Shifted Jack polynomials and multirectangular coordinates

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Symmetric functions

= "polynomials" in infinitely many variables $x_1, x_2, x_3, \dots$
that are invariant by permuting indices

- Augmented monomial basis:

$$\tilde{m}_\lambda = \sum_{\substack{i_1,\dots,i_\ell \geq 1 \\ \text{distinct}}} x_{i_1}^{\lambda_1} \cdots x_{i_\ell}^{\lambda_\ell}$$

Example: $\tilde{m}_{(2,1,1)} = 2x_1^2 x_2 x_3 + 2x_1 x_2^2 x_3 + 2x_1 x_2 x_3^2 + 2x_1^2 x_2 x_4 + \dots$

- Power-sum basis:

$$p_r = x_1^r + x_2^r + \dots, \quad p_\lambda = p_{\lambda_1} \cdots p_{\lambda_\ell}$$

Schur functions

(s_λ) is another basis of the symmetric function ring.

Several equivalent definitions:

- $s_\lambda = \sum_T x^T$, sum over semi standard Young tableaux ;
- orthogonal basis (for *Hall scalar product*) + triangular over (augmented) monomial basis ;
- with determinants. . .

-> Encode irreducible characters of symmetric and general linear groups.

Jack polynomials

Deformation of Schur functions with a positive real parameter α.

$$(J_\lambda^{(\alpha)}) \text{ basis,} \qquad J_\lambda^{(1)} = \mathrm{cst}_\lambda \cdot s_\lambda$$

Several equivalent definitions:

- $J_\lambda = \sum_T \psi_T(\alpha) x^T$, sum over semi standard Young tableaux ;
- orthogonal basis (for a deformation of *Hall scalar product*) + triangular over (augmented) monomial basis.

For $\alpha = 1/2, 2$, they also have a representation-theoretical interpretation (in terms of Gelfand pairs) but not in general !

Polynomiality in α with non-negative coefficients

Both definitions involve rational functions in α. Nevertheless, ...

Macdonald-Stanley conjecture ($\sim$ 90)

The coefficients of Jack polynomials in augmented monomial basis are polynomials in α with non-negative integer coefficients.

Notation: $[\tilde{m}_\tau]J_\lambda$.

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Knop-Sahi theorem (97)

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Notation: $[\tilde{m}_\tau]J_\lambda$.

KS give a combinatorial interpretation of $[\tilde{m}_\tau]J_\lambda$ as a weighted enumeration of *admissible* tableaux.

A function on the set of all Young diagrams

Definition

Let μ be a partition of k (without part equal to 1). Define

$$\mathrm{Ch}_\mu^{(\alpha)}(\lambda) = \begin{cases} \binom{n-k+m_1(\mu)}{m_1(\mu)} \cdot z_\mu \cdot [p_{\mu 1^{n-k}}] J_\lambda^{(\alpha)} & \text{if } n = |\lambda| \geq k; \\ 0 & \text{otherwise.} \end{cases}$$

$\mathrm{Ch}_\mu^{(\alpha)}$ is a function of all Young diagrams.

z_μ: standard explicit numerical factor.

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$\mathrm{Ch}_\mu^{(\alpha)}$ is a function of all Young diagrams.

Specialization: if $|\mu| < |\lambda|$,

$$\mathrm{Ch}_\mu^{(1)}(\lambda) = \frac{|\lambda|!}{(|\lambda| - |\mu|)!} \cdot \frac{\chi^\lambda_{\mu 1^{n-k}}}{\dim(V_\lambda)}.$$

Introduced by S. Kerov, G. Olshanski in the case $\alpha = 1$ (to study random diagrams with Plancherel measure), by M. Lassalle in the general case.

A function on the set of all Young diagrams

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Proposition (Kerov/Olshanski for $\alpha = 1$, Lassalle in general)

For any r, the application

$$(\lambda_1, \dots, \lambda_r) \mapsto \mathrm{Ch}_\mu^{(\alpha)}\left((\lambda_1, \dots, \lambda_r)\right)$$

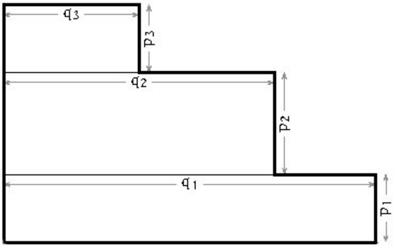
is a polynomial in λ_1, ..., λ_r. Besides, it is symmetric in $\lambda_1 - 1/\alpha$, ..., $\lambda_r - r/\alpha$.

In other words, $\mathrm{Ch}_\mu^{(\alpha)}$ is a shifted symmetric function.

Multirectangular coordinates (R. Stanley)

Consider two lists $\mathbf{p}$ and $\mathbf{q}$ of positive integers of the same size, with $\mathbf{q}$ non-decreasing.
We associate to them the partition

$$\lambda(\mathbf{p},\mathbf{q}) = \big(\underbrace{q_1,\dots,q_1}_{p_1 \text{ times}},\underbrace{q_2,\dots,q_2}_{p_2 \text{ times}},\dots\big).$$



Young diagram of $\lambda(\mathbf{p},\mathbf{q})$

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Proposition

Let μ be a partition of k. $\mathrm{Ch}_\mu^{(\alpha)}(\lambda(\mathbf{p},\mathbf{q}))$ is a polynomial in

$$p_1, p_2, \dots, q_1, q_2, \dots, \alpha$$

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Conjecture (M. Lassalle)

Let μ be a partition of k. $(-1)^k \operatorname{Ch}_\mu^{(\alpha)}(\lambda(\mathbf{p},\mathbf{q}))$ is a polynomial in

$$p_1, p_2, \dots, -q_1, -q_2, \dots, \alpha - 1$$

with non-negative integer coefficients.

Still open...

Link between the two questions ?

Knop-Sahi theorem and Lassalle conjecture do not seem related.

Two (main) differences:

- monomial coefficients *vs* power-sum coefficients ;
- look at some $J_\lambda^{(\alpha)}$ *vs* seen as a function of λ.

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- monomial coefficients *vs* power-sum coefficients ;
- look at some $J_\lambda^{(\alpha)}$ *vs* seen as a function of λ.

Idea: look at monomial coefficients as functions on Young diagrams.

Monomial coefficients as shifted symmetric functions

Definition

Let μ be a partition of k (without part equal to 1). Define

$$\mathrm{Ko}_\mu^{(\alpha)}(\lambda) = \begin{cases} \binom{n-k+m_1(\mu)}{m_1(\mu)} \cdot z_\mu \cdot [\tilde{m}_{\mu 1^{n-k}}] J_\lambda^{(\alpha)} & \text{if } n = |\lambda| \geq k; \\ 0 & \text{otherwise.} \end{cases}$$

Proposition

$\mathrm{Ko}_\mu^{(\alpha)}$ is a shifted symmetric function.

Proof: Uses $\mathrm{Ko}_\mu^{(\alpha)} = \sum_{\nu \vdash k} L_{\mu,\nu} \, \mathrm{Ch}_\nu^{(\alpha)}$ and Lassalle proposition.

($L_{\mu,\nu}$ is defined by $p_\nu = \sum_{\mu \vdash k} L_{\mu,\nu} \tilde{m}_\mu$).

A new conjecture

Proposition

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Conjecture (F., Alexandersson)

In the falling factorial basis in $\mathbf{p}$ and $\mathbf{q}$, $\mathrm{Ko}_\mu^{(\alpha)}(\mathbf{p} \times \mathbf{q})$ has non-negative integer coefficients.

falling factorial: $(n)_k := n(n-1)\dots(n-k+1)$.

falling factorial basis: $\left((p_1)_{i_1}(p_2)_{i_2}\dots(q_1)_{j_1}(q_2)_{j_2}\dots\alpha^k\right)$.

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It is stronger than positivity in Knop-Sahi theorem (and does not follow from their combinatorial interpretation) !

Another conjecture

Another interesting family of shifted symmetric function

Shifted Jack polynomials (Okounkov, Olshanski, 97)

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In the falling factorial basis in $\mathbf{p}$ and $\mathbf{q}$, $\alpha^{\ell(\mu)} J^{\sharp(\alpha)}_{\mu}(\mathbf{p} \times \mathbf{q})$ has non-negative integer coefficients.

For a fixed α, FF-positivity of $\alpha^{\ell(\mu)} J^{\sharp(\alpha)}_{\mu}(\mathbf{p} \times \mathbf{q})$ implies FF-positivity of $\mathrm{Ko}^{(\alpha)}_{\mu}(\mathbf{p} \times \mathbf{q})$.

Case $\alpha = 1$ (1/2)

For $\alpha = 1$, there is a combinatorial formula for $\mathrm{Ch}^{(1)}_\mu$:

Theorem (F. 2007; F., Śniady 2008 ; conj. by Stanley 2006)

Let μ a partition of k. Fix a permutation π in S_k of type μ. Then

$$(-1)^k \,\mathrm{Ch}_\mu(\mathbf{p} \times \mathbf{q}) = \sum_{\substack{\sigma,\tau \in S_k \\ \sigma\tau=\pi}} N_{\sigma,\tau}(\mathbf{p}, -\mathbf{q}).$$

$N_{\sigma,\tau}$: combinatorial polynomial with non-negative integer coefficients.
$\Rightarrow$ Lassalle conjecture holds for $\alpha = 1$.

Similar formula for $\alpha = 2$: replace permutations by pairings of $[2n]$ (F., Śniady, 2011).

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Proposition

Fix a set-partition Π whose block size are given by μ.

$$(-1)^k \mathrm{Ko}^{(1)}_\mu(\mathbf{p} \times \mathbf{q}) = \sum_{\substack{\sigma,\tau \in S_k \\ \sigma\tau\in S_\Pi}} N_{\sigma,\tau}(\mathbf{p}, -\mathbf{q}).$$

$$(-1)^k s^\sharp_{\lambda}{}_\mu(\mathbf{p} \times \mathbf{q}) = \sum_{\sigma,\tau \in S_k} \chi^\mu(\sigma\,\tau)\, N_{\sigma,\tau}(\mathbf{p}, -\mathbf{q})$$

Case $\alpha = 1$ (2/2)

…use explicit expression of $N_{\sigma,\tau}(\mathbf{p},\mathbf{q})$ + sum manipulations …
It is enough to prove

Question 1

For any three set partitions T, U and Π of the same set,

$$\sum_{\substack{\sigma \in S_T, \tau \in S_U \\ \sigma\tau \in S_\Pi}} \varepsilon(\tau) \geq 0.$$

Question 2

For any two set partitions T, U of $[n]$ and integer partition μ of n,

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Proof: representation theory + group algebra manipulation.

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Conclusion: Our second (and hence both) conjecture(s) hold(s) for $\alpha = 1$.

$\mathrm{Ko}_{(k)}$ is FF non-negative.

Observation: $$(-1)^k \, \mathrm{Ko}^{(1)}_{(k)}(\mathbf{p} \times \mathbf{q}) = \sum_{\substack{\sigma,\tau \in S_k \\ \text{no restriction}}} N_{\sigma,\tau}(\mathbf{p}, -\mathbf{q}).$$

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Proposition

For a general α,

$$(-1)^k \, \mathrm{Ko}^{(\alpha)}_{(k)}(\mathbf{p} \times \mathbf{q}) = \sum_{\sigma,\tau \in S_k} \alpha^{k - \#(\mathit{LR\text{-}max}(\sigma))} N_{\sigma,\tau}(\mathbf{p}, -\mathbf{q})$$

Proof: KS combinatorial interpretation + a new bijection.

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Proof: KS combinatorial interpretation + a new bijection.

Corollary (special case of our first conjecture)

The coefficients of $\mathrm{Ko}^{(\alpha)}_{(k)}$ in the falling factorial basis are non-negative.

Conclusion

A bridge between KS theorem and Lassalle's conjecture:

- Our conjecture involves shifted symmetric functions and multirectangular coordinates and implies KS theorem ;
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Other partial results?

- $\alpha = 2$ works similarly as $\alpha = 1$ with a bit more work ;
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An extension?

- What about (shifted) Macdonald polynomials and multirectangular coordinates?