Séminaire Lotharingien de Combinatoire **72** (2014), Article B72b

DUAL GARSIDE STRUCTURE OF BRAIDS AND FREE CUMULANTS OF PRODUCTS

PHILIPPE BIANE AND PATRICK DEHORNOY

Abstract. We count the n-strand braids whose normal decomposition has length at most 2 in the dual braid monoid B_n^{+*} by reducing the question to a computation of free cumulants for a product of independent variables, for which we establish a general formula.

1. Introduction

A Garside structure on a group G consists of a generating family that gives rise to distinguished decompositions of a particular type, leading in good cases to an automatic structure on G and, from there, to solutions of the word and conjugacy problems of G [7]. In the case of the n-strand Artin braid group B_n, two Garside structures are known: the so-called classical Garside structure, in which the distinguished generating family (the "Garside base") is a copy of the symmetric group $\mathfrak{S}_n$ [9, Chapter 9], and the so-called *dual Garside structure*, in which the Garside base is a copy of the family $\mathrm{NC}(n)$ of all size n noncrossing partitions [1].

Whenever a finite Garside base S is given on a group G, natural counting problems arise, namely the problem of counting how many elements of the group G or of the submonoid generated by S have length (at most) ℓ with respect to S. Call S-normal the distinguished decompositions associated with a Garside base S; as S-normal sequences happen to be geodesic, the above question amounts to counting the S-normal sequences of length ℓ. Moreover, since S-normality is a purely local property, the central question is to determine S-normal sequences of length two, the general case then corresponding to taking the ℓth power of the incidence matrix associated with length two.

Initially motivated by the investigation of the logical strength of certain statements involving the standard braid ordering [5], the above

2010 *Mathematics Subject Classification.* 20F36, 05A17, 46L54.

Key words and phrases. braid group; dual braid monoid; Garside normal form; generating function; free cumulants; independent random variables; noncrossing partitions.

mentioned counting questions in the case of the classical Garside structure of braids have been addressed in [6], leading to nontrivial results involving Solomon's descent algebra and to natural conjectures, like the one established in [11] using the theory of quasi-symmetric functions.

The aim of this paper is to address similar questions in the case of the dual Garside structure of braids and to obtain an explicit determination of the generating function for the number of normal sequences of length two. In the statement of the corresponding result below, the symbol Cat_n denotes the nth Catalan number $\frac{1}{n+1}\binom{2n}{n}$.

Theorem 1.1. *Let $b^*_{n,2}$ be the number of braids of length at most 2 in the dual braid monoid B_n^{+*}. Then the function $R(z) = 1 + \sum_{n\geqslant 1} b^*_{n,2} z^n$ is connected with $M(z) = 1 + \sum_{n\geqslant 1} \mathrm{Cat}_n^2 z^n$ by the equality*

$$R(zM(z)) = M(z). \tag{1.1}$$

This formula, which inductively determines the numbers $b^*_{n,2}$, will be deduced from a general formula for computing the free cumulants of a product of independent random variables.

Free cumulants were invented by Roland Speicher in order to perform computations with free random variables [14]. In particular, free cumulants give a simple way of computing the free additive convolution of two probability measures on the real line, namely the free cumulants of the sum of two free random variables are the sums of the free cumulants of the variables. Free cumulants also appear when one tries to compute the free cumulants of a product of free random variables, although in a more complicated way.

Here is the general formula we establish (the notations are standard and explained in Section 3):

Theorem 1.2. *Let $X_1, \ldots, X_k$ be a family of commuting independent random variables, and let $R_n^{(i)}$ be the free cumulants of X_i. Then the free cumulants of the product $X_1 X_2 \ldots X_k$ are given by*

$$R_n = \sum_{\pi_1 \vee \cdots \vee \pi_k = \mathbf{1}_n} \prod_i R^{(i)}_{\pi_i}, \tag{1.2}$$

the sum being over all k-tuples of noncrossing partitions in $\mathrm{NC}(n)$ *whose join is the largest partition.*

Since this result is purely combinatorial, we give also another formulation, without reference to free probability.

Theorem 1.3. *For $i = 1, \ldots, k$, let $R_n^{(i)}$ and $M_n^{(i)}$ be families of commuting indeterminates, with generating functions*

$$R^{(i)}(z) = 1 + \sum_{n \geqslant 1} R_n^{(i)} z^n \quad \text{and} \quad M^{(i)}(z) = 1 + \sum_{n \geqslant 1} M_n^{(i)} z^n$$

related by

$$R^{(i)}(zM^{(i)}(z)) = M^{(i)}(z).$$

Then the generating functions of

$$M_n = \prod_i M_n^{(i)} \quad \text{and} \quad R_n = \sum_{\pi_1 \vee \cdots \vee \pi_k = \mathbf{1}_n} \prod_i R_{\pi_i}^{(i)},$$

the latter sum being over all k-tuples of noncrossing partitions in $\mathrm{NC}(n)$ whose join is the largest partition, are related by

$$R(zM(z)) = M(z).$$

Free cumulants are specifically designed to deal with highly noncommuting objects. Therefore it may come as a surprise that there is also a simple formula for computing free cumulants of a product of independent commuting random variables, in terms of the free cumulants of the factors. As a matter of fact, such a formula also holds, with appropriate modifications, for classical and Boolean cumulants.

The organization of the paper is as follows. In Section 2, we recall the description of the dual Garside structure of braid groups and raise the induced counting questions. In Section 3, we review basic definitions about free cumulants and establish Theorem 1.2. This part can be read independently of the rest of the paper. In Section 4, we apply the result of Section 3 to braids and conclude with further questions and one additional result about the determinant of the involved incidence matrix.

2. The dual Garside structure of braids

Braid groups. For $n \geqslant 1$, the n-strand braid group B_n is the group defined by the presentation

$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{cl} \sigma_i \sigma_j = \sigma_j \sigma_i & \text{for } |i-j| \geqslant 2 \\ \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j & \text{for } |i-j| = 1 \end{array} \right\rangle. \tag{2.1}$$

The group B_n is both the group of isotopy classes of n-strand geometric braids, the mapping class group of an n-punctured disk, and the fundamental group of the configuration space obtained by letting the symmetric group act on the complement of the diagonal hyperplanes in $\mathbb{C}^n$ [3].

Garside structures. A *Garside base* in a group G is a subset S of G such that every element of G admits a decomposition of a certain syntactic form in terms of the elements of S, namely a symmetric S-normal decomposition in the following sense.

Definition 2.1 ([7]). Assume that G is a group and S is included in G.

(i) A finite S-sequence $(s_1, \dots, s_d)$ is called *S-normal* if, for $i < d$, every element of S left-dividing $s_i s_{i+1}$ left-divides s_i, where "f left-divides g" means "$f^{-1}g$ lies in the submonoid $\widehat{S}$ of G generated by S".

(ii) A pair of finite S-sequences $((s_1, \dots, s_d), (t_1, \dots, t_e))$ is called *symmetric S-normal* if $(s_1, \dots, s_d)$ and $(t_1, \dots, t_e)$ are S-normal and, in addition, the only element of S left-dividing s_1 and t_1 is 1.

(iii) A *Garside base* for G is a subfamily S of G such that every element g of G admits a symmetric S-normal decomposition, meaning that we have $g = t_e^{-1} \dots t_1^{-1} s_1 \dots s_d$ for some symmetric S-normal pair $((s_1, \dots, s_d), (t_1, \dots, t_e))$.

Every group G is trivially a Garside base in itself, and the notion is interesting only when S is small, typically when G is infinite and S is finite or, at least, is properly included in G. Under mild assumptions, the existence of a finite Garside base implies good properties for the group G such as the existence of an automatic structure or the decidability of the word and conjugacy problems.

Whenever S is a finite generating family in a group G, it is natural to consider the numbers

$$N_d^S = \#\{g \in G \mid \|g\|_S = d\},$$

where $\|g\|_S$ is the S-length of g, that is, the smallest ℓ such that g can be expressed as $s_1^{e_1} \dots s_\ell^{e_\ell}$ with $s_1, \dots, s_\ell$ in S and $e_1, \dots, e_\ell$ in $\{\pm 1\}$. In the case of a Garside base, symmetric S-normal decompositions are (essentially) unique, and they are geodesic. Therefore, N_d^S can be identified with the number of length d symmetric S-normal sequences. In such a context, the submonoid $\widehat{S}$ of G generated by S is the family of all elements of G whose symmetric S-normal decomposition has an empty denominator, that is, of all elements that admit an S-normal decomposition. Thus, it is also natural to introduce the number

$$N_{d,+}^S = \#\{g \in \widehat{S} \mid \|g\|_S = d\},$$

which, by the above remark, is the number of S-normal sequences of length d. It then follows from the definition that a sequence $(s_1, \dots, s_d)$ is S-normal if and only if every length two subsequence is S-normal, and the basic question is therefore to investigate the numbers

$$N^S_{2,+} = \#\{(s_1, s_2) \in S^2 \mid (s_1, s_2) \text{ is } S\text{-normal}\}. \tag{2.2}$$

The classical Garside structure of B_n. In the case of the braid group B_n, two Garside structures are known. The first one, often called *classical*, involves permutations. By (2.1), mapping σ_i to the transposition $(i, i{+}1)$ induces a surjective homomorphism $\mathrm{pr}_n : B_n \to \mathfrak{S}_n$. The positive braid diagrams in which any two strands cross at most once ("simple braids") provide a set-theoretic section for pr_n, whose image is a copy S_n of $\mathfrak{S}_n$ inside B_n. The family S_n is a Garside base in B_n [9, Chapter 9], the submonoid of B_n generated by S_n being the submonoid B_n^+ generated by $\sigma_1, \ldots, \sigma_{n-1}$ [10]. The associated numbers $N^{S_n}_{d,+}$ have been investigated in [6]. In particular, writing $b_{n,d}$ for $N^{S_n}_{d,+}$, it is shown that the numbers $b_{n,2}$ are determined by the recurrence

$$b_{n,2} = \sum_{i=0}^{n-1} (-1)^{n+i+1} \binom{n}{i}^2 b_{i,2}, \tag{2.3}$$

and that the double exponential series $\sum b_{n,2} z^n/(n!)^2$ is the inverse of the Bessel function $J_0(\sqrt{z})$.

The dual Garside structure of B_n. It is known since [4] that, for every n, there exists an alternative Garside structure on B_n, corresponding to the *dual* Garside base S_n^*, whose elements are in one-to-one correspondence with the noncrossing partitions of $\{1, 2, \ldots, n\}$. The question then naturally arises of determining the corresponding numbers $N^{S_n^*}_d$ and $N^{S_n^*}_{d,+}$. Here we shall concentrate on the latter, hereafter denoted $b^*_{n,d}$, and specifically on $b^*_{n,2}$. By (2.2), we have

$$b^*_{n,d} = \#\{(s_1, \ldots, s_d) \in (S_n^*)^d \mid (s_i, s_{i+1}) \text{ is } S_n^*\text{-normal for all } i < n\}. \tag{2.4}$$

In order to compute the numbers $b^*_{n,d}$, we shall describe the correspondence between S_n^* and noncrossing partitions and interpret the S_n^*-normality condition in terms of the latter.

We recall that a set partition of the set $\{1, \ldots, n\}$ is called noncrossing if there is no quadruple $1 \leqslant i < j < k < l \leqslant n$ such that i and k belong to some block of the partition, whereas j and l belong to another block. The set of noncrossing partitions of $\{1, \ldots, n\}$, denoted by $\mathrm{NC}(n)$, is a poset for the reverse refinement order: for two partitions π and π' we have $\pi \leqslant \pi'$ if and only if each block of π is included in some block of π'. With this order, $\mathrm{NC}(n)$ is a lattice, with largest element $\mathbf{1}_n$ — the partition with only one block — and smallest element $\mathbf{0}_n$ — the partition into n singletons. We denote by $\pi_1 \vee \cdots \vee \pi_d$ the join of the partitions $\pi_1, \ldots, \pi_d$.

Definition 2.2 ([4])**.** For $1 \leqslant i < j$, put

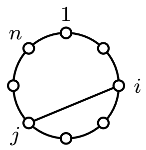
$$a_{i,j} = \sigma_i \ldots \sigma_{j-2}\, \sigma_{j-1}\, \sigma_{j-2}^{-1} \ldots \sigma_i^{-1}$$

in B_n. The *dual braid monoid* B_n^{+*} is the submonoid of B_n generated by all elements $a_{i,j}$, and S_n^* is the family of all left-divisors of Δ_n^* in B_n^{+*}, with $\Delta_n^* = \sigma_1\sigma_2 \ldots \sigma_{n-1}$.

Note that $\sigma_i = a_{i,i+1}$ holds for every i, hence B_n^{+*} includes B_n^+, a proper inclusion for $n \geqslant 3$.

Proposition 2.3 ([4])**.** *The family S_n^* is a Garside base for the group B_n.*

It is convenient to associate with the n-strand braid $a_{i,j}$ a graphical representation as the chord (i, j) in a disk with n marked vertices on the border as shown on the right.



Then the correspondence between the elements of S_n^* and noncrossing partitions stems from the following observation.

Lemma 2.4 ([1])**.** *For P a union of disjoint polygons in the n-marked disk, say $P = P_1 \cup \cdots \cup P_d$, let $a_P = a_{P_1} \ldots a_{P_d}$, with*

$$a_{P_k} = a_{i_1,i_2} a_{i_2,i_3} \ldots a_{i_{n_k-1},i_{n_k}},$$

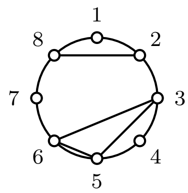
where $(i_1, \ldots, i_{n_k})$ is a clockwise enumeration of the vertices of P_k. Then:

(i) *The braid a_P only depends on P and not on the order of enumeration.*

(ii) *Mapping P to a_P establishes a bijection between unions of disjoint polygons in the n-marked disk and elements of S_n^*.*

It is standard to define a bijection between noncrossing partitions of the set $\{1, \ldots, n\}$ and unions of disjoint polygons in the n-marked disk, yielding the announced correspondence. Noncrossing partitions can be embedded into the symmetric group, using geodesics in the Cayley graph [2]: this amounts to mapping a union of polygons to the product of the cycles obtained by enumerating their vertices in clockwise order. Note that, although noncrossing partitions may be viewed as particular permutations, the associated braids need not coincide: for instance, the cycle (13), which corresponds to the partition $\{\{1, 3\}, \{2\}\}$, is associated in S_3^* with the braid $a_{1,3}$, that is, $\sigma_1\sigma_2\sigma_1^{-1}$, whereas it is associated with $\sigma_1\sigma_2\sigma_1$ in S_3.

For π a noncrossing partition of $\{1, \ldots, n\}$, we denote by a_π the associated element of S_n^*. For instance, for $\pi = \{\{1\}, \{2, 8\}, \{3, 5, 6\}, \{4\}, \{7\}\}$, we find $a_\pi = a_{2,8}a_{3,5}a_{5,6}$, as shown on the right.



Under the above correspondence, Δ_n^* corresponds to the (unique) n-gon in the n-marked disk, hence to the (noncrossing) partition $\mathbf{1}_n$ with one block. Due to (2.4), the numbers $b_{n,d}^*$ are determined by

$$b_{n,d}^* = \#\{(\pi_1, \ldots, \pi_d) \in \mathrm{NC}(n)^d \mid (a_{\pi_i}, a_{\pi_{i+1}}) \text{ is } S_n^*\text{-normal for all } i < d\}, \tag{2.5}$$

and we have to recognize when the braids associated with two partitions make an S_n^*-normal sequence.

By construction, the Garside base S_n^* is what is called bounded by the element Δ_n^*, that is, it exactly consists of the left-divisors of Δ_n^* in B_n^{+*}. In this case, the normality condition takes a simple form.

Lemma 2.5 ([7, Chapter VI]). *Assume that S is a Garside base for a group G, and S is bounded by an element Δ. Then, for s, t in S, the pair (s, t) is S-normal if and only if the only common left-divisor of ∂s and t in the monoid $\widehat{S}$ is 1, where ∂s is the element satisfying $s \cdot \partial s = \Delta$.*

We are thus left with the question of recognizing, in terms of noncrossing partitions (or of the corresponding unions of polygons), when a braid $a_{i,j}$ left-divides the braid a_π, and what the partition π' satisfying $a_{\pi'} = \partial a_\pi$ is.

Lemma 2.6 ([1]). *Assume that π lies in $\mathrm{NC}(n)$. Then:*

(i) *For $1 \leqslant i < j \leqslant n$, the braid $a_{i,j}$ left-divides a_π in B_n^{+*} if and only if the chord (i, j) lies inside the union of the convex hulls of the polygons associated with π.*

(ii) *We have $\partial a_\pi = a_{\overline{\pi}}$, where $\overline{\pi}$ is the Kreweras complement of π (as defined in [12]).*

Putting pieces together, we obtain:

Proposition 2.7. *For all n and d, the number $b_{n,d}^*$ of n-strand braids in B_n^{+*} that have length at most d with respect to the Garside base S_n^* is given by*

$$b_{n,d}^* = \#\{(\pi_1, \ldots, \pi_d) \in \mathrm{NC}(n)^d \mid \overline{\pi_i} \wedge \pi_{i+1} = \mathbf{0}_n \text{ for all } i < d\}. \tag{2.6}$$

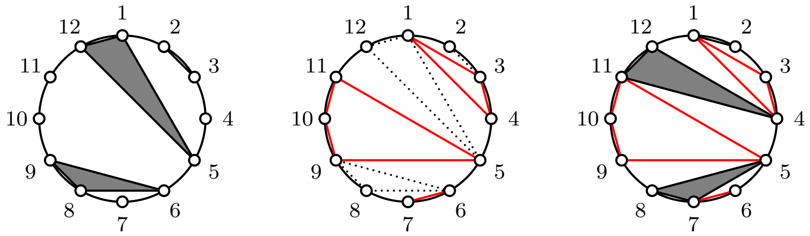


FIGURE 1. The polygons associated with a partition π and with its image under the complement map ∂ and ∂^2: the red partition is the Kreweras complement of π, and repeating the operation leads to the image of the initial partition under a rotation by $2\pi/n$, corresponding to conjugating under Δ_n^* in B_n^{+*}.

Proof. By Lemma 2.6 (i), saying that 1 is the only common left-divisor of a_π and $a_{\pi'}$ in B_n^{+*} amounts to saying that the convex hulls of the polygons associated with π and π' have no chord in common, hence are disjoint, so, in other words, that the meet of π and π' in the lattice $\mathrm{NC}(n)$ is the minimal partition $\mathbf{0}_n$. By Lemma 2.6 (ii), this condition has to be applied to $\overline{\pi_i}$ and π_{i+1} for every i. □

3. Free cumulants

In view of Proposition 2.7, we have to count sequences of noncrossing partitions satisfying lattice constraints involving adjacent entries. We shall derive partial results from a general formula expressing the free cumulants of a product of independent random variables.

3.1. **Noncrossing partitions and free cumulants.** We recall some basic facts. More details can be found in [14].

Given a sequence of indeterminates $T_1, \ldots, T_l, \ldots$ and a noncrossing partition π in $\mathrm{NC}(n)$, we define

$$T_\pi = \prod_{p\in\pi} T_{|p|}, \tag{3.1}$$

where the product ranges over all blocks p of π, and $|p|$ is the number of elements of p in $\{1, \ldots, n\}$.

Two sequences of indeterminates $M_1, \ldots, M_l, \ldots$ and $R_1, \ldots, R_l, \ldots$ are related by the moment-cumulant formula if, for every n, we have

$$M_n = \sum_{\pi\in\mathrm{NC}(n)} R_\pi. \tag{3.2}$$

It is easy to see that this equation can be inverted, and the R_l can be expressed as polynomials in the M_l. In fact, introducing the generating functions

$$M(z) = 1 + \sum_{l=1}^{\infty} z^l M_l, \qquad R(z) = 1 + \sum_{l=1}^{\infty} z^l R_l,$$

we can recast the relation (3.2) in the form

$$R(zM(z)) = M(z). \tag{3.3}$$

It follows from (3.3) and the Lagrange inversion formula that

(3.4) M_n is the coefficient of z^n in the expansion of $\dfrac{1}{n+1}R(z)^{n+1}$.

If the M_l are the moments of a probability measure (or a random variable), then the quantities R_l are called the *free cumulants* of the probability measure (or random variable).

Recall that a *semi-circular variable* is a random variable with moments $M_{2n+1} = 0$, $M_{2n} = \mathrm{Cat}_n$, and free cumulants $R_2 = 1$, $R_n = 0$ for $n \neq 2$. One can see that the square of a semi-circular variable has moments $M_n = \mathrm{Cat}_n$ and free cumulants $R_n = 1$.

Free cumulants of a product of independent random variables. Now we establish the general formula for free cumulants of products of independent random variables stated as Theorem 1.2 in the introduction.

Proof of Theorem 1.2. Let $M_n^{(i)}$ be the moments of X_i. Let us write the moment-cumulant formula for each of the variables in the product $X_1X_2\dots X_k$. As the nth moment of $X_1X_2\dots X_k$ is

$$M_n = M_n^{(1)} \dots M_n^{(k)},$$

we have

$$M_n = M_n^{(1)} \dots M_n^{(k)} = \sum_{\pi_1,\dots,\pi_k \in \mathrm{NC}(n)} \prod_{i=1}^{k} R^{(i)}_{\pi_i}.$$

Let us now decompose the sum on the right hand side according to the value of $\pi = \pi_1 \vee \dots \vee \pi_k$. Since $\pi_i \leqslant \pi$ holds, each block of π_i is included in some block of π. Let p be a block of π. The intersections $\pi_{i,p} := \pi_i \cap p$ form a partition of the set p. If we identify p with $\{1,\dots,|p|\}$ by the only increasing bijection, then the sets $\pi_{i,p}$ form a noncrossing partition of $\{1,\dots,|p|\}$. Furthermore we have

$$\pi_{1,p} \vee \dots \vee \pi_{n,p} = \mathbf{1}_{|p|},$$

whence

$$M_n = \sum_{\pi \in \mathrm{NC}(n)} \prod_{p \in \pi} \left[\sum_{\pi_{1,p} \vee \cdots \vee \pi_{n,p} = \mathbf{1}_{|p|}} \prod_i R^{(i)}_{\pi_{i,p}} \right].$$

Defining the sequence Q_n by

$$Q_n = \sum_{\pi_1 \vee \cdots \vee \pi_n = \mathbf{1}_n} \prod_i R^{(i)}_{\pi_i},$$

we obtain

$$M_n = \sum_{\pi \in \mathrm{NC}(n)} \prod_{p \in \pi} Q_{|p|} = \sum_{\pi \in \mathrm{NC}(n)} Q_\pi.$$

Then it follows from (3.2) that the quantities Q_n are the free cumulants of the sequence of moments $M_1, \ldots, M_n, \ldots$. □

The above argument is closely related to an argument in [17]. The first author would like to thank Roland Speicher for pointing out this reference.

Corollary 3.1. *The number of k-tuples $(\pi_1, \ldots, \pi_k)$ in $\mathrm{NC}(n)^k$ satisfying $\pi_1 \vee \cdots \vee \pi_k = \mathbf{1}_n$ is the nth free cumulant of the variable $X_1^2 \cdots X_k^2$, where $X_1, \ldots, X_k$ are independent centered semi-circular variables of variance* 1*.*

Proof. If we pick $R^{(i)}_m = 1$ for all i and m in Theorem 1.2, then R_n counts the k-tuples $(\pi_1, \ldots, \pi_k)$ in $\mathrm{NC}(n)^k$ satisfying $\pi_1 \vee \cdots \vee \pi_k = \mathbf{1}_n$, and the corresponding moments $M^{(i)}_n$ are the Catalan numbers (see the end of Section 3.1). □

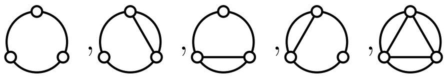
Classical and Boolean cumulants. Cumulants can also be defined using the lattice of all set partitions of $\{1, \ldots, n\}$ (this is the classical case, studied by Rota, Schützenberger, ...), or the lattice of interval partitions (these are the Boolean cumulants, see [16]). In both cases, it is immediate to check that the proof of Theorem 1.2 goes through and gives a formula for computing the corresponding cumulants of a product.

4. Back to braids

We now apply the result of Section 3 to braids.

Incidence matrices. For every n, there exists a binary relation on $\mathrm{NC}(n)$ that encodes S_n^*-normality. We introduce the associated incidence matrix.

Definition 4.1. For $n \geqslant 2$, we let A_n^* be the $\mathrm{Cat}_n \times \mathrm{Cat}_n$-matrix whose entries are indexed by pairs of noncrossing partitions, and such that $(A_n^*)_{\pi,\pi'}$ is 1 (respectively 0) if $\overline{\pi} \wedge \pi' = \mathbf{0}_n$ holds (respectively fails).

For instance, if the partitions of $\{1, 2, 3\}$ are enumerated in the order  , the matrix A_3^* is $\begin{pmatrix} 1&0&0&0&0 \\ 1&1&0&1&0 \\ 1&1&1&0&0 \\ 1&0&1&1&0 \\ 1&1&1&1&1 \end{pmatrix}$. It follows from the properties of $\mathbf{0}_n$ and $\mathbf{1}_n$ that the column of $\mathbf{0}_n$ and the row of $\mathbf{1}_n$ in A_n^* contain only ones, whereas the row of $\mathbf{0}_n$, with the exception of its $\mathbf{0}_n$-entry, and the column of $\mathbf{1}_n$, with the exception of its $\mathbf{1}_n$-entry, contain only zeroes.

Proposition 4.2. *For all n and $d \geqslant 1$, the number $b_{n,d}^*$ is the sum of all entries in the matrix $(A_n^*)^{d-1}$; in particular, $b_{n,2}^*$ is the number of positive entries in A_n^*.*

Proof. For π in $\mathrm{NC}(n)$ let $b_{n,d}^*(\pi)$ be the number of S_n^*-normal sequences of length d whose last entry is π. By Proposition 2.6, a length d sequence $(\pi_1, \dots, \pi_{d-1}, \pi)$ contributes to $b_{n,d}^*(\pi)$ if and only if $(\pi_1, \dots, \pi_{d-1})$ contributes to $b_{n,d-1}^*(\pi_{d-1})$ and (π_{d-1}, π) contributes to $b_{n,2}^*(\pi)$, that is, $(A_n^*)_{\pi_{d-1},\pi} = 1$ holds. We deduce

$$b_{n,d}^*(\pi) = \sum_{\pi' \in \mathrm{NC}(n)} b_{n,d-1}^*(\pi') \cdot (A_n^*)_{\pi',\pi}.$$

From there, an obvious induction shows that $b_{n,d}^*(\pi)$ is the πth entry in $(1, 1, \dots, 1)\,(A_n^*)^{d-1}$, and the result follows by summing over all π. $\square$

(Note that, for $d = 1$, Proposition 4.2 gives $b_{n,1}^* = \mathrm{Cat}_n$, which is indeed the sum of all entries in the size Cat_n identity-matrix.)

Using Theorem 1.2, we can now complete the proof of Theorem 1.1.

Proof of Theorem 1.1. By Proposition 4.2, $b_{n,2}^*$ is the number of positive entries in the matrix A_n^*, that is, the number of pairs (π, π') in $\mathrm{NC}(n)^2$ satisfying $\overline{\pi} \wedge \pi' = \mathbf{0}_n$. As the Kreweras complement is bijective, this number is also the number of pairs (π, π') in $\mathrm{NC}(n)^2$ satisfying $\pi \wedge \pi' = \mathbf{0}_n$. By complementation, the latter is also the number of pairs satisfying $\pi \vee \pi' = \mathbf{1}_n$. By Corollary 3.1, this number is the nth free cumulant of $X_1^2 X_2^2$, where X_1, X_2 are independent centered semi-circular variables of variance 1. The moments of the latter are the squares of the Catalan numbers, so (3.3) gives the expected result. $\square$

Further questions. It is easy to compute the numbers $b^*_{n,d}$ for small values of n and d, see Table 1.

d	1	2	3	4	5	6	7
$b^*_{1,d}$	1	1	1	1	1	1	1
$b^*_{2,d}$	2	3	4	5	6	7	8
$b^*_{3,d}$	5	15	83	177	367	749	1 515
$b^*_{4,d}$	14	99	556	2 856	14 122	68 927	334 632
$b^*_{5,d}$	42	773	11 124	147 855	1 917 046	24 672 817	
$b^*_{6,d}$	132	6 743	266 944	9 845 829	356 470 124		

TABLE 1. The number $b^*_{n,d}$ of n-strand braids of length at most d in the dual braid monoid B_n^{+*}: the first column ($d = 1$) contains the Catalan numbers, whereas the second column contains the sequence specified in Theorem 1.1, which is A168344 in [15].

It is natural to ask for a description of the columns in Table 1 beyond the first two ones. The characterization of Theorem 1.1 does not extend to $d \geqslant 3$. For instance, we have

$$b^*_{n,3} = \{(\pi_1, \pi_2, \pi_3) \in \mathrm{NC}(n)^3 \mid \overline{\pi_1} \wedge \pi_2 = \mathbf{0}_n \text{ and } \overline{\pi_2} \wedge \pi_3 = \mathbf{0}_n\}.$$

Replacement of $\wedge$ and $\mathbf{0}_n$ by $\vee$ and $\mathbf{1}_n$ is easy, but the Kreweras complement cannot be forgotten in this case.

On the other hand, attempts to describe the rows in Table 1 lead to further natural questions. By Proposition 4.2, the generating function of the numbers $b^*_{n,d}$ is rational for every n, and $b^*_{n,d}$ can be expressed in terms of the dth powers of the eigenvalues of the matrix A^*_n. For instance, we easily find $b^*_{3,d} = 6 \cdot 2^d - 2d - 5$ for every d, as well as $b^*_{3,d}(\pi) = 2^{d+1} - 1$ for $\pi \neq \mathbf{0}_n, \mathbf{1}_n$ (as above, we write $b^*_{n,d}(\pi)$ for the number of braids with a normal form finishing with π). Very little is known for $n \geqslant 4$.

It would be of interest to compute, or at at least approximate, the spectral radius of the matrix A^*_n. Here are the first values.

n	1	2	3	4	5	6	7
$\rho(A^*_n)$	1	1	2	$4.83\dots$	$12.83\dots$	$35.98\dots$	$104.87\dots$

TABLE 2. Spectral radius of the incidence matrix A^*_n

A determinant. Although we are not able to compute the eigenvalues of the matrix A_n^* explicitly, we can give a closed formula for its determinant.

Theorem 4.3. *For every n, we have*

$$|\det(A_n^*)| = \prod_{k=2}^{n} \mathrm{Cat}_{k-1}^{\binom{2n-k-1}{n-1}}.$$

Before proving this formula, we need an auxiliary result involving Möbius matrices, due to Lindström [13]. (Lindström's theorem is in fact slightly more general as it applies to semilattices.) By definition, if X is a finite poset, the associated Möbius matrix μ is the inverse of the order matrix ζ indexed by the elements of X and given by

$$\zeta(x,y) = \begin{cases} 1 & \text{for } x \leq y, \\ 0 & \text{otherwise.} \end{cases}$$

If the elements of X are ordered according to a linear extension of the partial order of X, then ζ is an upper triangular matrix, with ones on the diagonal. In particular, its determinant is 1. The same is true for μ.

Lemma 4.4 ([13])**.** *Assume that X is a lattice, and φ is a complex valued function on X. Let Φ be the matrix defined by $\Phi(x,y) = \varphi(x \wedge y)$. Then we have*

$$\det(\Phi) = \prod_{x \in X} \hat{\varphi}(x),$$

where $\hat{\varphi}$ is given by $\hat{\varphi}(x) = \sum_{y \leq x} \mu(y,x)\varphi(y)$.

Note that, under the above assumptions, we have $\varphi(x) = \sum_{y \leq x} \hat{\varphi}(y)$ by definition of the Möbius matrix.

Proof of Theorem 4.3. We apply Lemma 4.4 to $X = \mathrm{NC}(n)$ and φ defined by $\varphi(\pi) = 1$ for $\pi = \mathbf{0}_n$ and $\varphi(\pi) = 0$ for $\pi \neq \mathbf{0}_n$. We obtain $\hat{\varphi}(\pi) = \mu(\mathbf{0}_n, \pi)$. It is known that the Möbius function for the lattice of noncrossing partitions is multiplicative: if π is made of p_k blocks of size k for $k = 1, 2, \ldots$, then we have

$$\mu(\mathbf{0}_n, \pi) = \prod_k ((-1)^{k-1}\mathrm{Cat}_{k-1})^{p_k}.$$

The matrix A_n^* coincides with the corresponding matrix Φ up to a permutation of the columns (given by taking the Kreweras complement). We deduce

$$|\det(A_n^*)| = \prod_k \mathrm{Cat}_{k-1}^{a_{n,k}},$$

with

$$a_{n,k} = \sum_{\pi \in \mathrm{NC}(n)} p_k(\pi).$$

The numbers $a_{n,k}$ have been computed in [8] (see item 4 on page 218), and we have $a_{n,k} = \binom{2n-k-1}{n-1}$. We are grateful to an anonymous referee for this reference. For the convenience of the reader we give a short self-contained proof of this evaluation (our original proof has been greatly improved by a second anonymous referee, whom we also thank here).

Let us introduce the generating functions

$$f_{n,k}(y) = \sum_{\pi \in \mathrm{NC}(n)} y^{p_k(\pi)} \quad \text{and} \quad f(z,y,k) = \sum_{n \geq 0} z^n f_{n,k}(y).$$

Then we have

$$a_{n,k} = f'_{n,k}(1).$$

By the moment-cumulant formula, $f(z,y,k)$ is the moment generating function for the cumulant sequence $R_k = y, R_n = 1, n \neq k$. Since

$$R(z) = \frac{1}{1-z} + (y-1)z^k,$$

it follows from (3.4) that $f_{n,k}(y)$ is equal to the coefficient of x^n in the expansion of

$$\frac{1}{n+1}\left(\frac{1}{1-x} + (y-1)x^k\right)^{n+1}.$$

Taking the derivative with respect to y, at $y = 1$ we deduce that $f'_{n,k}(1)$ is the coefficient of x^n in $\frac{x^k}{(1-x)^n}$, which is $a_{n,k} = \binom{2n-k-1}{n-1}$, as expected. □

References

[1] D. Bessis, F. Digne, J. Michel, *Springer theory in braid groups and the Birman–Ko–Lee monoid*, Pacific J. Math **205** (2002) 287-309.

[2] P. Biane, *Some properties of crossings and partitions*, Discrete Math. **175** (1997) 41–53.

[3] J. Birman, *Braids, Links, and Mapping Class Groups*, Annals of Math. Studie, vol. 82, Princeton Univ. Press, Princeton, 1975.

[4] J. Birman, K.H. Ko & S.J. Lee, *A new approach to the word problem in the braid groups*, Adv. Math. **139** (1998) 322–353.

[5] L. Carlucci, P. Dehornoy & A. Weiermann, *Unprovability statements involving braids*, Proc. London Math. Soc. **102** (2011) 159–192.

[6] P. Dehornoy, *Combinatorics of normal sequences of braids*, J. Combin. Theory Ser. A **114** (2007) 389–409.

[7] P. Dehornoy, F. Digne, E. Godelle, D. Krammer & J. Michel, *Foundations of Garside Theory*, EMS Tracts in Mathematics, vol. 22 (2015); `http://www.math.unicaen.fr/~garside/Garside.pdf`.

[8] N. Dershowitz & S. Zaks, *Ordered trees and noncrossing partitions*, Discrete Math. **62** (1986) 215–218.
[9] D. Epstein, J. Cannon, D. Holt, S. Levy & M. Paterson & W. Thurston, *Word Processing in Groups*, Jones & Bartlett Publ. (1992).
[10] F.A. Garside, *The braid group and other groups*, Quart. J. Math. Oxford **20-78** (1969) 235–254.
[11] F. Hivert & J.-C. Novelli, J.-Y. Thibon, *Sur une conjecture de Dehornoy*, Comptes Rendus Math. **346** (2008) 375–378.
[12] G. Kreweras, *Sur les partitions non croisées d'un cycle*, Discr. Math. **1** (1972) 333–350.
[13] B. Lindström, *Determinants on semilattices*, Proc. Amer. Math. Soc. **20** (1969) 207–208.
[14] A. Nica & R. Speicher, *Lectures on the combinatorics of free probability*, London Mathematical Society Lecture Note Series, vol. 335. Cambridge University Press, Cambridge, 2006.
[15] N.J.A. Sloane, *On-Line Encyclopedia of Integer Sequences*, `http://oeis.org`.
[16] R. Speicher & R. Woroudi, *Boolean convolution*, Free probability theory, Fields Inst. Commun. **12** (1997) 267–279.
[17] R. Speicher, *A conceptual proof of a basic result in the combinatorial approach to freeness*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **3** (2000) 213–222.

Institut Gaspard-Monge, université Paris-Est Marne-la-Vallée, 5 Boulevard Descartes, Champs-sur-Marne, 77454, Marne-la-Vallée cedex 2, France
E-mail address: `biane@univ-mlv.fr`
URL: `monge.univ-mlv.fr/~biane/`

Laboratoire de Mathématiques Nicolas Oresme UMR 6139, Université de Caen, 14032 Caen, France
E-mail address: `dehornoy@math.unicaen.fr`
URL: `www.math.unicaen.fr/~dehornoy`