x) \ldots, p_d(x)$ with $p_d(x) \neq 0$, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + \cdots + p_d(n)a_{n+d} = 0,$$

for all $n \geq 0$.

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$$p_0(n)a_n + p_1(n)a_{n+1} + \cdots + p_d(n)a_{n+d} = 0,$$

for all $n\geq 0$.

A power series $f(x)$ is D-finite (short for *differentiably finite*) if there exist finitely many polynomials $p_0(x), p_1(x), \ldots, p_m(x)$ with $p_m(x)\neq 0$ such that

$$p_0(x)f(x) + p_1(x)f^{(1)}(x) + \cdots + p_m(x)f^{(m)}(x) = 0,$$

where $f^{(i)}(x) = d^i f/dx^i$.

Theorem (Stanley)
A sequence $(a_n)_{n\geq 0}$ is P-recursive if and only if its ordinary generating function $f(x) = \sum_{n\geq 0} a_n x^n$ is D-finite.

Corollary
A sequence $(a_n)_{n\geq 0}$ is P-recursive if and only if its exponential generating function $f(x) = \sum_{n\geq 0} a_n x^n/n!$ is D-finite.

A power series is said to be *algebraic* if there exist polynomials $p_0(x), \ldots, p_d(x)$, not all zero, such that

$$p_0(x) + p_1(x) f(x)^1 + \cdots + p_d(x) f(x)^d = 0.$$

Theorem (Stanley)

If $f(x)$ is an algebraic power series then $f(x)$ is D-finite

The converse of this result is false, since, for instance, the power series $f(x) = e^x$ is D-finite but not algebraic.

Theorem

If $f(x)$ and $g(x)$ are D-finite, then any linear combination $af(x) + bg(x)$ is also D-finite.
If $f(x)$ is D-finite and $g(x)$ is algebraic with $g(0) = 0$, then the composition $f(g(x))$ is D-finite.

Proposition

The sequence $\#\Pi'_n$, $n \geq 1$, is not P-recursive.

Proof.

By contradiction, assume that the sequence $\#\Pi'_n$ is P-recursive. Then, its generating function $F(x) = e^{e^x - 1 - x}$, must be D-finite, and so it must satisfy equation

$$p_0(x)F(x) + p_1(x)F^{(1)}(x) + \cdots + p_m(x)F^{(m)}(x) = 0.$$

A simple induction shows that

$$\frac{d^i}{dx^i}F(x) = F(x)\left(a_0^i + a_1^i e^x + a_2^i e^{2x} + \cdots + a_{i-1}^i e^{(i-1)x} + e^{ix}\right),$$

for constants a_j^i, $j = 0, 1, \ldots, i-1$. Thus, we get $q_0(x) + q_1(x)e^x + \cdots + q_d(x)e^{dx} = 0$, where $q_i(x) = p_i(x) + \sum_{k=i+1}^{d} a_i^k p_k(x)$. Moreover, since the $p_i(x)$ are not all zero, the same is true for the $q_i(x)$. But this imply that e^x is algebraic, a contradiction. $\square$

Theorem

For any $m \geq 1$, the following sequences are P-recursive, for $n \geq 1$:

$$\#\Pi'_n(12\cdots m), \quad \#\Pi'_n(\pi^i_m), \quad \#\Pi'_n(1/2/\cdots/m).$$

Furthermore, for any $\pi \vdash [3]$, the sequences $\#\Pi'_n(\pi)$, $\#E\Pi'_n(\pi)$ and $\#O\Pi'_n(\pi)$, $n \geq 1$, are P-recursive.

Proof.

The egf for $\#\Pi'_n(12\cdots m)$, $n \geq 1$, is given by $F_{12\cdots m}(x) = \exp(\exp_{m-1}(x) - 1 - x)$. Since $f(x) = e^x$ is D-finite, and $g(x) = \exp_{m-1}(x) - 1 - x$ is algebraic, the composition $f(g(x)) = F_{12\cdots m}(x)$ is D-finite.
The egf $\exp_{m-2}(e^x - 1 - x)$ and $\exp_{m-1}(e^x - 1 - x)$ for $\#\Pi'_n(\pi^i_m)$ and $\#\Pi'_n(1/2/\cdots/m)$, $n \geq 1$, are D-finite since this functions are linear combinations of series of the form $x^m e^{ax}$, with $m \in \mathbb{N}$ and $a \in \mathbb{R}$, and thus satisfy a linear homogeneous differential equation with constant coefficients. $\square$

Gray codes

A Gray code for a class of combinatorial objects is a list of these objects so that the transition from one object in the list to its successor takes only a "small change". The definition of "small change" depends on the particular class of objects.

In our case, we define the distance between two partitions π, ω of $[n]$ as the minimum number of letters that must be moved between blocks of π, possibly creating a new block, so that the resulting partition is ω.

$\Pi'_n(13/2)$

Definition

Let $\sigma = B_1/\cdots/B_{t-1}/B_t$ and π be layered singleton free partitions of $[n]$. We say that σ and π forms a *good pair* if whenever $\#B_{t-1} \geq 3$ and $B_t = \{n-1, n\}$, then $B_{t-1} \cup \{n-1, n\}$ is not a block of π.

Theorem

For each $n \geq 4$ there is a Gray code sequence with distance 2,

$$\pi_1, \pi_2 \ldots, \pi_s,$$

for $\Pi'_n(13/2)$ such that any two consecutive elements are good pairs, $\pi_1 = 12\cdots n$ and $\pi_s = 12\cdots(n-2)/(n-1)n$.

$\Pi'_n(1/2/3)$ and $\Pi'_{2k}(123)$

Theorem

For each $n \geq 4$ there is a Gray code sequence with distance 2 for $\Pi'_n(1/2/3)$ which starts with $12\cdots n$ and is followed by $1n/2\cdots(n-1)$.

Theorem

For each integer $k \geq 1$, there is a Gray code sequence for $\Pi'_{2k}(123)$ with distance 2.

$\Pi'_2(13/2)$	12
$\Pi'_3(13/2)$	123
$\Pi'_4(13/2)$	1234, 12/34
$\Pi'_5(13/2)$	12345, 12/345, 123/45
$\Pi'_6(13/2)$	123456, 12/3456, 123/456, 12/34/56, 1234/56
$\Pi'_7(13/2)$	1234567, 12/34567, 123/4567, 12/34/567, 1234/567, 123/45/67, 12/345/67, 12345/67
$\Pi'_8(13/2)$	12345678, 12/345678, 123/45678, 12/34/5678, 1234/5678, 123/45/678, 12/345/678, 12345/678, 1234/56/78, 12/34/56/78, 123/456/78, 12/3456/78, 123456/78

Table: Gray codes for $\Pi'_n(13/2)$, $n = 2, \ldots, 8$

$\Pi'_2(1/2/3)$	12
$\Pi'_3(1/2/3)$	123
$\Pi'_4(1/2/3)$	1234, 14/23, 24/13, 12/34
$\Pi'_5(1/2/3)$	12345, 15/234, 25/134, 35/124, 45/123, 14/235, 24/135, 12/345, 125/34, 245/13, 145/23
$\Pi'_6(1/2/3)$	123456, 16/2345, 26/1345, 36/1245, 46/1235, 56/1234, 15/2346, 25/1346, 35/1246, 45/1236, 14/2356, 24/1356, 12/3456, 125/346, 245/136, 145/236 1456/23, 2456/13, 1256/34, 126/345, 246/135, 146/235, 456/123, 356/124, 256/134, 156/234

Table: Gray codes for $\Pi'_n(1/2/3)$, $n = 2, 3, 4, 5, 6$

$\Pi_2'(123)$	12
$\Pi_4'(123)$	12/34, 13/24, 14/23
$\Pi_6'(123)$	12/34/56, 16/34/25, 26/34/15, 36/24/15, 46/23/15, 16/23/45, 16/24/35, 13/24/56, 13/26/45, 12/36/45, 12/46/35, 13/46/25, 14/36/25, 14/26/35, 14/23/45

Table: Gray codes for $\Pi_n'(123)$, $n = 2, 4, 6$