Lattice paths below a line of rational slope

74th Sï¿½minaire Lotharingien de Combinatoire (@Ellwangen)

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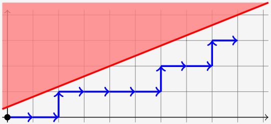
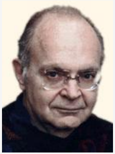
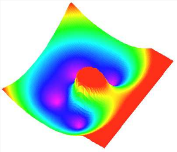






Knuth's AofA'14 problem #4

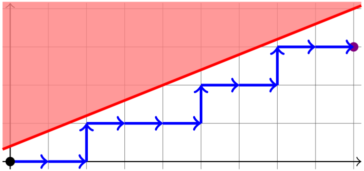
"Problems that Philippe Flajolet would have loved"[1] (Don Knuth)

Fourth problem: "Lattice paths of slope 2/5"

- Dyck paths, i.e. $\mathcal{S} = \{(1,0),(0,1)\}$,
- Under line of slope 2/5, i.e.

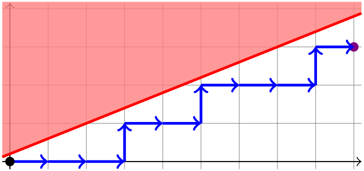
Model A

$y < \frac{2}{5}x + \frac{2}{5}$



Model B

$y < \frac{2}{5}x + \frac{1}{5}$



[1] http://www-cs-faculty.stanford.edu/~uno/flaj2014.pdf

Knuth's AofA'14 problem #4 - Original Slide 1

$$A[i,j] = \begin{cases} 0, & \text{if } j \geq 2i/5 + 2/5, \\ A[i-1,j] + A[i,j-1], & \text{if } j < 2i/5 + 2/5; \end{cases}$$

$$B[i,j] = \begin{cases} 0, & \text{if } j \geq 2i/5 + 1/5, \\ B[i-1,j] + B[i,j-1], & \text{if } j < 2i/5 + 1/5; \end{cases}$$

$A[i,0] = B[i,0] = 1$. When $0 \leq i \leq 4$ and $0 \leq j \leq 10$ we have:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 0 & 0 & 0 & 0 & 0 & 4 & 9 & 15 & 22 & 30 & 39 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 15 & 37 & 67 & 106 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 106 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 0 & 0 & 0 & 0 & 3 & 7 & 12 & 18 & 25 & 33 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 18 & 43 & 76 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 76 \end{pmatrix}$$

Knuth's AofA'14 problem #4 - Original Slide 2

Thus $A[x,y]$ enumerates lattice paths from $(0,0)$ that stay in the region $y < \frac{2}{5}x + \frac{2}{5}$, while $B[x,y]$ enumerates the paths that stay in the region $y < \frac{2}{5}x + \frac{1}{5}$.

Theorem (Nakamigawa, Tokushige, 2012)**:**

$$A[5t-1,2t-1]+B[5t-1,2t-1] = \frac{2}{7t-1}\binom{7t-1}{2t}, \quad \text{for all } t \geq 1.$$

Empirical observation:

$$\frac{A[5t-1,2t-1]}{B[5t-1,2t-1]} = a - \frac{b}{t} + O(t^{-2}),$$

where $a \approx 1.63026$ and $b \approx 0.159$ (I think).

Lattice paths below a rational slope are directed paths

Folklore proposition (bijection to directed paths)

Let $\boldsymbol{L : y = \frac{a}{c}x + \frac{b}{c}}$ be the barrier of rational slope. Assume $a, b, c \in \mathbb{N}$ s.t. $gcd(a, b, c) = 1$. There exists a bijection between

"lattice paths starting from the origin with North and East steps"

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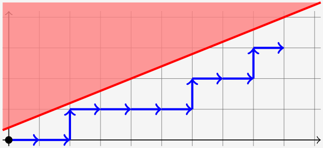
Staying *below L* is mapped to staying *above* the x-axis.



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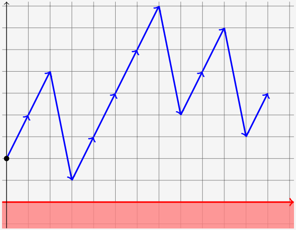

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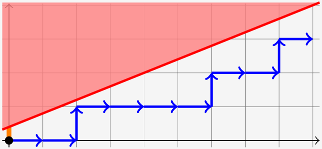

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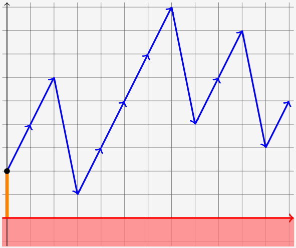

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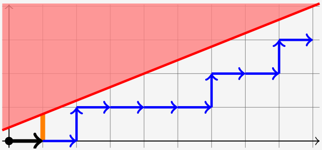

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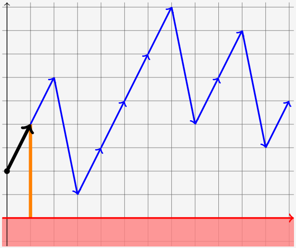

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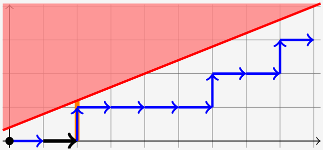

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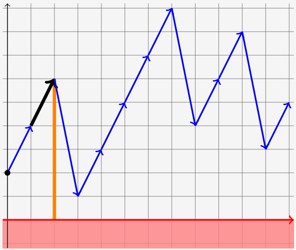

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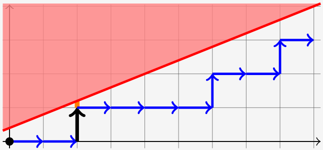

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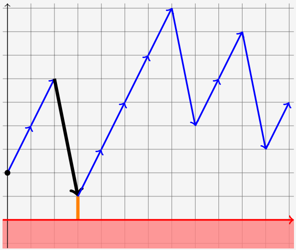

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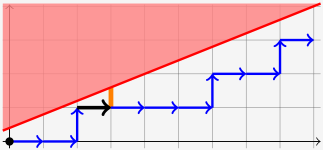

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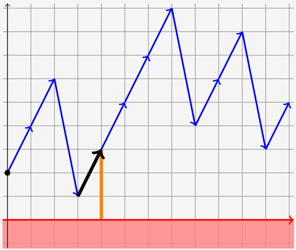

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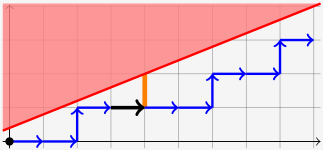

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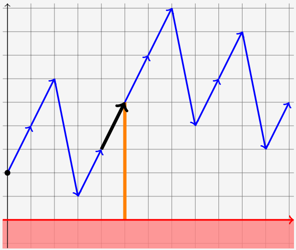

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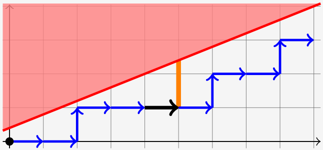

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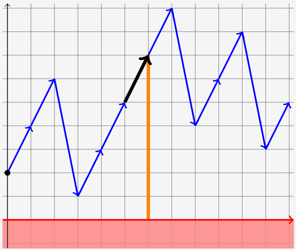

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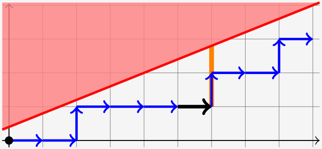

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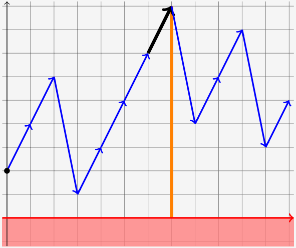

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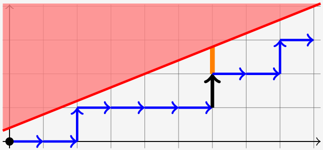

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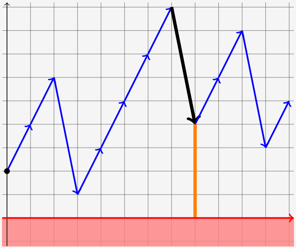

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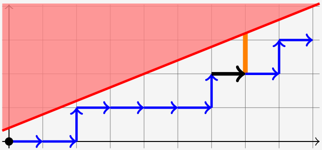

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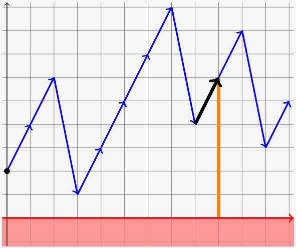

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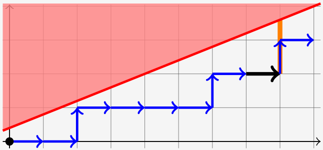

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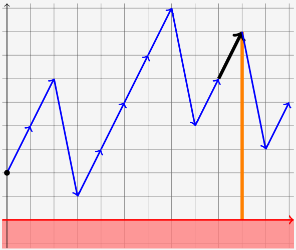

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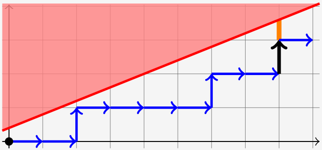

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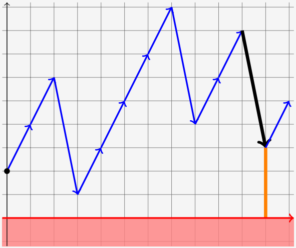

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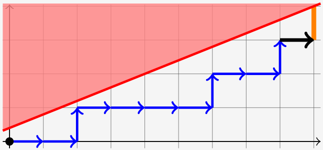

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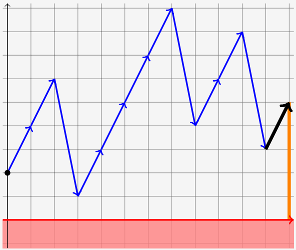

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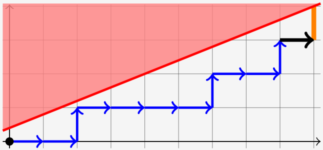

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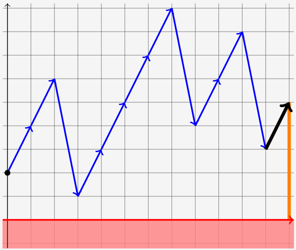
works for any jumps!



Generating functions - construction

1. *Bivariate generating function*

$$F(z,u) = \sum_{n,k\geq 0} \underbrace{F_{n,k}}_{\substack{\text{length: } n\\ \text{altitude: } k}} z^n u^k = \sum_{n\geq 0} \underbrace{f_n(u)}_{\text{length: } n} z^n = \sum_{k\geq 0} \underbrace{F_k(z)}_{\text{altitude: } k} u^n$$

2. Stepset $\mathcal{S} = \{-2, 5\}$ gives *jump polynomial*

$$P(u) = u^{-2} + u^5$$

3. *Recursive construction*

$$f_0(u) \in \mathbb{N}[u], \qquad f_{n+1}(u) = \{u^{\geq 0}\}\left[P(u) f_n(u)\right], \text{ for } n \geq 0$$

4. *One functional equation (with 3 unknowns!)*

$$(1 - zP(u))F(z,u) = f_0(u) - zu^{-2}F_0(z) - zu^{-1}F_1(z)$$

5. *Kernel equation*

$$1 - zP(u) = 0$$

For $z \sim 0$ we get:

- 2 small roots $u_1(z)$ and $u_2(z)$ ($u_i(z) \to 0$ for $z \to 0$)
- 5 large roots $v_1(z), \ldots, v_5(z)$ ($|v_j(z)| \to \infty$ for $z \to 0$)

Generating functions

6 Inserting the small branches gives linear system with 2 equations:

$$\underbrace{(1 - zP(u_i))}_{=0} F(z,u) = f_0(u_i) - z u_i^{-2} F_0(z) - z u_i^{-1} F_1(z), \quad \text{for } i = 1, 2.$$

Theorem (Banderier–Wallner)

$$F_0(z) = -\frac{u_1 u_2 \left(u_1 f_0(u_1) - u_2 f_0(u_2)\right)}{z(u_1 - u_2)}, \qquad F_1(z) = \frac{u_1^2 f_0(u_1) - u_2^2 f_0(u_2)}{z(u_1 - u_2)}$$

Generating functions

6 Inserting the small branches gives linear system with 2 equations:

$$\underbrace{(1 - zP(u_i))}_{=0} F(z,u) = f_0(u_i) - z u_i^{-2} F_0(z) - z u_i^{-1} F_1(z), \quad \text{for } i = 1,2.$$

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Model A

- Walk from $(0,4)$ to $(7n-2,1)$
- $f_0(u) := u^4$
- $\boldsymbol{A_1(z) := F_1(z) = \frac{u_1^6 - u_2^6}{z(u_1 - u_2)}}$
- $A[5n-1, 2n-1] = [z^{7n-2}]A_1(z)$ $(=: A_n)$

Model B

- Walk from $(0,3)$ to $(7n-2,0)$
- $f_0(u) := u^3$
- $\boldsymbol{B_0(z) := F_0(z) = -\frac{u_1 u_2 \left(u_1^4 - u_2^4\right)}{z(u_1 - u_2)}}$
- $B[5n-1, 2n-1] = [z^{7n-2}]B_0(z)$ $(=: B_n)$

Closed form for the sum of coefficients

Theorem [Nakamigawa and Tokushige (2012)]

$$A_n + B_n = \frac{2}{7n-1}\binom{7n-1}{2n}$$

See also: [Mohanty79, Sato89]. (here, clever use of cyclic lemma/Désiré André reflection principle).

No other linear combination $rA_n + sB_n$ leads to a hypergeometric solution (investigated by Manuel Kauers)

Knuth's conjecture

$$\frac{A_n}{B_n} = \kappa_1 - \frac{\kappa_2}{n} + \mathcal{O}(n^{-2}),$$

with

$$\kappa_1 \approx 1.63026 \text{ and } \kappa_2 \approx 0.159.$$

Universal square root behavior of u_1

Lemma (Banderier–Flajolet, 2002)

The principle small branch u_1 of the kernel equation $1 - zP(u) = 0$ possess the following asymptotic expansion as a Newton-Puiseux series:

$$u_1(z) = \tau - C\sqrt{1 - z/\rho} + \mathcal{O}(1 - z/\rho), \qquad \text{for } z \to \rho^-.$$

Constants

- *Structural constant* $\tau > 0$:
  unique positive real root of $P'(t) = 0$
- *Structural radius* $\rho > 0$: $\rho = \frac{1}{P(\tau)}$
- $C := \sqrt{2\frac{P(\tau)}{P''(\tau)}}$

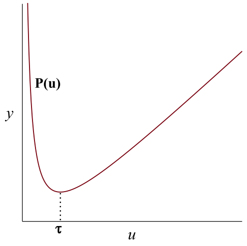


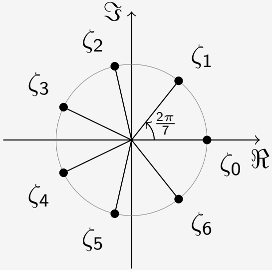
Figure: Jump polynomial $P(u)$ and unique saddle point $\tau > 0$

Periodic lattice paths

- **Periodic Lattice paths:** $\exists p \in \mathbb{N}, \exists H(u) \in \mathbb{R}[u]$ such that $P(u) = u^b H(u^p)$ with $b \in \mathbb{Z}$
- Here: period $p = 7$ for $P(u) = u^{-2} + u^5 = u^{-2}H(u^7)$ with $H(u) = 1 + u$.
- Singularity of u_i determined by $P'(t) = 0$, i.e. $H'(u^7(t)) = 0$

$\Rightarrow$ 7 possible singularities of the small branches u_1 and u_2 at

$$\zeta_k = \rho\omega^k, \qquad \text{with } \omega = e^{2\pi i/7}.$$



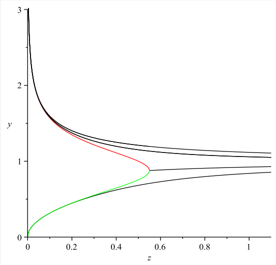


Figure: At ρ the small root u_1 (in green) meets the large root v_1 (in red), with a square root behavior. (In black, we also plotted $|u_2|, |v_2|, |v_3|, |v_4|, |v_5|$.)

Dominant singularities

Lemma - local behavior (short) (Banderier–Wallner)

Let $\omega = e^{2\pi i/7}$ and $\zeta_k = \rho\omega^k$. Then at every k exactly one small branch is singular and the other one is analytic. u_1 is singular at $k = 0, 2, 5$ and u_2 is singular at $k = 1, 3, 4, 6$.

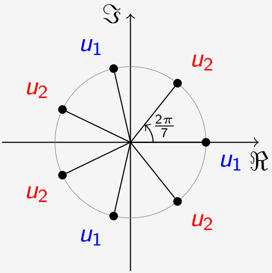


Figure: Singular branches

A rotation law (Banderier–Wallner)

For all $z \in \mathbb{C}$, with $|z| \leq \rho$ and $-\pi < \arg(z) < \pi - 2\pi/7$:

$$u_1(\omega z) = \omega^{-3} u_2(z),$$

$$u_2(\omega z) = \omega^{-3} u_1(z).$$

Rotation law: proof

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$$u_2(\omega z) = \omega^{-3} u_1(z).$$

Define $U(z) := \omega^3 u_1(\omega z)$ and consider

$$X(z) = U^2 - z\phi(U),$$

where $\phi(u) := u^2 P(u) = 1 + u^7$ from the kernel equation $1 - zP(u) = 0$. Next

$$\omega X(z/\omega) = u_1(z)^2 - z\phi(u_1(z)) = 0,$$

as we recognize the entire form of the kernel equation. Thus, U is a root of the kernel. Which one?

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as we recognize the entire form of the kernel equation. Thus, U is a root of the kernel. Which one?

- It must be a small one, as $U(z) \sim 0$ for $z \sim 0$.
- It is not $u_1(z)$ as it has a different Puiseux expansion.
- Hence, it is $u_2(z)$! (Analytic continuation as long as we avoid $\arg(z) = -\pi$.)

Local asymptotics: definition

Definition: Local asymptotics extractor $[z^n]_{\zeta_k}$

Let $F(z)$ be a GF with p dominant singularities ζ_k (for $k = 1, \dots, p$). Define

$$[z^n]_{\zeta_k} F(z) := [z^n](\textit{Puiseux expansion} \text{ of } F(z) \text{ at } z = \zeta_k)$$

Example

Let $F(z) = \frac{1}{1-z^2} = \frac{1}{(1-z)(1+z)}$. We have $p = 2$ dominant singularities $\zeta_1 = 1$ and $\zeta_2 = -1$. Then we get

$$[z^n]_{\zeta_1} F(z) = [z^n] \frac{1}{2(1-z)} = \frac{1}{2}, \qquad \text{for all } n \geq 0,$$

$$[z^n]_{\zeta_2} F(z) = [z^n] \frac{1}{2(1+z)} = \frac{(-1)^n}{2}, \qquad \text{for all } n \geq 0.$$

Then it holds:

$$[z^n]F(z) = [z^n]_{\zeta_1} F(z) + [z^n]_{\zeta_2} F(z) = \begin{cases} 1, & \text{for } n = 2k, \\ 0, & \text{otherwise.} \end{cases}$$

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Then it holds:

$$[z^n]F(z) = [z^n]_{\zeta_1} F(z) + [z^n]_{\zeta_2} F(z) = \begin{cases} 2[z^n]_{\zeta_1} F(z), & \text{for } n = 2k, \\ 0, & \text{otherwise.} \end{cases}$$

Local asymptotics: proposition

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Proposition (Banderier–Wallner)

Let $F(z)$ be a GF with non-negative coefficients. Let ρ be the positive real dominant singularity. When additionally the function $F(z)$ satisfies a rotation law $F(\omega z) = \omega^m F(z)$ (where $\omega = \exp(2\pi i/p)$), then it holds that

$$[z^n]F(z) = \begin{cases} \boldsymbol{p}[z^n]_\rho F(z)(1 + o(\rho^n)), & \text{if } p|(n-m), \\ 0, & \text{otherwise.} \end{cases}$$

Local asymptotics: proof

Proposition

Let $F(z)$ be a GF with non-negative coefficients. Let ρ be the positive real dominant singularity. When additionally the function $F(z)$ satisfies a rotation law $F(\omega z) = \omega^m F(z)$ (where $\omega = \exp(2\pi i/p)$), then it holds that

$$[z^n]F(z) = \boldsymbol{p}\, \chi_p(n-m)\, [z^n]_\rho F(z)(1 + o(\rho^n))$$

where $\chi_p(n-m)$ is 1 if $n-m$ is a multiple of p, 0 elsewhere.

Due to Pringsheim's Theorem a positive real dominant sing. ρ is guaranteed. Relabel ζ_k such that $\zeta_k = \omega^k \rho$, then

$$\begin{aligned}
[z^n]F(z) - o(\rho^n) &= \sum_{k=1}^{p} [z^n]_{\zeta_k} F(z) = \sum_{k=1}^{p} [z^n]_{\zeta_k} (\omega^m)^k F(\omega^{-k} z) \\
&= \sum_{k=1}^{p} (\omega^m)^k (\omega^{-k})^n [z^n]_\rho F(z) = \left(\sum_{k=1}^{p} (\omega^k)^{m-n} \right) [z^n]_\rho F(z) \\
&= p\, \chi_p(n-m)\, [z^n]_\rho F(z)
\end{aligned}$$

Application to Knuth's problem

From the local behavior of $u_1(z)$ and $u_2(z)$ we get the **rotation law**

$$A_1(\omega z) = \omega^{-2} A_1(z), \qquad B_0(\omega z) = \omega^{-2} B_0(z).$$

Hence, we have period $p = 7$ and $m = -2$. Thus, it is sufficient to compute the singular expansion of $A_1(z)$ and $B_0(z)$ at $z = \rho$ and multiply it with 7 to get:

$$A_n = [z^{7n-2}]A_1(z) = \alpha_1 \frac{\rho^{-7n}}{\sqrt{\pi(7n-2)^3}} + \frac{3\alpha_2}{2} \frac{\rho^{-7n}}{\sqrt{\pi(7n-2)^5}} + \mathcal{O}(n^{-7/2}),$$

$$B_n = [z^{7n-2}]B_0(z) = \beta_1 \frac{\rho^{-7n}}{\sqrt{\pi(7n-2)^3}} + \frac{3\beta_2}{2} \frac{\rho^{-7n}}{\sqrt{\pi(7n-2)^5}} + \mathcal{O}(n^{-7/2}),$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are some real constants.

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where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are some real constants.
Finally we directly get

$$\frac{A_n}{B_n} = \kappa_1 - \frac{\kappa_2}{n} + \mathcal{O}(n^{-2}) = \frac{\alpha_1}{\beta_1} + \frac{3}{14}\left(\frac{\alpha_1\beta_2 - \alpha_2\beta_1}{\beta_1^2}\right)\frac{1}{n} + \mathcal{O}(n^{-2}).$$

Hence, we have shown that

$\boldsymbol{\kappa_1} \approx$ **1.6302576629903501404248**, $\boldsymbol{\kappa_2} \approx$ **0.1586682269720227755147**.

Closed form Solution of Knuth's problem

Asymptotics of Knuth's problem

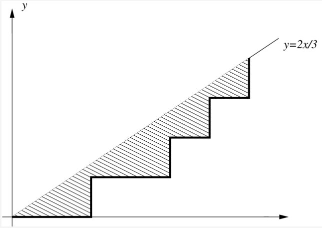
$$\frac{A_n}{B_n} = \kappa_1 - \frac{\kappa_2}{n} + \mathcal{O}(n^{-2}) \text{ with}$$
$$\kappa_1 \approx 1.6302576629903501404248,$$
$$\kappa_2 \approx 0.1586682269720227755147.$$

- κ_1 is the unique real root of the polynomial
$$23x^5 - 41x^4 + 10x^3 - 6x^2 - x - 1$$
- $(7/3)\kappa_2$ is the unique real root of the polynomial
$$11571875x^5 - 5363750x^4 + 628250x^3 - 97580x^2 + 5180x - 142$$
- The Galois group of each of these polynomials is S_5
$\Rightarrow$ No closed form formula in terms of basic operations on integers, and root of any degree.

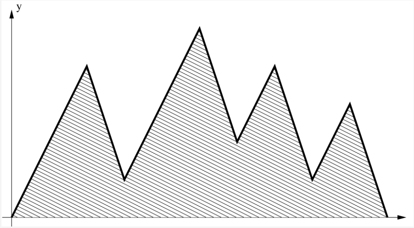
Duchon's club

Model

Number of "histories" of couples entering a club, and exiting by 3. What is the number of possible histories, if the club is closing empty?



(a) North-East model: Dyck paths below the line of slope 2/3



(b) Banderier–Flajolet model: excursions with $+2$ and -3 jumps

- Duchon conjectured average area as $Kn^{3/2}$ with $K \approx 3.43$
- $\boldsymbol{K = \frac{\sqrt{15\pi}}{2} \approx 3.432342124}$ (together with Bernhard Gittenberger).

Conclusion

- We solved Knuth's and Duchon's conjectures :-)
- We got a generic approach (any jumps!) to deal with enumeration and asymptotics of lattice paths below a rational slope.
- En passant, nice "closed form" formulae.
- Rigorous proof of the periodic case (more tricky).

Open (computer algebra) questions:

- a computer algebra package dealing with algebraic functions (and their asymptotics). Theory = Newton–Puiseux + Flajolet–Salvy ACA algorithm. Caveat: Maple (nested) "RootOf" not able to follow the right branch.
- How to efficiently go from the differential equation to the algebraic equation, and conversely?
- Is there a way to handle irrational slopes?