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GEODESICS IN A GRAPH OF PERFECT MATCHINGS

ROY H. JENNINGS

Abstract. Let $\mathscr{P}_m$ be the graph on the set of perfect matchings in the complete graph K_{2m}, where two perfect matchings are connected by an edge if their symmetric difference is a cycle of length four. This paper studies geodesics in $\mathscr{P}_m$. The diameter of $\mathscr{P}_m$, as well as the eccentricity of each vertex, are shown to be $m-1$. Two proofs are given to show that the number of geodesics between any two antipodes is m^{m-2}. The first is a direct proof via a recursive formula, and the second is via reduction to the number of minimal factorizations of a given m-cycle in the symmetric group S_m. An explicit formula for the number of geodesics between any two matchings in $\mathscr{P}_m$ is also given.

Let $\mathscr{M}_m$ be the graph on the set of non-crossing perfect matchings of $2m$ labeled points on a circle with the same adjacency condition as in $\mathscr{P}_m$. $\mathscr{M}_m$ is an induced subgraph of $\mathscr{P}_m$, and it is shown that $\mathscr{M}_m$ has exactly one pair of antipodes having the maximal number m^{m-2} of geodesics between them.

1. Introduction

Consider a set of $2m$ labeled points on a circle. Join its points in disjoint pairs by m straight line segments, such that no two lines intersect. Abstractly, this is a perfect matching in the complete graph K_{2m}. Such a matching is called a **non-crossing perfect matching**. Hernando, Hurtado and Noy defined in [8] the **graph of non-crossing perfect matchings** $\mathscr{M}_m$, in which two matchings are connected by an edge if their symmetric difference is a cycle of length four. They showed that the diameter and the eccentricity of every vertex in this graph are $m-1$.

This paper studies the graph $\mathscr{P}_m$ on the set of all perfect matchings in the complete graph K_{2m} where two matchings are connected by an edge if their symmetric difference is a cycle of length four.

The distance between any two matchings in $\mathscr{P}_m$ is shown to depend only on the number of components in their union.

Theorem (Theorem 3.4). *For any $M_1, M_2 \in \mathscr{P}_m$, $d(M_1, M_2) = m - l$, where l is the number of components in $M_1 \cup M_2$.*

Although $\mathscr{P}_m$ is larger than the graph of non-crossing perfect matchings $\mathscr{M}_m$, which is an induced subgraph of $\mathscr{P}_m$, it is shown that the two graphs still share the same diameter, and that it is equal to the eccentricity of all of their vertices.

Theorem (Corollary 3.5). *The diameter of $\mathscr{P}_m$, as well as the eccentricity of every vertex in it, is $m-1$.*

Enumeration of the number of geodesics (shortest paths) between antipodes in $\mathscr{P}_m$ reveals the following surprising result.

Theorem (Corollary 4.5). *The number of geodesics between any two antipodes in $\mathscr{P}_m$ is m^{m-2}.*

The expression m^{m-2} appears in Cayley's well known formula for the number of labeled trees on m vertices. On the other hand, Hurwitz [9] (see also [12]) showed that it gives as well the number of factorizations of a given m-cycle as a product of $m-1$ transpositions in S_m. (Bijections between these trees and factorizations appear, e.g., in [7, 11].) In Section 5, an alternative proof, suggested by Y. Roichman, for Corollary 4.5 is provided via reduction to the number of factorizations of a given m-cycle as a product of $m-1$ transpositions.

An explicit formula for the number of geodesics between any two matchings in $\mathscr{P}_m$ is given in Corollary 4.6.

Corollary 4.5 does not hold for the subgraph $\mathscr{M}_m$. Instead, we have the following result.

Theorem (Theorem 4.7). *The graph $\mathscr{M}_m$ of non-crossing perfect matchings has a unique pair of matchings with m^{m-2} geodesics between them. All other pairs have a smaller number of geodesics.*

This paper is based on the author's M.Sc. thesis. For generalizations and extensions of results in this paper see [2, 3, 6].

2. The Graph of Perfect Matchings

Consider the set of all perfect matchings in the complete graph K_{2m}. Denote the **graph of perfect matchings** on this set, in which two distinct matchings M_1 and M_2 are connected by an edge if their symmetric difference is a cycle of length four, by $\mathscr{P}_m$. Here, the symmetric difference of M_1 and M_2, is the graph consisting of the edges that belong to exactly one of these matchings. In this paper, all matchings are perfect, and thus the term *perfect* is freely omitted. Denote adjacent matchings in $\mathscr{P}_m$ by $M_1 \sim M_2$, and write $M_1 \simeq M_2$ if either $M_1 = M_2$ or $M_1 \sim M_2$.

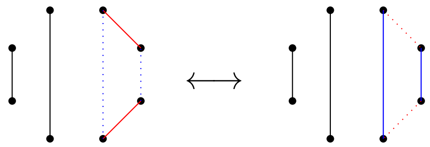

FIGURE 1. Adjacent matchings in $\mathscr{P}_4$

Fact 2.1. *The cardinality of* $\mathscr{P}_m$ *is the double factorial:*

$$|\mathscr{P}_m| = (2m-1)!! = (2m-1) \cdot (2m-3) \cdot \ldots \cdot 1.$$

For any edge $e = (v_1, v_2)$ in the complete graph K_{2m} and a matching M in $\mathscr{P}_m$, the **insertion of the edge** e **into the matching** M, denoted by $M * e$, is defined as follows:
If e is already in M, then $M * e = M$. Otherwise, if (v_1, v_3) and (v_2, v_4) are the edges in M incident with v_1 and v_2, then $M * (v_1, v_2)$ is the matching obtained from M by deleting (v_1, v_3) and (v_2, v_4) and adding (v_1, v_2) and (v_3, v_4).

Extend the definition recursively:

$$M * (e_1, \ldots, e_n) = (M * (e_1, \ldots, e_{n-1})) * e_n.$$

Observation 2.2. *Two distinct matchings* M_1 *and* M_2 *in* $\mathscr{P}_m$ *are adjacent if and only if* $M_2 = M_1 * e$*, for some edge* e *not in* M_1*.*

Remark 2.3. Let M be a matching in $\mathscr{P}_m$. By definition, every neighbor of M misses exactly two edges of M. Conversely, for each pair of distinct edges in M, there are exactly two neighbors of M not containing this pair. Therefore, $\mathscr{P}_m$ is a regular graph of degree $2\binom{m}{2}$.

Proposition 2.4. *Let* M *be a matching in* $\mathscr{P}_m$*, and let* $(e_1, \ldots, e_m)$ *be the edges of* M *arranged in any order. For any matching* M' *in* $\mathscr{P}_m$*,* $M = M' * (e_1, \ldots, e_m)$*.*

Indeed, the edges inserted into M' are vertex disjoint, and therefore they all belong to the final matching.

Corollary 2.5. *The graph of perfect matchings* $\mathscr{P}_m$ *is connected.*

3. GEODESICS AND DIAMETER

Recall that a **geodesic** between two vertices in a connected graph is a shortest path between them. The **distance** between two vertices u and v, denoted by $d(u, v)$, is the length of a geodesic between them. The **eccentricity** of a vertex is the maximal distance between this vertex and any other vertex. The maximal length of a geodesic in a connected graph is called the graph's **diameter**. If the distance between two vertices is the graph's diameter, then the vertices are called **antipodes**.

Hernando, Hurtado and Noy [8] found a formula for the distance between two matchings in the graph $\mathscr{M}_m$ of non-crossing perfect matchings with m edges, implying that the diameter of this graph and the eccentricity of every vertex in it are $m-1$. In this section, it is shown that, although $\mathscr{P}_m$ is larger than the graph of non-crossing perfect matchings, where the latter is an induced subgraph, the diameter and the eccentricity of every vertex in it remain $m-1$.

Fact 3.1. *The union of two matchings M_1 and M_2 in $\mathscr{P}_m$ is a vertex-disjoint union of alternating cycles C_i of even length (a common edge is considered as a cycle of length two):*

$$M_1 \cup M_2 = C_1 \,\dot{\cup}\, C_2 \,\dot{\cup}\, \ldots \,\dot{\cup}\, C_l \ .$$

Observation 3.2. *Denote the number of connected components in a graph G by $c(G)$. Given two matchings $M_1, M_2 \in \mathscr{P}_m$ and an edge $e \notin M_1$, exactly one of the following cases holds:*

(1) $c(M_1 * e \cup M_2) = c(M_1 \cup M_2) - 1$. *This is the case if and only if the vertices of e belong to two different cycles in $M_1 \cup M_2$.*
(2) $c(M_1 * e \cup M_2) = c(M_1 \cup M_2) + 1$. *This is the case if and only if the vertices of e belong to the same cycle in $M_1 \cup M_2$, and the insertion of e into M_1 splits this cycle into two cycles in $M_1 * e \cup M_2$.*
(3) $c(M_1 * e \cup M_2) = c(M_1 \cup M_2)$. *This is the case if and only if the vertices of e belong to the same cycle in $M_1 \cup M_2$, and the insertion of e into M_1 does not split this cycle in $M_1 * e \cup M_2$.*

Remark 3.3. In the settings of Observation 3.2, every pair of distinct edges in M_1 that belong to different components in $M_1 \cup M_2$, corresponds to two unique neighbors of M_1 that belong to the first case of the observation. Moreover, every pair of distinct edges in M_1 that belongs to the same component in $M_1 \cup M_2$ corresponds to one unique neighbor of M_1 that belongs to the second case of the observation, and one unique neighbor that belongs to the third case.

Theorem 3.4. *For any $M_1, M_2 \in \mathscr{P}_m$, $d(M_1, M_2) = m - l$, where l is the number of components in $M_1 \cup M_2$.*

Proof. Note that $m-l = 0$ if and only if $M_1 = M_2$. By Observation 3.2, $|c(M' \cup M_2) - c(M \cup M_2)| \leq 1$ for any two neighbors $M \sim M'$ in $\mathscr{P}_m$. Thus $d(M_1, M_2) \geq m - l$. For every matching $M \neq M_2$, there is some alternating cycle $C \subseteq M \cup M_2$ such that $|C| \geq 4$. Therefore, by Remark 3.3, for every pair of distinct edges in $M \cap C$, M has a unique neighbor M' such that $c(M' \cup M_2) - c(M \cup M_2) = 1$. Thus, $d(M_1, M_2) \leq m - l$. $\square$

By Theorem 3.4, antipodes in $\mathscr{P}_m$ are pairs of matchings whose union consists of one cycle.

Corollary 3.5. *The diameter of $\mathscr{P}_m$, as well as the eccentricity of every vertex in it, are $m-1$. The number of antipodes of every matching is $(2m-2)!!$.*

Remark 3.6. An analogue of Theorem 3.4 for the graph $\mathscr{M}_m$ of non-crossing perfect matchings appears in [8], in the equivalent form

$$d(M_1, M_2) = \frac{1}{2} \sum_{i=1}^{l} (\mathrm{length}(C_i) - 2) .$$

4. Counting Geodesics

In this section, the number of geodesics between any two matchings in $\mathscr{P}_m$ is given. In particular, the number of geodesics between antipodes is shown to be m^{m-2}. The section concludes by showing that in the subgraph $\mathscr{M}_m$ of non-crossing perfect matchings, there is exactly one pair of antipodes having the maximal number m^{m-2} of geodesics between them.

Definition 4.1. Denote the number of geodesics between two matchings, whose symmetric difference is one cycle of length $2k$, by P_{2k}, for $k \geq 2$. Define $P_2 = 1$.

Theorem 4.2. *For every positive integer k, we have*

$$P_{2k} = \frac{k}{2} \sum_{i=1}^{k-1} \binom{k-2}{i-1} P_{2i} P_{2k-2i}, \quad \textit{for } k \geq 2,$$

with $P_2 = 1$.

Proof. Let $M_1, M_2 \in \mathscr{P}_m$, $m \geq k$, be two matchings with a symmetric difference of one cycle of length $2k$. Denote this cycle by C. $M_1 \cup M_2$ has $m-k+1$ components ($m-k$ components of size 2 and one component of size $2k$). Therefore, by Theorem 3.4, $d(M_1, M_2) = k-1$.

By Remark 3.3 and Theorem 3.4, a neighbor M' of M_1 is in some geodesic between M_1 and M_2 if and only if C splits in $M' \cup M_2$ into two cycles. The lengths of these two cycles are $2i$ and $2k-2i$ for some i with $1 \leq i \leq \frac{k}{2}$. Therefore, the number of geodesics from M_1 to M_2, beginning with M', is $\binom{k-2}{i-1} P_{2i} P_{2k-2i}$, where the binomial coefficient $\binom{k-2}{i-1}$ counts the ways to interlace the remaining $k-2$ insertions between the two cycles.

For every $1 \leq i < \frac{k}{2}$, there are exactly k neighbors of M_1 in which C splits into two cycles of length $2i$ and $2k-2i$ as above. If k is even, then for $i = \frac{k}{2}$ there are exactly $\frac{k}{2}$ neighbors of M_1 in which C splits into two cycles of length k as above. Therefore, the formula $k\sum_{i=1}^{k-1}\binom{k-2}{i-1}P_{2i}P_{2k-2i}$ counts every geodesic twice, leading to the claimed result. $\square$

The following is a reformulation of Theorem 4.2, and its proof was suggested by R. Adin.

Lemma 4.3. *For every positive integer k, we have*

$$P_{2k} = \sum_{i=1}^{k-1} i\binom{k-2}{i-1}P_{2i}P_{2k-2i},$$

with $P_2 = 1$.

Proof. Write $a_i = i\binom{k-2}{i-1}P_{2i}P_{2k-2i}$. The symmetry of the binomial coefficients $\binom{k-2}{i-1} = \binom{k-2}{k-i-1}$ implies

$$\begin{aligned}
2\sum_{i=1}^{k-1} a_i &= \sum_{i=1}^{k-1} a_i + \sum_{i=1}^{k-1} a_{k-i} \\
&= \sum_{i=1}^{k-1}\Bigg(i\binom{k-2}{i-1}P_{2i}P_{2k-2i} \\
&\qquad\qquad +(k-i)\binom{k-2}{k-i-1}P_{2(k-i)}P_{2k-2(k-i)}\Bigg) \\
&= \sum_{i=1}^{k-1}\left(i\binom{k-2}{i-1}P_{2i}P_{2k-2i} + (k-i)\binom{k-2}{i-1}P_{2k-2i}P_{2i}\right) \\
&= k\sum_{i=1}^{k-1}\binom{k-2}{i-1}P_{2i}P_{2k-2i} \\
&= 2P_{2k}.
\end{aligned}$$

The last equation is essentially Theorem 4.2. $\square$

Corollary 4.4. *For every positive integer k, we have $P_{2k} = k^{k-2}$.*

Proof. By the well known Cayley Formula [5], the number of labeled trees on k vertices is $T_k = k^{k-2}$. By [10, Ex. 6, p. 34 and pp. 249–250], we have $T_k = \sum_{i=1}^{k-1} i\binom{k-2}{i-1}T_iT_{k-i}$, with $T_1 = 1$. Comparison with Lemma 4.3 concludes the proof. $\square$

By Theorem 3.4, antipodes in $\mathscr{P}_m$ are pairs of matchings with a symmetric difference of one cycle of length $2m$.

Corollary 4.5. *The number of geodesics between antipodes in $\mathscr{P}_m$ is m^{m-2}.*

Corollary 4.5 can be generalized to count the number of geodesics between any two matchings in $\mathscr{P}_m$.

Corollary 4.6. *Let $M_1, M_2 \in \mathscr{P}_m$ with $M_1 \cup M_2 = C_1 \dot\cup C_2 \dot\cup \cdots \dot\cup C_l$, as in Fact 3.1. The number of geodesics between M_1 and M_2 is*

$$\binom{m-l}{n(C_1), \dots, n(C_l)} \prod_{i=1}^{l} (n(C_i)+1)^{n(C_i)-1},$$

where $n(C_i) = \frac{\text{length}(C_i)}{2} - 1$.

Proof. $n(C_i)$ is the number of insertions within the cycle C_i. The binomial coefficient $\binom{m-l}{n(C_1),\dots,n(C_l)}$ counts the ways to interlace insertions between the cycles. □

The analogue of Corollary 4.5 for the induced subgraph in $\mathscr{P}_m$ of non-crossing perfect matchings $\mathscr{M}_m$, is the following.

Theorem 4.7. *The graph $\mathscr{M}_m$ of non-crossing perfect matchings has a unique pair of matchings with m^{m-2} geodesics between them. All other pairs have a smaller number of geodesics.*

Proof. $\mathscr{M}_m$ consists of matchings on a set of $2m$ points on a circle. Denote the convex hull of these points by H. We show that the two matchings having all their edges on the boundary of H are the only pair of antipodes in $\mathscr{M}_m$ with m^{m-2} geodesics between them. All other pairs have a smaller number of geodesics.

Let M_1 and M_2 be the two matchings in $\mathscr{M}_m$ having all their edges on the boundary of H. By Remark 3.6, M_1 and M_2 are antipodes in $\mathscr{M}_m$. By Theorem 3.4, they are also antipodes in $\mathscr{P}_m$. Let $P = (M_1 = M'_1, M'_2, \dots, M'_d = M_2)$ be a geodesic between M_1 and M_2. Since $M'_1 \cup M'_d$ has one component, by Theorem 3.4, $M'_2 \cup M'_d$ is a union of two vertex-disjoint alternating cycles. It is clear that these cycles are the boundaries of two disjoint polygons (including the case of a shared edge, which is a convex polygon with two vertices). Therefore, M'_2 is also in $\mathscr{M}_m$. Similarly, for every $M'_i \in P$, $2 \le i \le d$, some convex polygon in $M'_{i-1} \cup M'_d$, is split into two disjoint convex polygons in $M'_i \cup M'_d$. Thus, P is contained in $\mathscr{M}_m$, and the number of geodesics between M_1 and M_2 in $\mathscr{M}_m$ is m^{m-2}.

For any other pair M_1 and M_2 of antipodes in $\mathscr{M}_m$, one of the matchings, say M_1, has an edge e contained (except for its endpoints) in the interior of H. Denote the two components of $H \setminus e$ by H_1 and H_2. M_1 must have at least one edge e_1 contained in H_1 and another e_2 contained in H_2. Let e' be an edge incident with a vertex of e_1 and a vertex of e_2 such that the cycle $M_1 \cup M_2$ splits in $(M_1 * e') \cup M_2$ into two cycles. Then $M_1 * e'$ is a neighbor of M_1 in a geodesic between M_1 and

M_2 in $\mathscr{P}_m$. Since e and e' intersect, $M_1 * e' \notin \mathscr{M}_m$. Thus, the number of geodesics between M_1 and M_2 in $\mathscr{M}_m$ is smaller than m^{m-2}. □

5. Factorization of Permutations by Transpositions

Let S_m be the symmetric group on m elements. A **minimal factorization by transpositions** (or, simply, **minimal factorization**) of a permutation $\pi \in S_m$, is a product $\sigma_1 \cdots \sigma_n$ of a minimal number of transpositions, such that $\pi = \sigma_1 \cdots \sigma_n$. In this section, the number of geodesics between antipodes in $\mathscr{P}_m$ is shown to be equal to the number of minimal factorizations of an m-cycle in S_m. It follows from a result of Hurwitz [9], as explained by Strehl [12], and Cayley's classical formula [5] that the latter number is the same as the number of labeled trees on m vertices. Bijections which explain this coincidence directly can be, e.g., found in [7, 11].

Throughout this section, the vertices of the underlying complete graph K_{2m} of $\mathscr{P}_m$, are labeled by $\{1, -1, \dots, m, -m\}$. Denote the complete bipartite graph with bipartition of vertices $\{1, \dots, m\}$ and $\{-1, \dots, -m\}$ by $K_{m,m}$. Denote the subgraph of $\mathscr{P}_m$, induced by the perfect matchings of $K_{m,m}$, by $\mathscr{P}_{m,m}$. Matchings in $\mathscr{P}_{m,m}$ correspond naturally to permutations in S_m, by identifying a matching $\pi \in \mathscr{P}_{m,m}$ with a permutation $\pi' \in S_m$ such that $\pi'(i) = j$ for every $(i, -j) \in \pi$. Using this correspondence, a matching $\pi \in \mathscr{P}_{m,m}$ will also be referred to as a permutation in S_m (and vice versa).

Observation 5.1. *Let $\pi \in \mathscr{P}_{m,m}$. For $1 \le i \le j \le m$, we have*

$$\pi \circ (i, j) = \pi * (i, -\pi(j)) = \pi * (j, -\pi(i)),$$

where (i, i) is understood as the identity of S_m.

In Observation 5.1, the neighbor $\pi * (i, -\pi(j)) = \pi * (j, -\pi(i))$ of π in $\mathscr{P}_{m,m}$ is one of the two neighbors of π in $\mathscr{P}_m$, corresponding to the choice of the two edges $(i, -\pi(i))$ and $(j, -\pi(j))$, as in Remark 2.3. The other neighbor in $\mathscr{P}_m$, corresponding to this choice of two edges, is not in $\mathscr{P}_{m,m}$. Thus, every pair of edges in π corresponds to a unique neighbor of π in $\mathscr{P}_{m,m}$, which also corresponds to right multiplication of π by a unique transposition in S_m. In other words, adjacency in $\mathscr{P}_{m,m}$ can be understood as right multiplication by transpositions.

Let G be a group, and let S be a symmetric ($S^{-1} = S$) generating set of G. Recall that the (right) **Cayley graph** $X(G, S)$ is the directed graph on the elements of G, in which (g_1, g_2) is an edge if $g_1 s = g_2$ for some s in S.

Corollary 5.2. *$\mathscr{P}_{m,m}$ is the underlying simple graph of the (right) Cayley graph $X(S_m, S)$, where S is the set of all of the transpositions in S_m.*

Observation 5.3. *Let π and σ be two matchings in $\mathscr{P}_{m,m}$, and let $\pi * e$ be a neighbor of π (in $\mathscr{P}_m$) in a geodesic between π and σ. Then $\pi * e \in \mathscr{P}_{m,m}$.*

Indeed, the union of π and σ, in Observation 5.3, is a vertex-disjoint union of alternating cycles of even lengths. Clearly, the signs of the labels of the vertices within each cycle alternate. By Observation 3.2 and Theorem 3.4, a neighbor $\pi * e$ of π in $\mathscr{P}_m$ is in some geodesic between π and σ if and only if a cycle in $\pi \cup \sigma$ is split into two cycles in $\pi * e \cup \sigma$. This implies that the vertices of e belong to the same cycle in $\pi \cup \sigma$, and that their labels have different signs (see Figure 2).

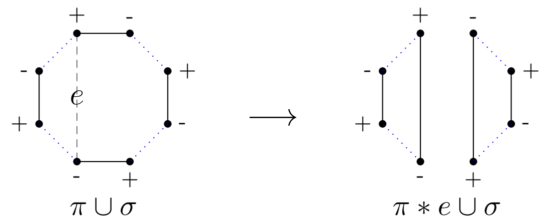

FIGURE 2. Neighbors in a geodesic in $\mathscr{P}_{m,m}$.
Here, the dotted edges belong to σ and the solid edges belong to π.

Corollary 5.4. *Let π and σ be two matchings in $\mathscr{P}_{m,m}$. Every geodesic between π and σ in $\mathscr{P}_m$ is contained in $\mathscr{P}_{m,m}$.*

By Corollaries 5.2 and 5.4, the number of geodesics in $\mathscr{P}_m$ between two elements π and σ in $\mathscr{P}_{m,m}$ is equal to the number of minimal factorizations of $\pi\sigma^{-1}$. In particular, by Theorem 3.4, an m-cycle π and the identity of S_m are antipodes in $\mathscr{P}_m$, and the number of geodesics between them is equal to the number of minimal factorizations of π.

Corollary 5.5. *The number of geodesics between antipodes in $\mathscr{P}_m$ is equal to the number of minimal factorizations of an m-cycle in S_m.*

Hurwitz's result was generalized to enumeration of maximal chains in the non-crossing partition lattice of any well-generated complex reflection group (see [4, Proposition 7.6]). Appropriate generalizations of Corollary 5.5 to other reflection groups are most desired.

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Department of Mathematics, Bar-Ilan University, Ramat-Gan 52900, Israel.

E-mail address: `RoyHJennings@gmail.com`