Stability of Plethysm Coefficients

Laura Colmenarejo
Joint with E. Briand

University of Seville

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Any (finite-dimensional, complex, analytic) linear representation V of $GL_n(\mathbb{C})$ decomposes as:

$$V \approx \bigoplus_{\lambda} m_\lambda S_\lambda(\mathbb{C}^n).$$

where

- m_λ are nonnegative integers
- $S_\lambda(\mathbb{C}^n)$ are irreducible representations indexed by the integer partitions λ of length at most n

Constructions

- $$\text{Tensor product} \longleftrightarrow \text{Littlewood-Richardson coefficients } c_{\mu\nu}^{\lambda}$$

  $$S_\mu(\mathbb{C}^k) \otimes S_\nu(\mathbb{C}^k) = \bigoplus c_{\mu\nu}^{\lambda} S_\lambda(\mathbb{C}^k)$$

  They count Littlewood-Richardson tableaux.

- $$\text{Restrictions of } GL_{mn}(\mathbb{C}) \text{ to } GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$$
  $$\updownarrow$$
  $$\text{Kronecker coefficients } g_{\lambda}^{\mu\nu}$$

  $$S_\lambda(\mathbb{C}^{mn}) = \bigoplus g_{\lambda}^{\mu\nu} S_\mu(\mathbb{C}^m) \otimes S_\nu(\mathbb{C}^n)$$

  Murnaghan and Littlewood observed some properties of stability.

Stability of Kronecker Coefficients

- Murnaghan (1938, 1955) proved that given any three partitions α, β, γ, the general term of Kronecker coefficients $g_{\alpha+(n),\beta+(n)}^{\gamma+(n)}$ is eventually constant.
- Also, Murnaghan proved that the sequence is weakly increasing as a function of n.
- Few years ago, E. Briand, R. Orellana and M. Rosas improved Murnaghan's bounds.
- Nowadays, J. R. Stembridge has given a more general result: Conditions on α, β, γ such that for all triples λ, μ, ν the sequences $g_{\lambda+n\alpha,\mu+n\beta}^{\nu+n\gamma}$ converge as $n \longrightarrow \infty$.

$$\text{Plethysm} \longleftrightarrow \text{Plethysm coefficients } a_{\pi\nu}^{\lambda}$$

Apply the Schur functor S_π to an irreducible representation S_ν

$$S_\pi\left(S_\nu(\mathbb{C}^k)\right) = \bigoplus a_{\pi\nu}^{\lambda} S_\lambda(\mathbb{C}^k)$$

TRANSLATION INTO SYMMETRIC FUNCTIONS

$$s_\pi[s_\nu] = \sum a_{\pi\nu}^{\lambda} s_\lambda$$

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Previous Results

P0 $\langle s_\lambda, s_n[s_m]\rangle \le \langle s_{\lambda+n}, s_n[s_{m+1}]\rangle$
Foulkes' conjecture
Weintraub: formulas for different cases of λ.

P1 $\left\langle s_\lambda[s_{\mu+(p)}], s_{\nu+(q)}\right\rangle$
Thibon and Carré: stability using Vertex Operators

P2 $\langle s_\pi[s_{\mu+n\lambda}], s_{\nu+np\lambda}\rangle$
Brion: stability and increase
Foulkes: $\lambda = (1)$ and $\ell(\mu) = \ell(\pi) = 1$
Weintraub: stacionary behaviour

Q1 $\left\langle s_{\lambda+(p)}[s_\mu], s_{\nu+(q)} \right\rangle$
Thibon and Carré : stability with Vertex Operators
Weintraub: stability

R1 $\langle s_{\pi+n}[s_\lambda], s_{\nu+n\lambda} \rangle$
Brion: stability and increase

We can see our plethysm coefficients as

$$a_{\mu\nu}^{\lambda} = \langle s_\mu[s_\nu], s_\lambda \rangle$$

We can write explicitly s_λ as a sum over the permutations σ in the symmetric group $\mathfrak{S}_N$

$$s_\lambda = \sum_{\sigma \in \mathfrak{S}_N} \varepsilon(\sigma) h_{\lambda+\omega(\sigma)}$$

where $\omega(\sigma)_j = \sigma(j) - j$ for all j between 1 and N.

Lemma

Let N and N' be positive integers. Let λ, μ and ν be partitions, such that μ has length at most N and λ has length at most N'. Then

$$a^{\lambda}_{\mu,\,\nu} = \sum_{\sigma,\tau} \varepsilon(\sigma)\varepsilon(\tau) \left\langle h_{\mu+\omega(\sigma)}[s_\nu] \mid h_{\lambda+\omega(\tau)} \right\rangle$$

where the sum is carried over all permutations $\sigma \in \mathfrak{S}_N$ and $\tau \in \mathfrak{S}_{N'}$.

For any partition ν and any finite sequences μ and λ of integers we set:

$$b^{\lambda}_{\mu,\,\nu} = \langle h_\mu[s_\nu] \mid h_\lambda \rangle \,.$$

They count the integer points in a polytope $Q(\mu, \nu, \lambda)$.

Proposition

The coefficient $b^{\lambda}_{\mu\nu}$ *is the cardinal of the set* $Q(\mu;\nu;\lambda;N)$ *of matrices* $\mathcal{M} = (m_{i,T})$ *with nonnegative integer entries whose rows are indexed by the integers* i *between* 1 *and* N *and whose columns are indexed by the semi-standard Young tableaux of shape* ν *with entries between* 1 *and* N, $T \in t(\nu;N)$, *such that:*

1. *The sum of the entries in row* i *of* $\mathcal{M}$ *is* μ_i.
2. *The sum of the entries in column* j *of* $\mathcal{M} \cdot \mathcal{P}_{\nu N}$ *is* λ_j, *where* $\mathcal{P}_{\nu N}$ *is the matrix of weights.*

Example: $\nu = (2)$, $\mu = (\mu_1, \mu_2)$, $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, $N = 3$

- Semi-standard Young tableaux of shape ν with entries between 1 and N

$$T_1 = \boxed{1\,|\,1} \quad T_2 = \boxed{1\,|\,2} \quad T_3 = \boxed{1\,|\,3}$$
$$T_4 = \boxed{2\,|\,2} \quad T_5 = \boxed{2\,|\,3} \quad T_6 = \boxed{3\,|\,3}$$

- Matrix of weights $\mathcal{P}_{\nu N}$ and matrix $\mathcal{M}$

$$\mathcal{M} = \begin{pmatrix} m_{1,T_1} & m_{1,T_2} & \dots & m_{1,T_6} \\ m_{2,T_1} & m_{2,T_2} & \dots & m_{2,T_6} \\ m_{3,T_1} & m_{3,T_2} & \dots & m_{3,T_6} \end{pmatrix} \qquad \mathcal{P}_{\nu N} = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 \end{pmatrix}^t$$

- Conditions

$$\underbrace{\begin{array}{c} \sum_j m_{1,T_j} = \mu_1 \\ \sum_j m_{2,T_j} = \mu_2 \\ \sum_j m_{3,T_j} = 0 \\ m_{i,T_j} \geq 0 \end{array}}_{\mathcal{M}} \qquad \underbrace{\begin{array}{rcl} \sum_i 2m_{i,T_1} + m_{i,T_2} + m_{i,T_3} & = & \lambda_1 \\ \sum_i m_{i,T_2} + 2m_{i,T_4} + m_{i,T_5} & = & \lambda_2 \\ \sum_i m_{i,T_3} + m_{i,T_5} + 2m_{i,T_6} & = & \lambda_3 \end{array}}_{\mathcal{M}\mathcal{P}_{\nu N}}$$

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Brion's result: $\langle s_{\pi+n}[s_\lambda], s_{\mu+n\lambda}\rangle$

- Stability of $a^{\mu+n\lambda}_{\pi+n,\lambda} \leftrightarrow$ Stability of $b^{\mu+n\lambda}_{\pi+n,\lambda}$
- $b^{\mu+n\lambda}_{\pi+n,\lambda} = \#Q(\pi+n,\lambda,\mu+n\lambda;N) = \#E(n)$
- $$\begin{array}{cccc} \iota(n): & E(n) & \hookrightarrow & E(n+1) \\ & \mathcal{M} & \longmapsto & \mathcal{M}' \end{array}$$

  where $\mathcal{M}'$ has $m'_{1,T_0} = m_{1,T_0}+1$.
- $\iota(n)$ is well defined and injective.
- $\iota(n)$ is surjective for $n >> 0$.

Let $\mathcal{M}' \in E(n+1)$. Let T_0 be the following tableau

$$\begin{array}{|c|c|c|c|}
\cline{1-1}
\cdots \\
\cline{1-2}
3 & \cdots \\
\cline{1-3}
2 & 2 & \cdots \\
\hline
1 & 1 & 1 & \cdots \\
\hline
\end{array}$$

Denote $\|\alpha\|$ for $\sum_{k=1}^{N} \sum_{j=1}^{k} \alpha_j$ and p_T for the row T of $\mathcal{P}_{\lambda;N}$.
Then,

$$\begin{cases} \|p_T\| \leq \|\lambda\| - 1 & \text{for } T \neq T_0 \\ \|p_T\| = \|\lambda\| & \text{for } T = T_0 \end{cases}$$

Using the row conditions on $\mathcal{M}'$, a few more elementary operations lead to

$$m'_{1,T_0} \geq \|\mu\| + \pi_1 - |\pi| \cdot \|\lambda\| + (n+1)$$

that proves that $m'_{1,T_0} > 0$ as soon as $n \geq |\pi| \cdot \|\lambda\| - \|\mu\| - \pi_1$

We need to prove that

$$\begin{gathered} Q_p = Q(\lambda; \mu + (p); \nu + (q); N) \\ \updownarrow \\ Q_{p+1} = Q(\lambda; \mu + (p+1); \nu + (q + |\mu|); N) \end{gathered}$$

FIRST STEP

$$\begin{array}{cccc} \varphi_p : & t(\mu + (p); N) & \longrightarrow & t(\mu + (p+1); N) \\ & T & \longrightarrow & \overline{T} \end{array}$$

where $\overline{T}$ is obtained from T adding one box in the first row and putting a one in the first box of the first row.

$$\begin{array}{cccc} \varphi_p : & t(\mu + (p); N) & \longrightarrow & t(\mu + (p+1); N) \\ & T & \longrightarrow & \overline{T} \end{array}$$

EXAMPLE: $\mu = (2) \longrightarrow \mu = (3)$

$\boxed{\mu = (2)}$

1 1	1 2	1 3	2 2	2 3	3 3

$\boxed{\mu = (3)}$

1 1 1	1 1 2	1 1 3	1 2 2	1 2 3	1 3 3

But there are more tableaux in case $\mu = (3)$

2 2 2	2 2 3	2 3 3	3 3 3

So we can separate the tableaux of $t(\mu + (p+1); N)$ with preimage in $t(\mu + p; N)$ and the new ones.

SECOND STEP

$$
\begin{array}{rccc}
\psi: & Q(\lambda;\mu+(p);\nu+(q);N) & \longrightarrow & Q(\lambda;\mu+(p+1);\nu+(q+|\mu|);N) \\
& \mathcal{M}_p & \longrightarrow & \mathcal{M}_{p+1} = \left(\mathcal{M}_p|\overline{0}\right)
\end{array}
$$

- ψ is well defined.
- ψ is injective.
- ψ is surjective for p big enough.

IDEA:

- Separate the tableaux of $t(\mu + (p+1); N)$ with preimage in $t(\mu + p; N)$ and the new ones.
- Estimate the number of ones that we can count on each type of tableaux.
- Apply these estimations on the first row condition for $\mathcal{M} \in Q_{p+1}$.

Using that idea, we prove that when $p > |\lambda| \cdot \mu_1 + \mu_2 - \nu_1 - \mu_1 - 1$, $\mathcal{M}$ is of the form $\left(\mathcal{M}_p | \overline{0}\right)$, with $\mathcal{M}_p \in Q_p$.

Thank you for coming!