Complements of Coxeter Group Quotients

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Main results and motivations

The quotients W^J of a Coxeter group W have been intensively studied from different points of view. In this talk will be shown that their complements $W \setminus W^J$ have analogous combinatorial and topological properties.

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Coxeter groups

Let S be a finite set and $m : S \times S \to \{ 1, 2, ..., \infty \}$ be a symmetric matrix such that

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The Coxeter group W relative to the Coxeter matrix m is defined by the presentation

$$\begin{cases} \text{Generators:} & S; \\ \text{Relations:} & (st)^{m(s,t)} = e \end{cases}$$

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We call (W, S) a Coxeter system.

Coxeter groups - Bruhat order

Given a Coxeter system (W, S), an element $w \in W$ is a word in the alphabet S:

$$w = s_1 s_2 \dots s_k, \quad s_i \in S.$$

If k is minimal among all such expressions for w then $\ell(w) = k$ is called the length of w and the word $s_1 s_2 \dots s_k$ is called a reduced word for w. Define

$$\ell(v, w) := \ell(w) - \ell(v).$$

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The Bruhat order of W is defined in the following way: given $u, v \in W$ and $v = s_1 s_2 ... s_q$ a reduced expression for v,

$$u \leqslant v \Leftrightarrow \text{there exists a reduced expression}$$
$$u = s_{i_1} ... s_{i_k}, \ 1 \leqslant i_1 < ... < i_k \leqslant q.$$

A right descent of $w \in W$ is an element $s \in S$ such that $\ell(ws) < \ell(w)$.

$$D_R(w) := \{ s \in S \mid \ell(ws) < \ell(w) \} .$$

Analogously is defined the left descent set of w, $D_L(w)$.

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One can show that for $J \subseteq S$, each element $w \in W$ has a unique expression

$$w = w_J w^J,$$

where $w^J \in W^J$ and $w_J \in W_J$, being $W_J \subseteq W$ the subgroup generated by the element of J.

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$$W^J \simeq W/W_J.$$

Example: the symmetric group

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(S_n, S) is a Coxeter system; in fact, it is straightforward to verify that the generators $s_1, s_2, ..., s_{n-1}$, where $s_i := (i, i+1)$ for $1 \leqslant i \leqslant n-1$, satisfy the Coxeter relations

$$\begin{cases} s_i^2 = e, & \\ s_i s_j s_i = s_j s_i s_j & \text{if } |i-j| = 1 \\ s_i s_j = s_j s_i & \text{if } |i-j| > 1. \end{cases}$$

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$$W^J = \{ w \in S_n \mid s_i \in J \Rightarrow w(i) < w(i+1) \}$$

The quotient W^J can be ordered by the induced Bruhat order. With $[v,w]^J$ it's denoted an interval in W^J, i.e., if $v, w \in W^J$ and $v \leqslant w$,

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Coxeter groups - Quotients and projections

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Define the canonical projection $P^J : W \to W^J$ by

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It's known that P^J is a morphism of posets, i.e.

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Theorem (Deodhar, 1977)

The Möbius function of the poset W^J is

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Theorem (Björner and Wachs, 1982)

The order complex of $[v,w]^J$ is shellable.

Combinatorial properties of $W \setminus W^J$

We denote by $[u, v]^{\setminus J}$ the Bruhat intervals in $W \setminus W^J$, i.e., if $u, v \in W \setminus W^J$ and $u \leqslant v$,

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Theorem (S., 2014)

The set $W \setminus W^J$, with the induced Bruhat order, is a graded poset with the length minus one as rank function.

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The Möbius function of the poset $W \setminus W^J$ is

$$\mu^{\setminus J}(u,v) = \begin{cases} (-1)^{\ell(u,v)} & \text{if } u \not< v^J, \\ 0, & \text{otherwise.} \end{cases}$$

Equivalently,

$$\mu^{\setminus J}(u,v) = \begin{cases} (-1)^{\ell(u,v)} & \text{if } [u,v]^{\setminus J} = [u,v], \\ 0, & \text{otherwise.} \end{cases}$$

Let $A := \mathbb{Z}[q^{-1/2}, q^{1/2}]$ be the ring of Laurent polynomials in the indeterminate $q^{1/2}$. Recall that the Hecke algebra $\mathcal{H}(W)$ is the free A-module generated by the set $\{ T_w \mid w \in W \}$ with product defined by

$$T_w T_s = \begin{cases} T_{ws}, & \text{if } s \notin D_R(w), \\ qT_{ws} + (q-1)T_w, & \text{otherwise,} \end{cases}$$

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For $s \in S$ one can easily see that

$$T_s^{-1} = (q^{-1} - 1)T_e + q^{-1}T_s$$

and then use this to invert all the elements T_w, where $w \in W$.

Algebraic properties of $W \setminus W^J$ - Hecke algebras

On $\mathcal{H}(W)$ there is an involution ι defined by

$$\iota(q^{\frac{1}{2}}) = q^{-\frac{1}{2}}, \quad \iota(T_w) = T_{w^{-1}}^{-1},$$

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Theorem (Kazhdan and Lusztig, 1979)

There is an ι-invariant basis $\{ \ C_w \ \}_{w \in W}$ of the Hecke algebra $\mathcal{H}(W)$, where

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The polynomials $\{ P_{v,w} \}_{v,w \in W} \subseteq \mathbb{Z}[q]$ are called the Kazhdan-Lusztig polynomials of W.

Algebraic properties of $W \setminus W^J$ - The Hecke modules M^J (Deodhar, 1987)

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For $J \subseteq S$ let

$$M^J := \operatorname{span}_A \left\{ m_v^J \,\middle|\, v \in W^J \right\}.$$

There is an A-module morphism $\phi^{J,x} : \mathcal{H} \to M^J$ defined by

$$\phi^{J,x}(T_w) = x^{\ell(w_J)} m^J_{w^J},$$

where $x \in \{ -1, q \}$.

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We call $M^{J,-1}$ and $M^{J,q}$ these two right $\mathcal{H}$-modules and $\left\{ m_v^{J,x} \right\}_{v \in W^J}$ the elements of their basis, for $x \in \{ -1, q \}$.

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for all $v \in W^J$

Algebraic properties of $W \setminus W^J$ - The Hecke modules M^J

Theorem (Deodhar, 1987)

There is an ι^x-invariant basis $\left\{ C_w^{J,x} \right\}_{w \in W^J}$ of the Hecke module $M^{J,x}$, where

$$C_w^{J,x} = q^{\frac{\ell(w)}{2}} \sum_{y \in [e,w]^J} (-1)^{\ell(y,w)} q^{-\ell(y)} P_{y,w}^{J,x}(q^{-1}) m_y^{J,x}.$$

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The polynomials $\left\{ P_{v,w}^{J,x} \right\}_{v,w \in W^J} \subseteq \mathbb{Z}[q]$ are called the parabolic Kazhdan-Lusztig polynomials of W^J of type x.

Note that the modules $M^{J,x}$ are cyclic; in fact

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Let $\mathrm{ann}_e^{J,x} := \left\{ a \in \mathcal{H} \;\middle|\; m_e^{J,x} a = 0 \right\}$ be the annihilator of $m_e^{J,x}$. In particular $\mathrm{ann}_e^{J,x} = \ker(\phi^{J,x})$.

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It's easy to see that the right ideal $\operatorname{ann}_e^{J,x}$ is ι-invariant.

Algebraic properties of $W \setminus W^J$ - Polynomials

Let $\left\{ b_w^{J,x} \right\}_{w \in W} \subset \mathcal{H}(W)$ be elements defined by

$$b_w^{J,x} := x^{\ell(w_J)} T_{w^J} - T_w \in \mathcal{H}(W).$$

Note that $b_w^{J,x} = 0$ if and only if $w \in W^J$. Then

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Proposition (S., 2014)

The set $\mathcal{B}^{J,x} := \left\{ b_w^{J,x} \;\middle|\; w \in W \setminus W^J \right\}$ *is an* A*-basis of* $\mathrm{ann}_e^{J,x}$*, for every* $J \subseteq S$, $x \in \{ -1, q \}$.

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Theorem (S., 2014)

There is an ι-invariant basis $\left\{ c_w^{J,x} \right\}_{w \in W \setminus W^J}$ *of the annihilator* $\mathrm{ann}_e^{J,x}$, *where*

$$c_w^{J,x} = q^{\frac{\ell(w)}{2}} \sum_{y \in (W \setminus W^J) \cap [e,w]} (-1)^{\ell(y,w)} q^{-\ell(y)} \tilde{P}^{J,x}_{y,w}(q^{-1}) b_y^{J,x}.$$

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The polynomials $\left\{ \tilde{P}_{v,w}^{J,x} \right\}_{v,w \in W \setminus W^J} \subseteq \mathbb{Z}[q]$ are the parabolic Kazhdan-Lusztig polynomials of $W \setminus W^J$ of type x.

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We have that

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Example

Take $v = 324156$ *and* $w = 546132$ *in* $W \setminus W^{S \setminus \{s_3\}}$.

We have that

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We don't know any expression of $\tilde{P}^{J,-1}$ in terms of known polynomials.

Example

Take $v = 324156$ *and* $w = 546132$ *in* $W \setminus W^{S \setminus \{ s_3 \}}$. *Then*

$$\tilde{P}^{S \setminus \{ s_3 \},-1}_{v,w} = -5q^2 + 2q.$$

Algebraic properties of $W \setminus W^J$ - Polynomials

The polynomial $\tilde{P}^{S\setminus\{s_3\},-1}_{v,w}$ in the previous example was computed thanks to the recursion

$$q^{\ell(v,w)}\tilde{P}^{J,x}_{v,w}(q^{-1}) = \sum_{z\in[v,w]^{\setminus J}} Z^{J,x}_{v,z}(q)\tilde{P}^{J,x}_{z,w}(q),$$

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The polynomials $Z^{J,x}$ are related to the R-polynomials of W by the following simple formula:

$$Z^{J,x}_{v,w} = R_{v,w} - (q-1-x)^{\ell(w_J)} R_{v,w^J},$$

for all $v, w \in W \setminus W^J$ and $x \in \{ -1, q \}$.

Topological properties of $W \setminus W^J$

Theorem (S., 2014)

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Corollary

The order complex of $(u,v)^{\setminus J}$ is PL homeomorphic to

1. *the sphere $\mathbb{S}^{\ell(u,v)-2}$, if $u \not\leq v^J$;*
2. *the ball $\mathbb{B}^{\ell(u,v)-2}$, otherwise.*