Coxeter elements in well-generated reflection groups

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joint work with
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Context and motivation

$\mathrm{NC}(n) := \{w \in \mathfrak{S}_n \mid \ell_T(w) + \ell_T(w^{-1}c) = \ell_T(c)\}$, where

- $T := \{$all transpositions of $\mathfrak{S}_n\}$, ℓ_T associated length function ("absolute length");
- c is a long cycle (n-cycle).

$\mathrm{NC}(n)$ is

- equipped with a natural partial order ("absolute order"), and is a lattice;
- isomorphic to the poset of NonCrossing partitions of an n-gon ("noncrossing partition lattice"), so it is counted by the Catalan number $\mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}$.

Generalization to finite Coxeter groups (or reflection groups):

- replace $\mathfrak{S}_n$ with a Coxeter group W;
- replace T with $R := \{$all reflections of $W\}$, and ℓ_T with ℓ_R;
- replace c with a Coxeter element of W.

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$$\mathrm{NC}(W, c) := \{w \in W \mid \ell_R(w) + \ell_R(w^{-1}c) = \ell_R(c)\},$$

- also equipped with a "W-absolute order";
- counted by the W-Catalan number $\mathrm{Cat}(W) := \prod_{i=1}^{n} \frac{d_i + h}{d_i}$.

$\mathrm{Cat}(W)$ appears in other combinatorial objects attached to (W, c): cluster complexes, generalized associahedra...
$\rightsquigarrow$ "Coxeter-Catalan combinatorics".

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Outline

1. Coxeter elements in real reflection groups — *via* Coxeter systems
   - Classical definition
   - Extended definition

2. Coxeter elements in well-generated complex reflection groups — *via* eigenvalues
   - Classical definition
   - Extended definition

3. Reflection automorphisms and main results

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Coxeter element of a Coxeter system

Definition

A Coxeter system (W, S) is a group W equipped with a generating set S of involutions, such that W has a presentation of the form:

$$W = \left\langle S \;\middle|\; s^2 = 1 \ (\forall s \in S);\ (st)^{m_{s,t}} = 1 \ (\forall s \neq t \in S) \right\rangle \ ,$$

with $m_{s,t} \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ for $s \neq t$.

Definition ("Definition 0")

Write $S := \{s_1, \dots, s_n\}$. A Coxeter element of (W, S) is a product of all the generators:

$$c = s_{\pi(1)} \dots s_{\pi(n)} \quad \text{for } \pi \in \mathfrak{S}_n.$$

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Coxeter element of a real reflection group

- V real vector space of dimension n
- W finite subgroup of $\mathrm{GL}(V)$ generated by reflections

$\rightsquigarrow$ W admits a structure of **Coxeter system**.

Take for S the set of reflections through the walls of a fixed chamber of the hyperplane arrangement of W.

Definition ("Classical definition")

Let W be a finite real reflection group. A Coxeter element of W is a product (in any order) of all the reflections through the walls of a chamber of W.

Proposition

The set of Coxeter elements of W forms a conjugacy class.

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Alternative Coxeter structures

In general a real reflection group does not have a unique Coxeter structure.

Example

Symmetry group of the regular hexagon $= I_2(6) \simeq A_1 \times A_2$

But "unicity if S consists of reflections":

Proposition (Observation/Folklore?)

Let W be a finite real reflection group, R the set of all reflections of W. Let $S, S' \subseteq R$ be such that (W, S) and (W, S') are both Coxeter systems.
Then (W, S) and (W, S') are isomorphic Coxeter systems.

proof not enlightening! (case-by-case check on the classification)

$\rightsquigarrow$ Do you have a better proof?

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New Coxeter elements

For a real reflection group W, one may be able to construct Coxeter structures which do not come from a chamber of the arrangement...

$\rightsquigarrow$ Isomorphic, but not conjugate structures!

Example of $I_2(5)$.

Definition

We call generalized Coxeter element of W a product (in any order) of the elements of some set S, where S is such that:

- S consists of reflections;
- (W, S) is a Coxeter system.

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Complex reflection group

- V complex vector space of dimension n
- W finite subgroup of $\mathrm{GL}(V)$ generated by "reflections" ($r \in \mathrm{GL}(V)$ of finite order and fixing pointwise a hyperplane)
- assume W is well-generated, i.e., can be generated by n reflections.

Finite *real* reflection groups can be seen as complex reflection groups.

But there are much more. In general: no Coxeter structure, no privileged (natural, canonical) set of n generating reflections.

$\rightsquigarrow$ how to define a Coxeter element of W?

Recall: Geometry of Coxeter elements in *real* groups

Assume W is a finite, **real** reflection group (irreducible). Let c be a Coxeter element of W, h the order of c (Coxeter number).

Facts

- $h = d_n$, the highest invariant degree of W: $d_1 \leq \cdots \leq d_n$ degrees of homogeneous polynomials $f_1, \ldots, f_n \in \mathbb{C}[V]$ such that $\mathbb{C}[V]^W = \mathbb{C}[f_1, \ldots, f_n]$.
- There exists a plane $P \subseteq V$ stable by c and on which c acts as a rotation of angle $\frac{2\pi}{h}$.
- Thus, c admits $e^{\frac{2i\pi}{h}}$ as an eigenvalue.
- The elements of W having $e^{\frac{2i\pi}{h}}$ as an eigenvalue form a conjugacy class of W. [Springer's theory of regular elements]

Proposition

c is a Coxeter element of W iff c admits $e^{\frac{2i\pi}{h}}$ as an eigenvalue.

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Outline

Coxeter element in a complex reflection group

Back to W well-generated complex reflection group (irreductible).
$\rightsquigarrow$ how to define a Coxeter element of W?

Define the Coxeter number h of W as the highest invariant degree:
$h := d_n$.

[Springer] $\Rightarrow$ the set of elements of W having $e^{\frac{2i\pi}{h}}$ as eigenvalue

- is non-empty;
- forms a conjugacy class of W.

Definition ("classical definition", after Bessis '06)

Let W be a well-generated, irreducible complex reflection group. We call Coxeter element of W an element that admits $e^{\frac{2i\pi}{h}}$ as an eigenvalue.

Bessis' seminal work related to Coxeter-Catalan combinatorics for complex groups uses this definition.

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Replace $e^{2i\pi/h}$ by another h-th root of unity

Natural generalization: "Galois twist".

Definition ("Extended definition")

Let W be a well-generated, irreducible complex reflection group, and h its Coxeter number.

We call generalized Coxeter element an element of W that admits a primitive h-th root of unity as an eigenvalue.

Equivalently, c is a generalized Coxeter element if and only if $c = w^k$ where w is a *classical* Coxeter element and $k \wedge h = 1$.

Is this definition compatible with the extended definition for real groups ?

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Four definitions is too much to remember!

	Classical definition	Extended definition
W real	Product of reflections through the walls of a chamber	$\prod_{s \in S} s$, for some $S \subseteq R$, with (W, S) Coxeter
W complex	$e^{\frac{2i\pi}{h}}$ is eigenvalue	$e^{\frac{2ik\pi}{h}}$ is eigenvalue for some k, $k \wedge h = 1$

Outline

Stability by reflection automorphisms

Definition

A reflection automorphism of W is an automorphism of W that preserves the set R of all reflections of W.

Theorem (Reiner-R.-Stump)

Let $c \in W$. The following are equivalent:

(i) *c has an eigenvalue of order h;*

(ii) *$c = \psi(w)$ where w is a classical Coxeter element and ψ is a reflection automorphism of W;*

(iii) *(c is a Springer-regular element of order h).*

If W is real, this is also equivalent to:

(iv) *There exists $S \subseteq R$ such that (W, S) is a Coxeter system and c is the product (in any order) of elements of S.*

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Stability by reflection automorphisms

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A reflection automorphism of W is an automorphism of W that preserves the set R of all reflections of W.

Theorem (Reiner-R.-Stump)

Let $c \in W$. The following are equivalent:

(i) *c has an eigenvalue of order h;*

(ii) *$c = \psi(w)$ where w is a classical Coxeter element and ψ is a reflection automorphism of W;*

(iii) *(c is a Springer-regular element of order h).*

If W is real, this is also equivalent to:

(iv) *There exists $S \subseteq R$ such that (W, S) is a Coxeter system and c is the product (in any order) of elements of S.*

Application to Coxeter-Catalan combinatorics

Corollary

Let W be a well-generated, irreducible complex reflection group, and $R = \mathrm{Refs}(W)$.
Then, for all generalized Coxeter elements c, the sets

$$\mathrm{NC}(W, c) := \{w \in W \mid \ell_R(w) + \ell_R(w^{-1}c) = \ell_R(c)\}$$

are all isomorphic posets (so can be called W-noncrossing partition lattices).

More generally, any property

- known for classical Coxeter elements, and
- depending only on the combinatorics of the couple (W, R),
- $\rightsquigarrow$ extends to generalized Coxeter elements.

Applies to properties related to Coxeter-Catalan combinatorics. For example, the number of reduced decompositions of a generalized Coxeter element into reflections is $\frac{n!h^n}{|W|}$.

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How many new Coxeter elements?

Definition

The field of definition K_W of W is the smallest field over which one can write all matrices of W.

Examples: $K_W = \mathbb{Q}$ iff W crystallographic (Weyl group).
For $W = I_2(m)$, $K_W = \mathbb{Q}(\cos\frac{2\pi}{m})$.

Theorem (RRS)

- *The number of conjugacy classes of generalized Coxeter elements is* $[K_W : \mathbb{Q}]$.
  (only 1 for Weyl groups; $\varphi(m)/2$ *for dihedral group* $I_2(m)$*...)*
- *(More precisely, there is a natural action of the Galois group* $\mathrm{Gal}(K_W/\mathbb{Q})$ *on the set of conjugacy classes of generalized Coxeter elements of* W*, and this action is simply transitive.*

$$\forall C, C' \in \mathrm{Cox}(W), \exists! \gamma \in \Gamma, C' = \gamma \cdot C.\ )$$

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Ingredients of the proofs

- a spoonful of classical Springer's theory of regular elements
- a big chunk of Galois automorphisms and reflection automorphisms of W [Marin-Michel '10]
- a pinch of case-by-case checks ☹

Further results and questions

- Some results extends to more general elements of W, namely Springer's regular elements of arbitrary order.

- the characterization of generalized Coxeter elements for real groups extends to Shephard groups (those nicer complex groups with presentations “à la Coxeter”).

- for the other well-generated complex groups, there is no canonical form of presentation, and not (yet?) a “combinatorial” vision of Coxeter elements.

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Thank you!