Hall-Littlewood symmetric polynomials via “chip firing game”

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Hall-Littlewood polynomials

Hall-Littlewood polynomials: the $R_\lambda(x; t)$-normalization

If $\lambda$ is a partition with $l \leq n$ parts and $x = \{x_1, x_2, \ldots, x_n\}$ then we set

$$R_\lambda(x; t) = \sum_{\sigma \in S_n} \sigma \left( x^\lambda \prod_{1 \leq i < j \leq n} \frac{x_i - tx_j}{x_i - x_j} \right).$$
If $G = (V, E)$ then a **configuration** of $G$ is a map

$$\alpha : v \in V \rightarrow \alpha(v) \in \mathbb{Z}.$$
Configurations and toppling

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**Firing** a vertex $v$ produces a **toppling** of its weight,
If $G = (V, E)$ then a **configuration** of $G$ is a map

$$\alpha: v \in V \rightarrow \alpha(v) \in \mathbb{Z}.$$  

**Firing** a vertex $v$ produces a **toppling** of its weight, namely

- the weight $\alpha(v)$ of $v$ **decreases** by $\text{deg}(v)$;
- the weight $\alpha(w)$ of each neighbour $w$ of $v$ **increases** by 1.
Firing a vertex: an example

Consider the following configuration
Firing a vertex: an example

Let’s fire a vertex
Firing a vertex: an example

new labels are
Firing a vertex: an example

and a new configuration is obtained
Toppling equivalence

If, for a fixed graph $G$, there exists a sequence of firings that change $\alpha$ in $\beta$ then we say that $\alpha$ and $\beta$ are equivalent, written $\alpha \equiv_G \beta$. 

Toppling on a graph
Toppling equivalence

If, for a fixed graph $G$, there exists a sequence of firings that change $\alpha$ in $\beta$ then we say that $\alpha$ and $\beta$ are equivalent, written

$$\alpha \equiv_G \beta.$$ 

Toppling equivalence has relations with parking functions and $q, t$-Catalan numbers [Cori and Le Borgne]
The toppling group

Fix a graph $G$ and label its vertices so that

$$V = \{1, 2, \ldots, n\}.$$
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Set $\alpha_i = \alpha(i)$ and identify

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n).$$
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Set $\alpha_i = \alpha(i)$ and identify

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Define the map $T_i : \mathbb{Z}^n \to \mathbb{Z}^n$ so that

$$T_i(\alpha) = \beta$$

iff

$$\beta \text{ comes from } \alpha \text{ by firing the vertex } i$$
The toppling group

The toppling group of $G$ is the group $\mathcal{T} = \mathcal{T}^G$ generated by $T_1, T_2, \ldots, T_n$. 
The toppling group

The **toppling group** of $G$ is the group $\mathcal{T} = \mathcal{T}^G$ generated by $T_1, T_2, \ldots, T_n$.

The **orbits** of its action on $\mathbb{Z}^n$ are the class of equivalences of $\equiv_G$:

$$\alpha \equiv_G \beta \iff \beta = T^\lambda(\alpha),$$

where $\lambda \in \mathbb{N}^n$ and $T^\lambda = T_1^{\lambda_1} T_2^{\lambda_2} \cdots T_n^{\lambda_n}$. 
The partial order $\leq_G$

We set

$$\beta \leq_G \alpha \text{ iff } \beta = T^\lambda(\alpha)$$

for some $\lambda_1 \geq \lambda_2 \ldots \geq \lambda_n \geq 0$.

Theorem

1. $\leq_G$ is a partial order
2. if $\beta \leq_G \alpha$ then there exists a unique $\lambda$ such that $\lambda_n = 0$ and $\beta = T^\lambda(\alpha)$
The principal order ideal $\mathcal{H}_\alpha$

Instead of studying orbits

$$\mathcal{O}_\alpha = \{ \beta \mid \beta \equiv_G \alpha \}$$

we focus our attention on ideals

$$\mathcal{H}_\alpha = \{ \beta \mid \beta \leq_G \alpha \}$$
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In particular we can give a description of the series

$$\mathcal{H}_\alpha(x; t) = \sum_{\beta \leq_G \alpha} t^{dist(\lambda')} x^\beta,$$

with $\lambda$ being the unique such that $\lambda_n = 0$ and $\beta = T^\lambda(\alpha)$
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In particular we can give a description of the the series

$$\mathcal{H}_\alpha(x; t) = \sum_{\beta \leq_G \alpha} t^{\text{dist}(\lambda')} x^\beta,$$

with $\lambda$ being the unique such that $\lambda_n = 0$ and $\beta = T^\lambda(\alpha)$ and $\text{dist}(\lambda')$ is the number of distinct parts of its conjugate.
A first result

\( \mathcal{T} \) acts on monomials \( x^\alpha \) by following

\[ T_i \cdot x^\alpha = x^{T_i(\alpha)}. \]
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$\mathcal{T}$ acts on monomials $x^\alpha$ by following

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Theorem

We have

$$\mathcal{H}_\alpha(x; t) = \prod_{1 \leq i \leq n-1} \frac{1 - (1 - t)[i]}{1 - [i]} \cdot x^\alpha,$$

where

$$[i] = T_1 T_2 \cdots T_i.$$
A first result

$\mathcal{T}$ acts on monomials $x^\alpha$ by following

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**Theorem**

We have

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where

$$[i] = T_1 T_2 \cdots T_i.$$
A first result: a sketch proof

Since

\[ T^\lambda = \begin{array}{ccccccc}
R_1 & R_1 & \ldots & R_1 & R_1 \\
R_2 & R_2 & \ldots & R_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
R_n & \ldots & R_n
\end{array} \]

\lambda_1 \text{ times} \\
\lambda_2 \text{ times} \\
\lambda_n \text{ times}

by multiplying along columns

\[ T^\lambda = [\lambda'_1][\lambda'_2] \cdots \]
A first result: a sketch proof

Since

\[ T^\lambda = \begin{bmatrix} R_1 & R_1 & \ldots & R_1 & R_1 \\ R_2 & R_2 & \ldots & R_2 \\ \vdots & \vdots & & \vdots & \vdots \\ R_n & \ldots & R_n \end{bmatrix}, \]

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\lambda_2 \text{ times} \\
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by multiplying along columns

\[ T^\lambda = [\lambda'_1][\lambda'_2] \cdots \]

Finally, by expanding \( \tau \) as a series

\[ \tau \cdot x^{\alpha} = \sum_{\lambda' \leq n-1} t^{dist(\lambda')} [\lambda'_1][\lambda'_2] \cdots = \sum_{\ell(\lambda) \leq n-1} t^{dist(\lambda')} T^\lambda \cdot x^{\alpha}. \]
An action of $\mathcal{T}$ on $\mathbb{Z}[x_1, x_2, \ldots, x_n]$ is obtained by means of

$$T_i \cdot x^\alpha = \begin{cases} x^{T_i(\alpha)} & \text{if } T_i(\alpha) \in \mathbb{N}^n \\ 0 & \text{otherwise} \end{cases}.$$
Acting on polynomials: basis attached on each graph

An action of $\mathcal{T}$ on $\mathbb{Z}[x_1, x_2, \ldots, x_n]$ is obtained by means of

$$T_i \cdot x^\alpha = \begin{cases} x^{T_i(\alpha)} & \text{if } T_i(\alpha) \in \mathbb{N}^n \\ 0 & \text{otherwise} \end{cases}.$$ 

In this case $(\mathcal{H}_\alpha(x; t))_\alpha$ is a basis (a basis for each $G$).
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In this case $(\mathcal{H}_\alpha(x; t))_\alpha$ is a basis (a basis for each $G$).

By symmetrizing a basis (attached on each $G$) of symmetric polynomials is obtained

$$\mathcal{H}^*_\alpha(x; t) = \sum_{\sigma \in \mathfrak{S}_n} \sigma(\mathcal{H}_\alpha(x; t)).$$
Some manipulation

Note that

\[ T_i \cdot x^\alpha = x^\alpha \frac{1}{x_i^{\text{deg}(i)}} \prod_{k \text{ neig. of } i} x_k, \]
Note that
\[ T_i \cdot x^\alpha = x^\alpha \prod_{k \text{ neig. of } i} x_{\deg(i)} \prod_{k \text{ neig. of } i} x_k, \]
and in particular
\[ \tau \cdot x^\alpha = \tau(x; t) x^\alpha, \]
for some \( \tau(x, t) \in \mathbb{Z}[t] \left[ \left[ x_1^{\pm 1}, x_2^{\pm 2}, \ldots, x_n^{\pm 1} \right] \right] \) uniquely determined by \( G \).
A special case

If $G = L_n$ is the graph

![Graph with $n$ nodes and $n-1$ edges]

then $[i] = T_1 T_2 \cdots T_i$ “produces” $\alpha_i - 1$ and $\alpha_{i+1} + 1$ and then

$$\prod_{1 \leq i \leq n-1} \frac{1 - (1 - t)[i]}{1 - [i]} \cdot x^\alpha = \prod_{1 \leq i \leq n} \frac{x_i - (1 - t)x_{i+1}}{x_i - x_{i+1}} x^\alpha.$$
A special case

If $G = L_n$ is the graph

\[
\begin{array}{c}
\circ \\
1 \\
\mid \\
2 \\
\mid \\
\vdots \\
n-1 \\
\mid \\
n \\
\end{array}
\]

then $[i] = T_1 T_2 \cdots T_i$ “produces” $\alpha_i - 1$ and $\alpha_{i+1} + 1$ and then

\[
\prod_{1 \leq i \leq n-1} \frac{1 - (1 - t)[i]}{1 - [i]} \cdot x^\alpha = \prod_{1 \leq i \leq n} \frac{x_i - (1 - t)x_{i+1}}{x_i - x_{i+1}} x^\alpha.
\]

N.B. $\tau(x; t)$ is obtained by expanding in powers of $x_{i+1}/x_i$
From toppling to $R_\lambda(x; t)$: looking for elements $[i, j]$’s

The idea is to find elements $[i, j]$ in the toppling group that for $L_n$ give

$$
\prod_{1 \leq i < j \leq n} \frac{1 - (1 - t)[i, j]}{1 - [i, j]} \cdot x^\alpha = \prod_{1 \leq i < j \leq n} \frac{x_i - (1 - t)x_j}{x_i - x_j} x^\alpha,
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From toppling to $R_\lambda(x; t)$: looking for elements $[i, j]$’s

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$$\prod_{1 \leq i < j \leq n} \frac{1 - (1 - t)[i, j]}{1 - [i, j]} \cdot x^\alpha = \prod_{1 \leq i < j \leq n} \frac{x_i - (1 - t)x_j}{x_i - x_j} x^\alpha,$$

so that if

$$\hat{H}_\alpha(x; t) = \prod_{1 \leq i < j \leq n} \frac{1 - (1 - t)[i, j]}{1 - [i, j]} \cdot x^\alpha,$$

then

$$\hat{H}_\alpha^*(x; t) = \sum_{\sigma \in S_n} \sigma \left( \hat{H}_\alpha(x; t) \right) = R_\alpha(x, 1 - t).$$
From toppling to $R_\lambda(x; t)$

This is obtained by setting

$$[i, j] = [i][i + 1] \cdots [j - 1].$$
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Also we have the following combinatorial interpretation:

$$\hat{H}_\alpha(x; t) = \prod_{1 \leq i < j \leq n} \frac{1 - (1 - t)[i, j]}{1 - [i, j]} \cdot x^\alpha = \sum_{\beta \leq_G \alpha} K_{\alpha, \beta}(t)x^\beta,$$
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\hat{H}_\alpha(x; t) = \prod_{1 \leq i < j \leq n} \frac{1 - (1 - t)[i, j]}{1 - [i, j]} \cdot x^\alpha = \sum_{\beta \leq G \alpha} K_{\alpha, \beta}(t)x^\beta,
\]

where

\[
K_{\alpha, \beta}(t) = \sum_{\beta = [i_1, j_1]^{a_1}[i_2, j_2]^{a_2} \cdots [i_k, j_k]^{a_k}(\alpha)} t^k,
\]

with the \([i_1, j_1], [i_2, j_2], \ldots, [i_k, j_k]\) pairwise distinct.
We have

1. via topplings, for each graph we define a partial order $\leq_G$
2. (for each graph!) we construct a basis of symmetric polynomials $(\hat{H}_\alpha(x; t))_\alpha$
3. such basis encodes principal order ideals of $\leq_G$ and reduces to Hall-Littlewood symmetric polynomials when $G = L$
Hall-Littlewood symmetric polynomials via “chip firing game”

Reassuming

What’s more?

1. more parameters $t, q, z$ related to further statistics of the unique $\lambda$ and more general polynomials $(\hat{H}_\alpha^*(x; t, q, z))_\alpha$ may be constructed (do they share some property with Macdonald polynomials?)
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2. explicit connection between $\leq_G$ and standard tableaux are possible:
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   **starting configuration**: $\alpha$
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   **PLAYING TOPPLING GAME:**
   
   **starting configuration:** $\alpha$
   
   **goal:** you want to move from $\alpha$ towards a given $\beta \leq_G \alpha$
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1. more parameters $t, q, z$ related to further statistics of the unique $\lambda$ and more general polynomials $(\hat{\mathcal{H}}^*_\alpha(x; t, q, z))_\alpha$ may be constructed (do they share some property with Macdonald polynomials?)

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   - **starting configuration**: $\alpha$
   - **goal**: you want to move from $\alpha$ towards a given $\beta \leq_G \alpha$
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   - **question**: in how many ways you may get $\beta$?
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   **answer:** number of standard Young tableaux of shape $\lambda$
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   **answer:** number of standard Young tableaux of shape $\lambda$

3. do special choices of $G$ (trees, cyclic graphs, . . . ) give rise to interesting basis?
Thanks

Many thanks for your attention!