cut-and-project sets: geometry and combinatorics

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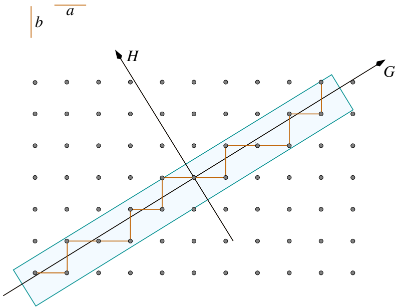
cut-and-project scheme: Fibonacci chain as 1d quasicrystal



- *G physical space*, *H internal space*, lattice in $G \times H$
- *window* $W \subset H$ defines strip, chain $\widehat{=}$ lattice points inside strip
- projection onto G yields intervals of lengths $\tau = \frac{1+\sqrt{5}}{2}$, 1 for a, b
- chain $abaababa\ldots$ also via substitution rule: $a \to ab$, $b \to a$
- Beatty sequences Λ_α with irrational slope α and $W = [0, 1)$

model sets (Meyer 72)

cut-and-project scheme with *star map* $()^\star : L \to L^\star$

$$
\begin{array}{ccccc}
G & \xleftarrow{\pi_G} & G \times H & \xrightarrow{\pi_H} & H \\
\cup & & \cup & & \cup \\
L & \xleftarrow{1\text{–}1} & \text{lattice } \mathcal{L} & \xrightarrow{\text{dense}} & L^\star
\end{array}
$$

projection set via *window* $W \subset H$

$$\curlywedge(W) = \{x \in L \,|\, x^\star \in W\}$$

- *weak model set*: W relatively cpct
- *model set*: in addition $\mathring{W} \neq \varnothing$
- *generic*: $L^\star \cap \partial W = \varnothing$
- *regular model set*: model set with $\mathrm{vol}(\partial W) = 0$

assumptions: G, H σ-cpct LCA groups, H metrisable

properties of weak model sets

every lattice $\Lambda \subset G$ is

- uniformly discrete: $\exists U$ nbhd $\forall t \in G : |tU \cap \Lambda| \leq 1$
- relatively dense: $\exists K$ cpct: $K\Lambda = G$
- periodic
- "pure point diffractive"

weak model sets generalise lattices:

- W relatively cpct $\Longrightarrow$ ⅄(W) uniformly discrete
- $\mathring{W} \neq \varnothing \Longrightarrow$ ⅄(W) relatively dense
- W generic $\Longrightarrow$ ⅄(W) repetitive
- $\mathrm{vol}(\partial W) = 0 \Longrightarrow$ ⅄(W) pure point diffractive

weak model sets are uniformly discrete

- note $(\{e\} \times W^{-1}W) \cap \mathcal{L} = \{e\}$ since $\pi_G|_{\mathcal{L}}$ one-to-one
- as $\mathcal{L}$ discrete and W rel cpct, we find small unit nbhd U with

$$(U \times W^{-1}W) \cap \mathcal{L} = \{e\}$$

- hence $\{e\} = U \cap \curlywedge(W^{-1}W) = U \cap \curlywedge(W)^{-1}\curlywedge(W)$
- now assume $y \in xU$ for $x, y \in \curlywedge(W)$
- then $x^{-1}y \in U \cap \curlywedge(W)^{-1}\curlywedge(W)$
- hence $x = y$

□

fundamental domains for cp schemes

- lattice projects densely into H
- hence we have "arbitarily thin" fundamental domains

Lemma

Let $(G, H, \mathcal{L})$ be a cut-and-project scheme. Then for any non-empty open $U \subset H$ there exists compact $F \subset G$ satisfying

$$(F \times U)\mathcal{L} = G \times H.$$

thin fundamental domains

- Let $\mathcal{F}$ be some relatively cpct fundamental domain of $\mathcal{L}$.
- For non-empty open $U \subset H$, use $\mathcal{F}$ to find cpct $F \subset G$ such that
$$(F \times U)\mathcal{L} = G \times H$$
- Since $\pi_H(\mathcal{F})$ is compact and $\pi_H(\mathcal{L})$ is dense in H, there exist $\ell_1, \ldots, \ell_n \in \pi_G(\mathcal{L})$ such that
$$\mathcal{F} \subset \pi_G(\mathcal{F}) \times \pi_H(\mathcal{F}) \subset \pi_G(\mathcal{F}) \times \bigcup_{i=1}^{n} \ell_i^\star U$$
- statement follows with $F := \bigcup_{i=1}^{n} \ell_i^{-1} \pi_G(\mathcal{F})$

□

model sets are relatively dense

since $\mathring{W} \neq \varnothing$, we can apply the previous lemma

- there is cpct $F \subset G$ such that

$$(F \times W^{-1})\mathcal{L} = G \times H$$

In fact $F \curlywedge(W) = G$:

- shift (x, e) to fundamental domain:

$$(y, w^{-1})(\ell, \ell^{\star}) = (x, e)$$

  for some $y \in F$, $w \in W$, $(\ell, \ell^{\star}) \in \mathcal{L}$
- hence $\ell^{\star} = w$ and $\ell \in \curlywedge(W)$, which means

$$x = y\ell \in F \curlywedge(W)$$

□

repetition of patterns in model sets

For nonempty Λ and compact K such that $\Lambda \cap K \neq \varnothing$ consider

$$T_K(\Lambda) = \{t \in G : \Lambda \cap K = t^{-1}\Lambda \cap K\},$$

the set of *K-periods of Λ*

Proposition

Λ non-empty weak model set $\Longrightarrow$ $T_K(\Lambda)$ non-empty weak model set

- remember

$$T_K(\Lambda) = \{t \in G : \Lambda \cap K = t^{-1}\Lambda \cap K\} \subset \Lambda\Lambda^{-1}$$

- for a model set $\Lambda = 人(W)$ we have

$$T_K(人(W)) = \{\ell_K \in L \,:\, 人(W) \cap K = 人({\ell_K^\star}^{-1} W) \cap K\}$$

- hence $\ell_K \in T_K(人(W))$ iff

$$\ell_K^\star \in {\ell^\star}^{-1} W \qquad \forall \ell \in 人(W) \cap K$$

$$\ell_K^\star \notin {\ell^\star}^{-1} W \qquad \forall \ell \in 人(W^c) \cap K$$

- hence $T_K(人(W)) = 人(W_K)$ with

$$W_K = \bigcap_{\ell \in 人(W) \cap K} {\ell^\star}^{-1} W \;\setminus\; \bigcup_{\ell \in 人(W^c) \cap K} {\ell^\star}^{-1} W$$

- W_K rel cpct since W rel cpct and $人(W) \cap K$ nonempty finite

□

Proposition

Let $\curlywedge(W)$ be a (non-empty weak) model set with generic window

$$L^\star \cap \partial W = \varnothing.$$

Then $\curlywedge(W)$ is repetitive, i.e., $T_K(\curlywedge(W))$ rel dense for all cpct K.

In that case the above W_K is a unit neighborhood:

- $L^\star \cap \partial W = \varnothing$ implies $e \in \operatorname{int}(\ell^{\star -1} W)$ for all $\ell \in \curlywedge(W)$
- $L^\star \cap \partial W = \varnothing$ implies $e \in \operatorname{int}(\ell^{\star -1} W^c)$ for all $\ell \in \curlywedge(W^c)$
- hence $W' = \bigcap_{\ell \in \curlywedge(W) \cap K} \ell^{\star -1} W$ rel cpct unit neighborhood
- But W' intersects only finitely many $\ell^{\star -1} W$ where $\ell \in L \cap K$.
  (note $W' \cap \ell^{\star -1} W \neq \varnothing$ implies $\ell \in \curlywedge(WW'^{-1})$ uniformly discrete)
- Hence W_K unit neighborhood due to

$$W_K = W' \setminus \bigcup_{\ell \in \curlywedge(W) \cap K} \ell^{\star -1} W = W' \cap \bigcap_{\ell \in \curlywedge(W^c) \cap K} \ell^{\star -1} W^c \qquad \square$$

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Proposition

If W is any window, then there exists c such that cW is generic.

S is nowhere dense if $\mathring{\bar{S}} = \varnothing$.
M is meagre if it is a countable union of nowhere dense sets.

Lemma (Baire)

Any meagre set has nonempty interiour.

Proof of Proposition.

∂W nowhere dense, $L^\star$ countable, hence $L^\star \partial W$ meagre.
Baire: $L^\star \partial W$ has nonempty interiour, in particular $L^\star \partial W \neq H$.
Hence $c^{-1} \notin L^\star \partial W$, hence $c\partial W \cap L^\star = \varnothing$ □

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holes in weak model sets

"weak model sets may have arbitrarily large holes"

$\Lambda \subset G$ is *hole-repetitive* if for every compact set $K \subset G$ the set

$$\{t \in G \mid t^{-1}K \cap \Lambda = \varnothing\}$$

is relatively dense in G.

Proposition (R–Huck 14)

Let $(G, H, \mathcal{L})$ be a cut-and-project scheme. If W is nowhere dense, i.e., $\overline{\mathring{W}} = \varnothing$, then $⅄(W)$ is hole-repetitive.

Example: If W cpct and $\mathring{W} = \varnothing$, then $W = \partial W$ is nowhere dense.

proof of hole-repetitivity

- Baire: since $L^\star$ is countable, there is $c \in H$ such that

$$L^\star \cap cW = \varnothing \iff (G \times cW) \cap \mathcal{L} = \varnothing$$

- hence $(K \times cW) \cap \mathcal{L} = \varnothing$ for any compact $K \subset G$
- take small unit nbhd U such that still $(K \times UcW) \cap \mathcal{L} = \varnothing$
- for any ℓ from the relatively dense $\curlywedge(Uc)$ we have

$$\varnothing = (K \times \ell^\star W) \cap \mathcal{L} = (\ell^{-1} K \times W) \cap \mathcal{L}$$

- hence $\ell^{-1} K \cap \curlywedge(W) = \varnothing$

□

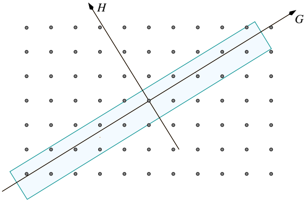
density formula (Meyer 72, Schlottmann 98, Moody 02)

count points within balls or van Hove sequence $(B_r)_{r\in\mathbb{N}}$:

- Λ lattice:

$$|\Lambda \cap B_r| = \mathrm{dens}(\Lambda)\cdot \mathrm{vol}(B_r) + o(\mathrm{vol}(B_r))$$

- $\curlywedge(W)$ regular model set with measurable W:



$$|\curlywedge(W)\cap B_r| = \mathrm{dens}(\mathcal{L})\cdot \mathrm{vol}(W)\mathrm{vol}(B_r) + o(\mathrm{vol}(B_r))$$

(convergence uniform in shifts of W and center of balls)

density formula for weak model sets

- consider relative point frequencies

$$f_r = \frac{1}{\mathrm{vol}(B_r)} \, |\curlywedge(W) \cap B_r|$$

- average with "van Hove sequences" $(B_r)_{r\in\mathbb{N}}$:
  compact sets of positive volume such that for all compact K

$$\lim_{n\to\infty} \frac{\mathrm{vol}(\partial^K A_n)}{\mathrm{vol}(A_n)} = 0,$$

- with the *(generalised) van Hove boundary*

$$\partial^U W = (U\overline{W} \cap \overline{W^c}) \cup (U\overline{W^c} \cap \overline{W}).$$

- e.g. balls, rectangles of diverging inradius, Følner sequences

density formula for weak model sets

Lemma (density formula for weak model sets)

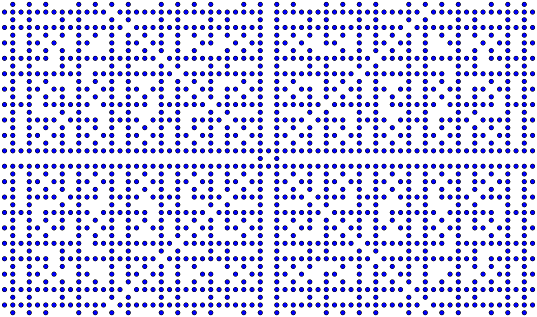
$\curlywedge(W)$ weak model set, $(B_r)_{r\in\mathbb{N}}$ van Hove sequence. Then

$$\mathrm{dens}(\mathcal{L})\,\mathrm{vol}(\mathring{W}) \leq \liminf_{r\to\infty} f_r \leq \limsup_{r\to\infty} f_r = \mathrm{dens}(\mathcal{L})\,\mathrm{vol}(\overline{W}).$$

- regular model sets with measurable W: $f_r \to \mathrm{dens}(\mathcal{L})\,\mathrm{vol}(W)$
- later: proof for regular model sets via harmonic analysis
- general case by approximation with regular model sets

a number theory quasicrystal

visible lattice points



arbitrarily large holes, positive pattern entropy, pp diffraction!

visible lattice points $V = \mathbb{Z}^2 \setminus \bigcup_p p\mathbb{Z}^2$

$$\begin{aligned}(r,s) \text{ visible } &\Leftrightarrow (r,s) \neq p(r',s') \text{ for all primes}\\ &\Leftrightarrow (r,s) \mod p\mathbb{Z}^2 \neq 0 \text{ for all primes}\end{aligned}$$

cut-and-project scheme (Sing 05)

- number-theoretic sieve $H = \prod_p \mathbb{Z}^2/p\mathbb{Z}^2$
- star map $x^\star = (x \mod p\mathbb{Z}^2)_p$

window $W = \prod_p (\mathbb{Z}^2/p\mathbb{Z}^2) \setminus \{0\}$

- $\overline{W} = W$ since every component is closed
- $\mathring{W} = \varnothing$ since no component is maximal
- hence $W = \partial W$

visible lattice points

hence volume of the window is

$$\mathrm{vol}(W) = \prod_p \left(1 - \frac{1}{p^2}\right) = \frac{1}{\zeta(2)}$$

- for a sequence $(B_r)_r$ of balls about 0 one computes

$$\mathrm{dens}(V) = \mathrm{vol}(W)$$

- V is a weak model set. For the above averaging sequence, it has maximal density!
- This is similar to regular model sets, which have maximal density for every van Hove sequence.

pattern entropy

- count point configurations in translates of $B \subset \mathbb{R}^2$

$$N_B^*(V) = \left|\{x^{-1}V \cap B \,|\, x \in V\}\right|$$

- configurational entropy

$$h^*(V) = \limsup_{r\to\infty} \frac{1}{\mathrm{vol}(B_r)} \log N_{B_r}^*(V)$$

- alternatively, V may be viewed as a 01-colouring of $\mathbb{Z}^2$

$$N_B^*(V, \mathbb{Z}^2) = |\{1_{x^{-1}V\cap B} \,|\, x \in V\}|$$

- for a sequence $(B_r)_r$ of balls about 0 one calculates

$$h^*(V, \mathbb{Z}^2) = \mathrm{vol}(\partial W) \log 2!$$

pattern entropy of weak model sets

Theorem (R-Huck 14)

Let $\curlywedge(W)$ be a weak model set, and let $\curlywedge(W) \subset \Lambda_0$ for some regular model set Λ_0. Then

$$h^*(\curlywedge(W)) \leq h^*(\curlywedge(W), \Lambda_0) \leq \mathrm{dens}(\mathcal{L}) \cdot \mathrm{vol}(\partial W) \cdot \log 2$$

- conjectured by Moody–Pleasants 06
- geometric proof with standard estimate
- also for non-commutative cp-schemes with $\mathcal{L}$ normal in $G \times H$

note:

- regular model sets have 0 entropy
- visible lattice points have maximal entropy

step 1: lift centered patterns to $G \times H$

- we bound

$$N_B^*(\curlywedge(W)) = \left|\{x^{-1}\curlywedge(W) \cap B \,|\, x \in \curlywedge(W)\}\right|$$

(proof for $N_B^*(\curlywedge(W), \Lambda_0)$ analogous)

- bound number of G-inequivalent centered patterns in G

$$xB \cap \curlywedge(W), \qquad x \in \curlywedge(W)$$

- bound no of $(G \times H)$-inequivalent centered patterns in $G \times H$

$$(xB \times W) \cap \mathcal{L} = \pi_G^{-1}(xB \cap \curlywedge(W)), \qquad x \in \curlywedge(W)$$

step 2: shift pattern centers to fundamental domain

fundamental domain $\mathcal{F}$

- choose cpct $F \times U$ such that $(F \times U)\mathcal{L} = G \times H$
- $F \times U$ contains fundamental domain $\mathcal{F}$ of $\mathcal{L}$

shift (x, e) to fundamental domain $\mathcal{F}$

- $\ell(x, e) = (y, u)$ for some $(y, u) \in \mathcal{F}$ and $\ell \in \mathcal{L}$
- pattern center $(x, x^\star) \in (G \times W) \cap \mathcal{L}$ gets shifted to

$$\ell(x, x^\star) = (y, ux^\star) \in (F \times UW) \cap \mathcal{L}$$

shifted patterns

- F compact, hence only finitely many values $y = y(x)$
- shifted patterns with same $y = y(x)$ are similar:

$$xB \cap 人(W) \text{ shifted to some subset of } yB \cap 人(UW)$$

step 2: shift pattern centers to fundamental domain

- all such patterns differ only "near the boundary" of W, i.e., on

$$yB \cap \curlywedge(\partial^U W)$$

- hence a standard estimate yields

$$N_B^*(\curlywedge(W)) \leq |(F \times UW) \cap \mathcal{L}| \cdot 2^{|FB \cap \curlywedge(\partial^U W)|}$$

- for $r \to \infty$ the density formula yields

$$h^*(\curlywedge(W)) \leq \limsup_{r \to \infty} \frac{1}{\mathrm{vol}(B_r)} \left| FB_r \cap \curlywedge(\partial^U W) \right| \cdot \log 2$$
$$\leq \mathrm{dens}(\mathcal{L}) \cdot \mathrm{vol}(\partial^U W) \cdot \log 2$$

step 3: choose arbitrarily thin fundamental domains

remember: as $\pi_H(\mathcal{L})$ is dense in H, we have

Lemma

Let $(G, H, \mathcal{L})$ be a cut-and-project scheme. Then for any non-empty open $U \subset H$ there exists compact $F \subset G$ satisfying

$$(F \times U)\mathcal{L} = G \times H.$$

as H is metrisable, we can use dominated convergence to infer

$$\lim_{U \to \{e\}} \mathrm{vol}(\partial^U W) \to \mathrm{vol}(\partial^{\{e\}} W) = \mathrm{vol}(\partial W),$$

and the entropy estimate follows. $\square$

outlook

observations:

- visible lattice points are *hereditary systems*:
  every subset of a pattern is a translated pattern
- visible lattice points have maximal density
- pattern entropy $h^*(V, \mathbb{Z}^2)$ equals topological entropy of the hull $\mathbb{X}_V = \overline{\{tV \mid t \in \mathbb{R}^2\}}$ of V

hence:

- study hereditary systems!
- study weak model sets of maximal density!
- study relation to topological entropy of $\mathbb{X}_{⅄(W)}$!

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