A complexity theorem for the Novelli–Pak–Stoyanovskii-algorithm

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joint work with Christoph Neumann

Universität Wien

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Outline

1. The Novelli–Pak–Stoyanovskii-algorithm
2. Complexity
3. The theorem

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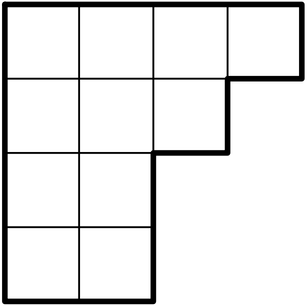
Partitions



The partition $4 + 3 + 2 + 2$.

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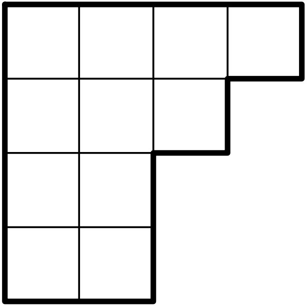

The partition $4 + 3 + 2 + 2$.

A partition λ of n is a left-justified array of cells with λ_i cells in the i-th row, where $\lambda_1 \geq \lambda_2 \geq \ldots$ and $\sum \lambda_i = n$.

Tabloids

2	9	5	6
1	4	3	
7	11		
8	10		

A tabloid.

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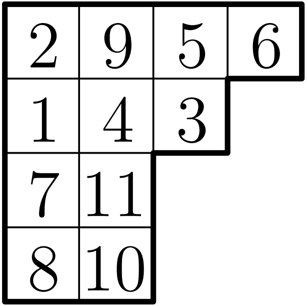

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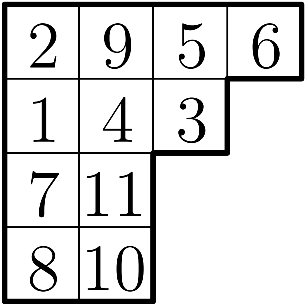

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A tabloid of shape λ is a bijection $T : \lambda \to \{1, \ldots, n\}$.

Let $\mathrm{T}(\lambda)$ denote the set of all tabloids of shape lambda.

Standard Young tableaux

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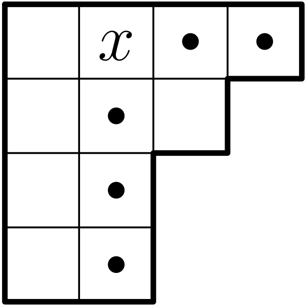
A tabloid $T \in \mathrm{T}(\lambda)$ is called standard Young tableau if T increases from left to right and from top to bottom.

Let $\mathrm{SYT}(\lambda)$ denote the set of all standard young tableaux of shape lambda.

Hooks



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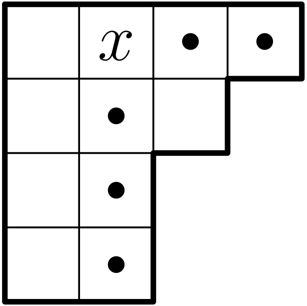

We have $h_\lambda(x) = 6$.

The hook-length $h_\lambda(x)$ of a cell $x \in \lambda$ is the number of cells directly below or to right of x plus one.

How many SYT are there?

The hook-length formula (Frame, Robinson, Thrall)

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For any partition λ we have

$$\#\,\mathsf{SYT}(\lambda) = \frac{n!}{\prod_{x\in\lambda} h_\lambda(x)}.$$

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A bijective proof is due to Novelli, Pak and Stoyanovskii.

They show that

$$n! = \#\,\mathrm{SYT}(\lambda)\cdot\prod_{x\in\lambda} h_\lambda(x)$$

by assigning to each tabloid $T \in \mathrm{T}(\lambda)$ a pair of a SYT W and a hook function H.

$$T \mapsto (W, H)$$

Ordering the cells

We define an order $\prec_U$ on the cells of λ where $U \in \mathrm{SYT}(\lambda)$ as

$$x \prec_U y :\Leftrightarrow U(x) < U(y).$$

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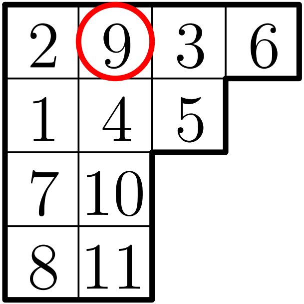

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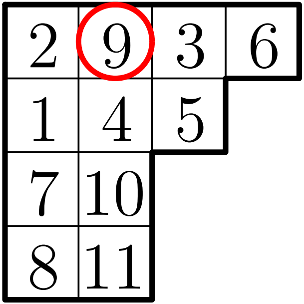

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We sort by exchanging entries.

Let $s = T(x)$. Exchange s with the least entry among its right or bottom neighbours.

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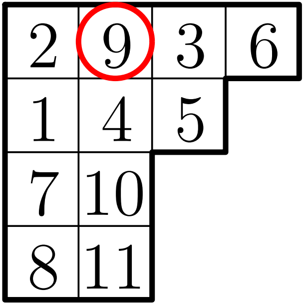

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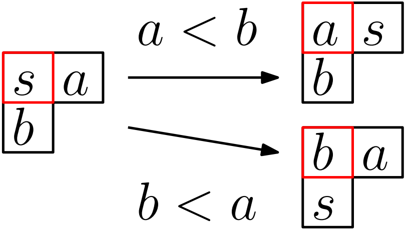

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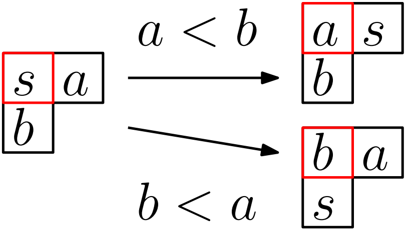

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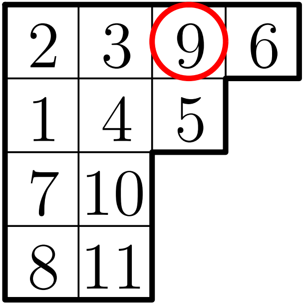

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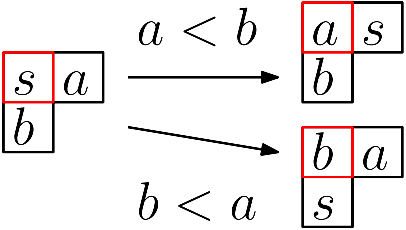

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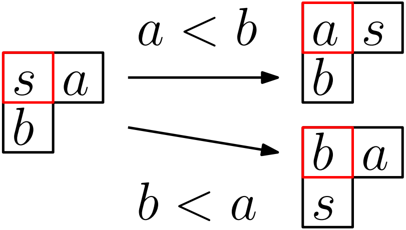

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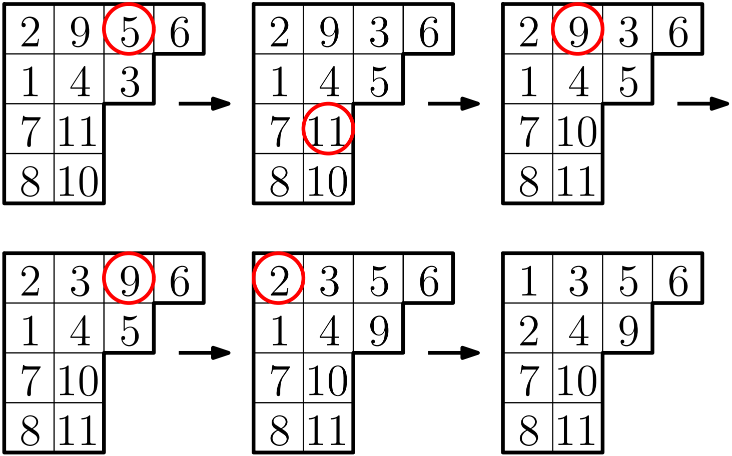
An example



We need five exchanges.

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Complexity

Definition

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Definition

We define the complexity of the Novelli–Pak–Stoyanovskii-algorithm with respect to the order $\prec_U$ as the average number of exchanges, i.e.,

$$C(U) := \frac{1}{n!} \sum_{T \in \mathrm{T}(\lambda)} N_U(T),$$

where $N_U(T)$ is the number of exchanges needed to sort the tabloid T.

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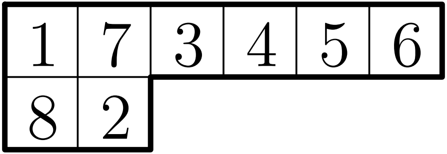

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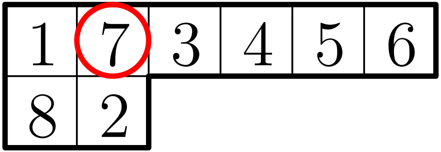

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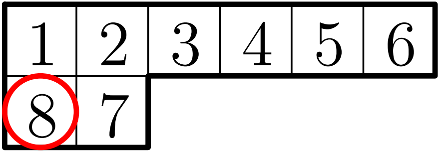

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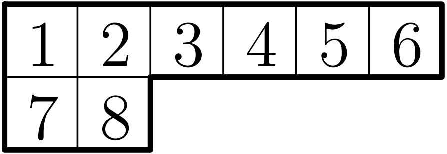

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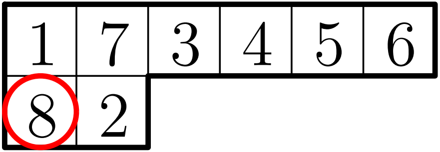

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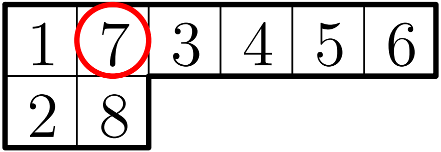

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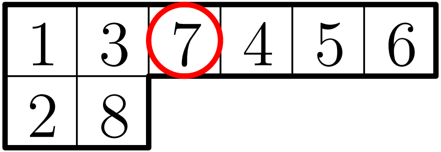

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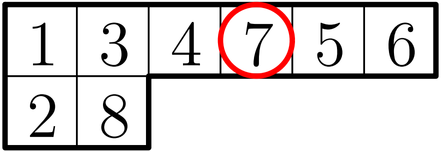

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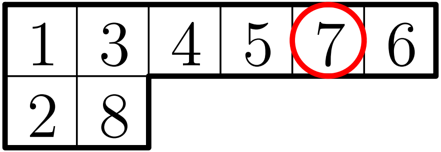

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Lemma

Let a, b, c be entries such that $a < b < c$. Then

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Therefore, we just write $m_U(a)$.

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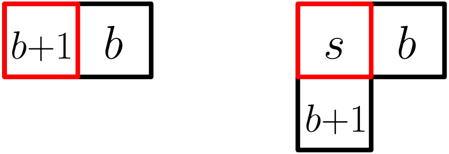

Both situations may appear only after b and $b+1$ have dropped.

Distribution

Definition (Distribution vector)

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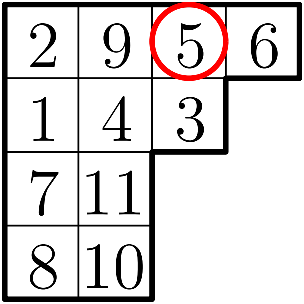
Let $U, W \in \mathsf{SYT}(\lambda)$. Denote by $z_U(W)$ the number of tabloids $T \in \mathsf{T}(\lambda)$ such that sorting with respect to $\prec_U$ transforms T into W. We call

$$\mathbf{z}_U := (z_U(W))_{W \in \mathsf{SYT}(\lambda)}$$

the distribution vector of U.

Height



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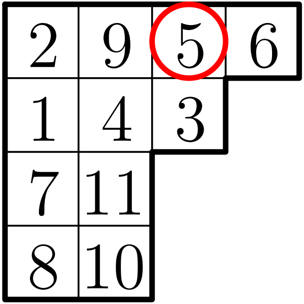

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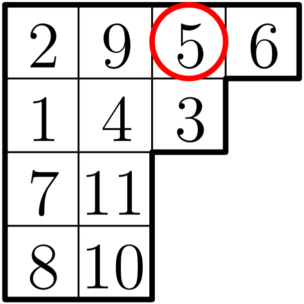

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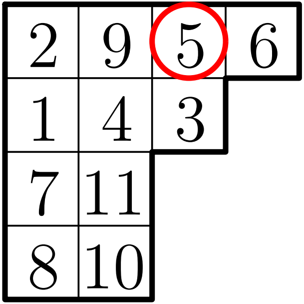

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Suppose during sorting we exchange a and b where $a < b$. Then $h'(a, T_i) = h'(a, T_{i-1}) - 1$ and $h'(b, T_i) = h'(b, T_{i-1}) + 1$.

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$$\sum_{c=b+1}^{n} m_U(b) = (n-b) m_U(b).$$

Initial and terminal height

Fix an entry $1 \leq b \leq n$.

Consider the total initial height of this entry

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Let $\prec_U$ be given by $U \in \mathrm{SYT}(\lambda)$. For all $1 \leq b \leq n$ we have

$$(n-b)\, m_U(b) = (n-1)! \sum_{x \in \lambda} h'(x) + \sum_{a=1}^{b-1} m_U(a) - \sum_{W \in \mathrm{SYT}(\lambda)} z_U(W) h'(b, W).$$

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Corollary

The exchange numbers $m_U(a)$ only depend on the distribution vector $\mathbf{z}_U$.

The conjecture follows

We have $C(U) = \frac{1}{n!} \sum_{a=1}^{n} (n-a)\, m_U(a)$.

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Let $\prec_U$ be the column-wise order. Due to the bijection of Novelli, Pak and Stoyanovskii we have

$$z_U(W) = \prod_{x \in \lambda} h_\lambda(x).$$

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Let $\prec_V$ be the row-wise order, then $\mathbf{z}_U = \mathbf{z}_V$. The conjecture follows.

The end

Thanks for your attention!