Gelfand-Tsetlin Polytopes

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Based on *Gelfand-Tsetlin patterns, integrally closedness and compressed polytopes*, arXiv:1405.4718

Skew Gelfand-Tsetlin patterns

A *Gelfand-Tsetlin pattern*, or GT-patterns for short, is a triangular or parallelogram arrangement of non-negative numbers,

$$\begin{array}{ccccccccccccc}
x_1^m & & x_2^m & & \cdots & & \cdots & & x_n^m & & & & \\
& \ddots & & \ddots & & & & & & \ddots & & & \\
& & x_1^2 & & x_2^2 & & \cdots & & \cdots & & x_n^2 & & \\
& & & x_1^1 & & x_2^1 & & \cdots & & \cdots & & x_n^1 &
\end{array}$$

satisfying

$$x_j^{i+1} \geq x_j^i \text{ and } x_j^i \geq x_{j+1}^{i+1}$$

for all values of i, j where the indexing is defined.

A BIJECTION

The skew shape defined by row j and $j+1$ in a GT-pattern G describes which boxes in a tableau T that have content j. In particular, if the bottom row in G is $\boldsymbol{\mu}$ and the top row is $\boldsymbol{\lambda}$, then T has shape $\boldsymbol{\lambda}/\boldsymbol{\mu}$. Here is an example of this correspondence:

$$
\begin{array}{ccccccccccc}
4 & & 3 & & 2 & & 1 & & & & \\
& 4 & & 3 & & 1 & & 1 & & & \\
& & 3 & & 3 & & 1 & & 1 & & \\
& & & 3 & & 2 & & 1 & & 0 & \\
& & & & 2 & & 1 & & 1 & & 0
\end{array}
\quad \longleftrightarrow \quad
\begin{ytableau}
\none & \none & 1 & 3 \\
\none & 1 & 2 \\
\none & 4 \\
2
\end{ytableau}
$$

$$
\begin{array}{ccccccccccc}
5 & & 4 & & 2 & & 1 & & 1 & & 0 \\
& 5 & & 3 & & 2 & & 1 & & 0 & \\
& & 3 & & 3 & & 2 & & 1 & & \\
& & & 3 & & 3 & & 1 & & & \\
& & & & 3 & & 2 & & & & \\
& & & & & 3 & & & & &
\end{array}
\quad \longleftrightarrow \quad
\begin{ytableau}
1 & 1 & 1 & 5 & 5 \\
2 & 2 & 3 & 6 \\
3 & 4 \\
4 \\
6
\end{ytableau}
$$

The concatenation operator $\boxtimes$

The $\boxtimes$ operator denotes the elementwise addition of GT-patterns. Hence, the $\boxtimes$-sum of any two Young tableaux is a new Young tableau.

Observation: Every skew semi-standard Young tableaux of shape $k\boldsymbol{\lambda}/k\boldsymbol{\mu}$ can be "decomposed" as k tableaux of shape $\boldsymbol{\lambda}/\boldsymbol{\mu}$:

$$\begin{array}{cccccccccccc} & & & & & & 1 & 1 & 1 & 1 & 1 & 5 \\ & & & 1 & 1 & 1 & 3 & 3 & 3 & & & \\ 1 & 2 & 2 & 2 & 2 & 2 & 4 & 4 & 5 & & & \\ 2 & 4 & 5 & & & & & & & & & \end{array}$$

$$=$$

$$\begin{array}{cccc} & & 1 & 1 \\ & 1 & 3 & \\ 1 & 2 & 4 & \\ 2 & & & \end{array} \boxtimes \begin{array}{cccc} & & 1 & 1 \\ & 1 & 3 & \\ 2 & 2 & 4 & \\ 4 & & & \end{array} \boxtimes \begin{array}{cccc} & & 1 & 5 \\ & 1 & 3 & \\ 2 & 2 & 5 & \\ 5 & & & \end{array}$$

Here, $\boldsymbol{\lambda}/\boldsymbol{\mu} = (4,3,3,1)/(2,1)$ and $k = 3$.

Gelfand-Tsetlin polytopes

Consider an $m \times n$ GT-pattern, with top and bottom row $\boldsymbol{\lambda}$ resp. $\boldsymbol{\mu}$. The GT-inequalities defines a convex polytope, $\mathcal{P}_{\boldsymbol{\lambda}/\boldsymbol{\mu}} \subset \mathbb{R}^{mn}$.

The integer points in $\mathcal{P}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ corresponds to the Young tableaux with shape $\boldsymbol{\lambda}/\boldsymbol{\mu}$, where the entries are in the set $1, 2, \dots, m-1$.

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The observation that a tableau of shape $k\boldsymbol{\lambda}/k\boldsymbol{\mu}$ can be represented as a $\boxtimes$-sum of k tableaux of shape $\boldsymbol{\lambda}/\boldsymbol{\mu}$ corresponds to $\mathcal{P}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ being integrally closed.

Integrally closed polytopes

A convex polytope $\mathcal{P}$ is *integrally closed* if for every positive integer k and integer point $p \in k\mathcal{P}$, there are integer points $p_j \in \mathcal{P}$ such that

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All integrally closed polytopes are *integral*, that is, all vertices of the polytope are integer points.

Hence all $\mathcal{P}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ are integral.

Gelfand-Tsetlin polytopes II

Warning: New notation!

Let $\mathcal{P}_{\boldsymbol{\lambda}/\boldsymbol{\mu},\mathbf{w}}$ be the Gelfand-Tsetlin polytope defined by the same inequalities and equalities before, *with the addition that the sum of the entries in row j resp. row $j+1$ in the pattern differ by exactly w_j.*

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\quad \longleftrightarrow \quad
\begin{array}{cccc}
 & & 1 & 3 \\
 & 1 & 2 & \\
 & 4 & & \\
2 & & &
\end{array}
$$

Here, $\mathbf{w} = (2, 2, 1, 1)$ and $\mathbf{w}$ is the *type* of the tableau; w_j counts the number of boxes with content j.

Properties of $\mathcal{P}_{\boldsymbol{\lambda}/\boldsymbol{\mu},\mathbf{w}}$

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Let $\overline{\mathbf{w}}$ be a permutation of the entries in $\mathbf{w}$. Then

- $\mathcal{P}_{\boldsymbol{\lambda}/\boldsymbol{\mu},\mathbf{w}}$ might be integral while $\mathcal{P}_{\boldsymbol{\lambda}/\boldsymbol{\mu},\overline{\mathbf{w}}}$ is non-integral.
- The number of integer points in $\mathcal{P}_{\boldsymbol{\lambda}/\boldsymbol{\mu},\mathbf{w}}$ and $\mathcal{P}_{\boldsymbol{\lambda}/\boldsymbol{\mu},\overline{\mathbf{w}}}$ are always the same.

Main conjectures

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Note $\mathcal{P}^{\diamond}_{\boldsymbol{\lambda}/\boldsymbol{\mu},\mathbf{w}} \subseteq \mathcal{P}_{\boldsymbol{\lambda}/\boldsymbol{\mu},\mathbf{w}}$.

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Conjecture

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Results so far (arXiv:1405.4718)

Theorem (A. 2014)

All $\mathcal{P}^{\diamond}_{\boldsymbol{\lambda}/\boldsymbol{\mu},\mathbf{1}}$ are *compressed*. This implies integrally closedness.

Note that the polytope $\mathcal{P}^{\diamond}_{\boldsymbol{\lambda}/\boldsymbol{\mu},\mathbf{1}}$ is always non-empty and that integer points in this polytope correspond to *standard* Young tableaux of shape $\boldsymbol{\lambda}/\boldsymbol{\mu}$.
An integral polytope is compressed if all *pulling triangulations* are *unimodular*.

Corollary: All integral points in $\mathcal{P}^{\diamond}_{\boldsymbol{\lambda}/\boldsymbol{\mu},\mathbf{1}}$ are vertices.

Special cases

Proposition (A. 2014)

If $\boldsymbol{\lambda}/\boldsymbol{\mu}$ is a skew Young diagram without any 2×2-arrangement of boxes, then $\mathcal{P}_{\boldsymbol{\lambda}/\boldsymbol{\mu},\mathbf{w}}$ is integral and integrally closed.

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$\mathcal{P}_{\boldsymbol{\lambda},\mathbf{1}}$ is non-integral whenever $\lambda_1 \geq \lambda_2 > \lambda_3 \geq 1$.

Refinement results

Partial order $<_{\text{ref}}$ with respect to composition refinements of $\mathbf{w}$:

Proposition (A. 2014)

Let $\mathbf{w}' <_{\text{ref}} \mathbf{w}$ and let $P = \mathcal{P}_{\boldsymbol{\lambda}/\boldsymbol{\mu},\mathbf{w}} \subset \mathbb{R}^d$ and $P' = \mathcal{P}_{\boldsymbol{\lambda}/\boldsymbol{\mu},\mathbf{w}'} \subset \mathbb{R}^{d'}$. Then

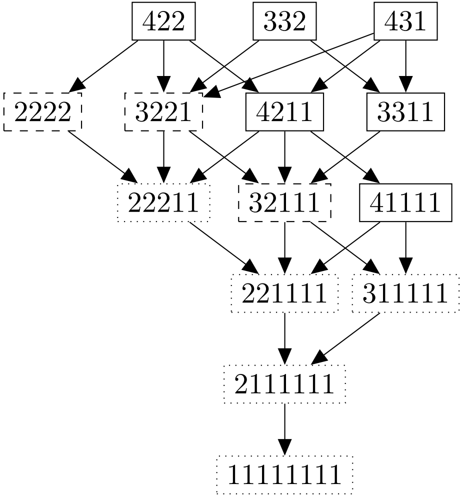
1. $|P' \cap \mathbb{Z}^{d'}|$ is greater or equal to $|P \cap \mathbb{Z}^d|$. (Trivial)
2. If P' is empty, then P is empty. (Trivial)
3. If P' is integral, then P is integral.
4. If P' is integrally closed, then so is P.

Conjecture

5. If P' is a unimodular simplex, then P is a unimodular simplex.

(Part of the) general picture

Non-skew case $\boldsymbol{\lambda} = 431$, and $\mathbf{w}$ in the boxes.



Further questions

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4. Hint about Kronecker coefficients?

The end

Thank you for your time