New Foundations of Combinatorial Theory

Part 2. What is a combinatorial interpretation?

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What is a combinatorial interpretation?

You have: a combinatorial sequence $\{a_n\}$, such that $a_n \in \mathbb{N}$.

You want: a set of objects a_n enumerates described algorithmically (a *formula*, see Lecture 1).

Examples: Permutations, partitions, words, trees, tableaux, lattice walks, etc.

Note: No formal definition is usually used in Combinatorics context.

Open problems on combinatorial interpretations

Problem 1. *Super Catalan numbers* [Gessel, 1992] :

$$C(m,n) \,=\, \frac{(2m)!\,(2n)!}{2m!n!\,(m+n)!}\,.$$

These are Catalan numbers for $m = 1$.

For $m = 2$, see Gessel-Xin, Fusy-Schaeffer-Poulalhon, etc.

Problem 2. *Kronecker coefficients* $g(\lambda, \mu, \nu)$ [Murnaghan, 1938] :

$$\chi^\lambda \otimes \chi^\mu \,=\, \sum_{\nu \vdash n} g(\lambda,\mu,\nu)\,\chi^\nu\,, \quad \text{where} \ \ \lambda, \mu \vdash n, \tag{1}$$

where χ^α denotes the irreducible character of S_n indexed by $\alpha \vdash n$.

Known for two-row partitions, hooks, some assorted examples (see Remmel, Rosas, Vallejo, Ballantine-Orellana, Briand-Orellana-Rosas, Blasiak, P.-Panova, etc.)

Unimodality problems

Theorem [P.-Panova, Vallejo] Let

$$a_k(\lambda, \mu) \,=\, \sum_{\alpha \vdash k,\, \beta \vdash n-k} c^{\lambda}_{\alpha\beta}\, c^{\mu}_{\alpha\beta}\,,$$

where $c^{\nu}_{\pi\theta}$ are the *Littlewood–Richardson coefficients.*

For any two partitions $\lambda, \mu \vdash n$, the sequence

$$a_0(\lambda, \mu), \ldots, a_n(\lambda, \mu)$$

is symmetric and unimodal.

Problem 2′ Find a combinatorial interpretation for

$$a_k(\lambda, \mu) \,-\, a_{k-1}(\lambda, \mu) \,=\, g\big(\lambda, \mu, (n-k, k)\big).$$

Restricted partitions: $\lambda = \mu = (m^\ell)$. Then $a_k(\lambda,\mu) \, = \, p_k(\ell,m)$, where

$$\binom{m+\ell}{m}_q \; = \; \frac{(q^{m+1}-1)\cdots(q^{m+\ell}-1)}{(q-1)\cdots(q^\ell-1)} \; = \; \sum_{k=0}^{\ell m} p_k(\ell,m)\, q^k .$$

In this case we DO have a combinatorial interpretation via KOH [O'Hara, 1990].

Formalizing this is due to P.-Panova (2014+), see also [Zanello] and [Dhand].

Theorem (P.-Panova, 2014)

There is a universal constant $A > 0$, such that for all $m \ge \ell \ge 8$ and $2 \le k \le \ell m/2$, we have:

$$p_k(\ell,m) - p_{k-1}(\ell,m) \; > \; A\, \frac{2^{\sqrt{s}}}{s^{9/4}}\,, \qquad \text{where} \quad s = \min\{2k, \ell^2\}.$$

$A = 0.00449$ works. The proof uses Almkvist's results on asymptotics of partitions

\+ Manivel's extension of the semigroup property of Kronecker coefficients.

Back to combinatorial interpretations

Question: What does that mean if there is NO combinatorial interpretation?

Can we *formally state* that? Prove in some cases? No such results are known.

Conjecture 1. (Mulmuley, 2007; modified by P.)
Kronecker coefficients $g(\lambda, \mu, \nu)$ count the number of integer points in a polytope $P(\lambda, \mu, \nu) \subset \mathbb{R}^d$ where $d = O(n^c)$ and the constraints are linear in (λ, μ, ν).

Kronecker coefficients are quasi-polynomial, so no contradiction here so far.

Conjecture 2. (Mulmulley, 2007)
Decision problem whether $g(\lambda, \mu, \nu) > 0$ is in P.

New question:

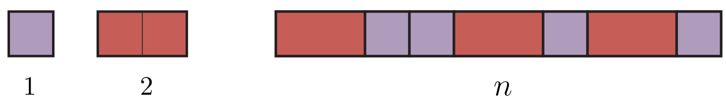
Can we perhaps expand the set of possible *combinatorial interpretations* to include objects from discrete geometry?

Back to tilings of of $[1 \times n]$ rectangles

Fix a finite set $T = \{\tau_1, \ldots, \tau_k\}$ of *rational* tiles of height 1.

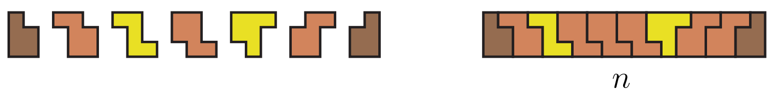
Let $a_n = a_n(T)$ the number of tilings of $[1 \times n]$ with T.

Transfer-matrix Method: $\mathcal{A}_T(t) = \sum_n a_n t^n = P(t)/Q(t)$, where $P, Q \in \mathbb{Z}[t]$.



$a_n = F_n$

$\mathcal{A}(t) = \frac{1}{1-t-t^2}$



$a_n = \binom{n-2}{2}$

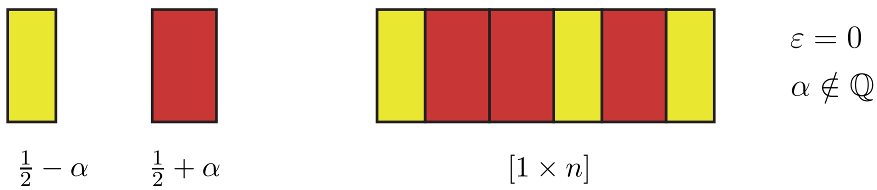
$\mathcal{A}(t) = \frac{t^4}{(1-t)^3}$

Irrational Tilings of $[1 \times (n+\varepsilon)]$ rectangles

Fix $\varepsilon \geq 0$ and a finite set $T = \{\tau_1, \ldots, \tau_k\}$ of *irrational tiles* of height 1.

Let $a_n = a_n(T, \varepsilon)$ the number of tilings of $[1 \times (n+\varepsilon)]$ with T.

Observe: we can get *algebraic* g.f.'s $\mathcal{A}_T(t)$.



Here $a_n = \binom{2n}{n}$, $\mathcal{A}(t) = \frac{1}{\sqrt{1-4t}}$.

$\mathbb{N}$-Rational Functions $\mathcal{R}_1$

Definition: Let $\mathcal{R}_1$ be the smallest set of functions $F(x)$ which satisfies

(1) $0, x \in \mathcal{R}_1$,

(2) $F, G \in \mathcal{R}_1 \implies F + G, F \cdot G \in \mathcal{R}_k$,

(3) $F \in \mathcal{R}_1,\ F(0) = 0 \implies 1/(1 - F) \in \mathcal{R}_1$.

Note that all $F \in \mathcal{R}_1$ satisfy: $F \in \mathbb{N}[[x]]$, and $F = P/Q$, for some $P, Q \in \mathbb{Z}[x]$.

For example,

$$\frac{1}{1 - x - x^2} \quad \text{and} \quad \frac{x^3}{(1-x)^4} \ \in \mathcal{R}_1 \, .$$

Theorem [Schützenberger + folklore]
For every *rational* T, we have $\mathcal{A}_T(x) \in \mathcal{R}_1$.
Conversely, for every $F(x) \in \mathcal{R}_1$ there is a *rational* T s.t. $F(x) = \mathcal{A}_T(x)$.

$\mathbb{N}$-rational functions of one variable:

Word of caution: $\mathcal{R}_1$ is already quite complicated.
The following example is from [Gessel, 2003].

For example, take the following $F, G \in \mathbb{N}[[t]]$:

$$F(t) \;=\; \frac{t+5t^2}{1+t-5t^2-125t^3}\,, \qquad G(t) \;=\; \frac{1+t}{1+t-2t^2-3t^3}\,.$$

Then $F \notin \mathcal{R}_1$ and $G \in \mathcal{R}_1$; neither of these are obvious.

The proof follows from results in [Berstel, 1971] and [Soittola, 1976],
see also [Katayama–Okamoto–Enomoto, 1978].

Diagonals of Rational Functions

Let $G \in \mathbb{Z}[[x_1, \ldots, x_k]]$. A *diagonal* is a g.f. $\mathcal{B}(t) = \sum_n b_n t^n$, where

$$b_n = \left[x_1^n, \ldots, x_k^n\right] G(x_1, \ldots, x_k).$$

Theorem: *Every $\mathcal{A}_T(t) \in \mathcal{F}$ is a diagonal of a rational function P/Q, for some polynomials $P, Q \in \mathbb{Z}[x_1, \ldots, x_k]$.*

For example,

$$\binom{2n}{n} = [x^n y^n] \frac{1}{1 - x - y}.$$

Proof idea: Say, $\tau_i = [1 \times \alpha_i]$, $\alpha_i \in \mathbb{R}$. Let $V = \mathbb{Q}\langle \alpha_1, \ldots, \alpha_k \rangle$, $d = \dim(V)$.
We have natural maps $\varepsilon \mapsto (c_1, \ldots, c_d)$, $\alpha_i \mapsto v_i \in \mathbb{Z}^d \subset V$.
Interpret irrational tilings as walks $O \to (n + c_1, \ldots, n + c_d)$ with steps $\{v_1, \ldots, v_k\}$.

Properties of Diagonals of Rational Functions

(1) must be *D-finite*, see [Stanley, 1980], [Gessel, 1981].

(2) when $k = 2$, must be *algebraic*, and

(2′) every algebraic $\mathcal{B}(t)$ is a diagonal of $P(x,y)/Q(x,y)$, see [Furstenberg, 1967].

No surprise now that Catalan g.f. $C(t)$, $tC(t)^2 - C(t) + 1 = 0$, is a diagonal:

$$C_n = [x^n y^n] \, \frac{y(1-2xy-2xy^2)}{1-x-2xy-xy^2}, \qquad C_n = [x^n y^n] \, \frac{1-x/y}{1-x-y}.$$

For the first formula, see [Rowland–Yassawi, 2014].

$\mathbb{N}$-Rational Functions in many variables

Definition: Let $\mathcal{R}_k$ be the smallest set of functions $F(x_1, \ldots, x_k)$ which satisfies

(1) $0, x_1, \ldots, x_k \in \mathcal{R}_k$,

(2) $F, G \in \mathcal{R}_k \implies F + G, F \cdot G \in \mathcal{R}_k$,

(3) $F \in \mathcal{R}_k,\ F(0) = 0 \implies 1/(1 - F) \in \mathcal{R}_k$.

Note that all $F \in \mathcal{R}_k$ satisfy: $F \in \mathbb{N}[[x_1, \ldots, x_k]]$, and $F = P/Q$, for some $P, Q \in \mathbb{Z}[x_1, \ldots, x_k]$.

Let $\mathcal{D}$ be a class of diagonals of $F \in \mathcal{R}_k$, for some $k \geq 1$. For example,

$$\sum_n \binom{2n}{n} t^n \in \mathcal{D} \qquad \text{because} \qquad \frac{1}{1 - x - y} \in \mathcal{R}_2 .$$

Main Theorem: $\mathcal{F} = \mathcal{D}$ [Garrabrant, P., 2014]

Here $\mathcal{F}$ denote the class of g.f. $\mathcal{A}_T(t)$ enumerating irrational tilings. In other words, every tile counting function $\mathcal{A}_T \in \mathcal{F}$ is a diagonal of an $\mathbb{N}$-rational function $F \in \mathcal{R}_k$, $k \geq 1$, and vice versa.

Key Lemma: Both $\mathcal{F}$ and $\mathcal{D}$ coincide with a class $\mathcal{B}$ of g.f. $F(t) = \sum_n f(n)t^n$, where $f : \mathbb{N} \to \mathbb{N}$ is given as finite sums $f = \sum g_j$, and each g_j is of the form

$$g_j(m) = \begin{cases} \displaystyle\sum_{v \in \mathbb{Z}^{d_j}} \prod_{i=1}^{r_j} \binom{\alpha_{ij}(v,n)}{\beta_{ij}(v,n)} & \text{if} \quad m = p_j n + k_j, \\ 0 & \text{otherwise}, \end{cases}$$

for some $\alpha_{ij} = a_{ij}v + a'_{ij}n + a''_{ij}$, $\beta_{ij} = b_{ij}v + b'_{ij}n + b''_{ij}$, and $p_j, k_j, r_j, d_j \in \mathbb{N}$.

Asymptotic applications

Corollary: There exist $\sum_n f_n, \sum_n g_n \in \mathcal{F}$, s.t.

$$f_n \sim \frac{\sqrt{\pi}}{\Gamma\left(\frac{5}{8}\right)\Gamma\left(\frac{7}{8}\right)}\,128^n, \qquad g_n \sim \frac{\Gamma\left(\frac{3}{4}\right)^3}{\sqrt[3]{2}\,\pi^{5/2}}\,n^{-3/2}\,384^n$$

Proof idea: Take

$$f_n := \sum_{k=0}^{n} 128^{n-k}\binom{4k}{k}\binom{3k}{k}.$$

Note: We have $b_n \sim \mathrm{B}\,n^\beta\gamma^n$, where $\beta \in \mathbb{N}$, and $\mathrm{B}, \gamma \in \mathbb{A}$, for all $\sum_n b_n t^n = P/Q$.

Conjecture: *For every $\sum_n f_n \in \mathcal{F}$, we have $f_n \sim \mathrm{B}n^\beta\gamma^n$, where $\beta \in \mathbb{Z}/2, \gamma \in \mathbb{A}$, and B is spanned by values of ${}_p\Phi_q(\cdot)$ at rational points, cf.* [Kontsevich–Zagier, 2001].

Curious Conjecture on Catalan numbers:

We have:

$$C(t) \notin \mathcal{F}\,, \qquad \textit{where} \qquad C(t) = \frac{1-\sqrt{1-4t}}{2t}\,.$$

In other words, there is no set T of irrational tiles and $\varepsilon \geq 0$, s.t.

$$a_n(T,\varepsilon) \, = \, C_n \quad \text{for all} \;\; n \geq 1, \quad \text{where} \quad C_n \, = \, \frac{1}{n+1}\binom{2n}{n}.$$

More on Catalan numbers

Recall
$$C_n \sim \frac{1}{\sqrt{\pi}} n^{-3/2} 4^n.$$

Corollary: *There exists* $\sum_n f_n t^n \in \mathcal{F}$, *s.t.* $f_n \sim \frac{3\sqrt{3}}{\pi} C_n$. *Furthermore,* $\forall \epsilon > 0$, *there exists* $\sum_n f_n t^n \in \mathcal{F}$, *s.t.* $f_n \sim \lambda C_n$ *for some* $\lambda \in [1-\epsilon, 1+\epsilon]$.

Moral: Curious Conjecture cannot be proved via rough asymptotics. However:

Conjecture: *There is no* $\sum_n f_n t^n \in \mathcal{F}$, *s.t.* $f_n \sim C_n$.

Note: This conjecture probably involves deep number theory.

More applications

Proposition: *For every* $m \geq 2$*, there is* $\sum_n f_n t^n \in \mathcal{F}$*, s.t.*

$$f_n = C_n \mod m, \quad \textit{for all} \;\; n \geq 1.$$

Proposition *For every prime* $p \geq 2$*, there is* $\sum_n g_n t^n \in \mathcal{F}$*, s.t.*

$$\operatorname{ord}_p(g_n) = \operatorname{ord}_p(C_n), \quad \textit{for all} \;\; n \geq 1,$$

where $\operatorname{ord}_p(N)$ *is the largest power of* p *which divides* N*.*

Moral: Elementary number theory does not help to prove the Curious Conjecture.

Note: For $\operatorname{ord}_p(C_n)$, see [Kummer, 1852], [Deutsch–Sagan, 2006].

Proof idea: Take

$$f_n = \binom{2n}{n} + (m-1)\binom{2n}{n-1}.$$

Schützenberger's principle

There is a general metamathematical principle that goes back to M.-P. Schützenberger and that states the following: whenever a rational series in one variable counts a class of objects, then the series is $\mathbb{N}$*-rational. This phenomenon has been observed on a large number of examples: generating series and zeta functions in combinatorics, Hilbert series of graded or filtered algebras, growth series of monoids or of groups.*

[Berstel, Reutenauer; 2008]

Open Problem: Suppose $F \in \mathcal{F}$ is rational. Does this imply that $F \in \mathcal{R}_1$?

If NO, this implies that Schützenberger's principle is FALSE, i.e. there is a set of *irrational tiles* which gives a combinatorial interpretation to a non-negative rational functions, which nonetheless is not $\mathbb{N}$-rational.

Thank you!