Evolution equations in Combinatorial Physics

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INTRODUCTION & CONTEXT

Context

This is a starting project with a group of physicists (second group of names). The aim is to provide methods in order to compute Dyson series based on a tractable (combinatorial) indexing. The advantage of the presented method are the following

- Easy implementation
- Eased combinatorial analysis of the outputs, the basic indexing being provided by noncommutative and (perspective) partially commutative words
- Possible factorization in infinite products (Schützenberger's factorization)

Linear differential equations and Dyson series

Let us start with the simplest (linear) case:
∂_z denotes d/dz and $a_0, \ldots, a_n \in \mathcal{A}$ (some full algebra of $\mathbb{C}$-valued functions).

We are in search of solutions of

$$a_n(z)\partial_z^n y(z) + \ldots + a_1(z)\partial_z y(z) + a_0(z)y(z) = 0 \tag{1}$$

Which can be written in matrix form as

$$(ED) \quad \left\{ \begin{array}{rcl} \partial_z q(z) & = & A(z)q(z), \\ q(z_0) & = & \eta, \\ y(z) & = & \lambda q(z), \end{array} \right. \tag{2}$$

with

$$A(z) = (A_{i,j}(z))_{i,j=1..n} \in \mathcal{M}_{n,n}(\mathcal{A}), \lambda \in \mathcal{M}_{1,n}(k), \eta \in \mathcal{M}_{n,1}(k).$$

... and implies

$$q(z) = q(z_0) + \int_{z_0}^{z} A(s)q(s)ds \tag{3}$$

Linear differential equations and Dyson series II

By successive Picard iterations w.r.t. the integral eq.3, we get

$$q_0 \equiv q(z_0);\ \ q_k(z) = q(z_0) + \int_{z_0}^{z} A(s) q_{k-1}(s) ds$$

with the initial point $q_0(z) = \eta$, one gets $y(z) = \lambda U(z_0; z)\eta$,

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with the initial point $q_0(z) = \eta$, one gets $y(z) = \lambda U(z_0; z)\eta$, where $U(z_0; z)$ is the solution of the differential system

$$(EDR) \qquad \left\{ \begin{array}{rcl} \partial_z U(z_0, z) & = & A(z) U(z_0, z), \\ U(z_0, z_0) & = & I_{n \times n} \end{array} \right. \tag{4}$$

and U satisfies the functional expansion

$$U(z_0; z) = \sum_{k \geq 0} \int_{z_0}^{z} A(s_1) ds_1 \int_{z_0}^{s_1} A(s_2) ds_2 \ldots \int_{z_0}^{s_{k-1}} A(s_k) ds_k \tag{5}$$

also called Dyson series.

An example with singularities (familiar to combinatorists)

Example (Hypergeometric equation)

Let a, b, c be parameters and

$$z(1-z)\partial_z^2 y(z) + [c-(a+b+1)z]\partial_z y(z) - aby(z) = 0.$$

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Let $q_1(z) = y(z)$ and $q_2(z) = (1-z)\partial_z y(z)$. One has

$$\begin{aligned} A\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} &= \begin{pmatrix} 0 & \dfrac{1}{1-z} \\ \dfrac{ab}{z} & \dfrac{a+b-c}{1-z} - \dfrac{c}{z} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \\ &= \left[\begin{pmatrix} 0 & 0 \\ -ab & -c \end{pmatrix} \frac{1}{z} - \begin{pmatrix} 0 & 1 \\ 0 & c-a-b \end{pmatrix} \frac{1}{1-z} \right] \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}. \end{aligned}$$

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A_0 &= \begin{pmatrix} 0 & 0 \\ -ab & -c \end{pmatrix},\ A_1 = \begin{pmatrix} 0 & 1 \\ 0 & c-a-b \end{pmatrix}.
\end{aligned}$$

FORMAL SETTING

Drinfel'd's symbolic calculus (iterated integrals and exponential form)

The trick is to replace the matrices by (formal, noncommuting) letters (in our example, it is $\{x_0, x_1\}$).
From the Dyson series of eq. 5, this replacement provides a formal power series

$$S = \sum_{w \in X} \langle S|w\rangle w \tag{6}$$

satisfying

$$S' = (\frac{x_0}{z} + \frac{x_1}{1-z})S = MS \tag{7}$$

where the coefficients $\langle S|w\rangle$ of S are analytic functions on a (open, connected and simply connected) domain $\Omega \subset \mathbb{C}$ ($\Omega = \mathbb{C} - (]-\infty, 0] \cup [1, +\infty[)$ is usually considered).

Chen's iterated integral along a path and polylogarithms

The iterated integral, along $z_0 \rightsquigarrow z$ in $\mathbb{C} - (]-\infty, 0] \cup [1, +\infty[)$ and associated to $w = x_{i_1} \cdots x_{i_k}$, over $\omega_0(z) = z^{-1}dz$ and $\omega_1(z) = (1-z)^{-1}dz$ is defined by

$$\alpha_{z_0}^{z}(1_{X^*}) = 1 \quad \text{and} \quad \alpha_{z_0}^{z}(x_{i_1} \ldots x_{i_k}) = \int_{z_0}^{z} \omega_{i_1}(z_1) \ldots \int_{z_0}^{z_{k-1}} \omega_{i_k}(z_k).$$

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For any $w = x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1 \in X^*x_1$,

$$\alpha_{z_0}^{z}(w) = \sum_{n_1 > \ldots > n_r > 0} \frac{z^{n_1}}{n_1^{s_1} \ldots n_r^{s_r}} = \mathrm{Li}_{s_1,\ldots,s_r}(z), \qquad |z| < 1.$$

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Example (In this case, the word ending by x_1, one can take $z_0 = 0$)

$$\alpha_0^{z}(x_0x_1) = \mathrm{Li}_2(z) = \int_0^z \frac{ds_1}{s_1} \int_0^{s_1} \frac{ds_2}{1-s_2}$$

$$= \int_0^z \frac{ds_1}{s_1} \int_0^{s_1} ds_2 \sum_{k \geq 0} s_2^k = \sum_{k \geq 1} \int_0^z ds_1 \frac{s_1^{k-1}}{k} = \sum_{k \geq 1} \frac{z^k}{k^2}.$$

Iterated integrals and shuffle products

The symbols $\alpha_{z_0}^{z}$ have a marvelous property: they are compatible with the shuffle product of words.

Three equivalent ways to define shuffle products

- Recursive definition $\mathcal{A}\langle X\rangle \otimes \mathcal{A}\langle X\rangle \to \mathcal{A}\langle X\rangle$

$$\begin{aligned} \forall w \in X^*, \qquad & w \sqcup\!\sqcup 1_{X^*} = 1_{X^*} \sqcup\!\sqcup w = w, \\ \forall x, y \in X, \forall u, v \in X^*, \qquad & xu \sqcup\!\sqcup yv = x(u \sqcup\!\sqcup yv) + y(xu \sqcup\!\sqcup v). \end{aligned}$$

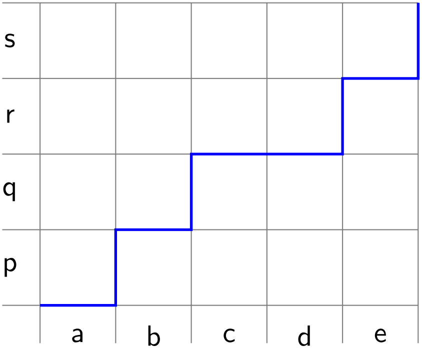
- Comultiplication $\mathcal{A}\langle X\rangle \to \mathcal{A}\langle X\rangle \otimes \mathcal{A}\langle X\rangle$

$$\Delta_{\sqcup\!\sqcup}(x) = x \otimes 1 + 1 \otimes x \tag{8}$$

  Extension by morphism, one finds for $w \in X^*$

$$\Delta_{\sqcup\!\sqcup}(x) = \sum_{I+J=[1\cdots|w|]} w[I] \otimes w[J] \tag{9}$$

- Evaluation of paths



Path which contributes $apbqcdres$ in the shuffle product $abcde \sqcup\!\sqcup pqrs$.

$$u \sqcup\!\sqcup v = \sum_{\pi \in \mathcal{D}(|u|,|v|)} ev(\pi, u, v)$$

$$\mathcal{D}(p,q) = \{\pi \in \{n,e\}^* \mid |\pi|_e = p,\ |\pi|_n = q\}$$

Iterated integrals and shuffle products (replay)

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- Evaluation of paths

... all these objects extend to the ring of formal power series $\mathcal{A}\langle\langle X\rangle\rangle$. Let us put on them a bit of structure.

ALGEBRAIC COMBINATORICS
OF NONCOMMUTATIVE GENERATING SERIES

Polynomials and power series on noncommutative variables

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A f.p.s S with coefficients in A over X (resp. Y) is the following map which can be identified to its graph

$$\begin{array}{rcl} S : X^* \text{ (resp. } Y^*) & \longrightarrow & A, \\ w & \longmapsto & \langle S|w\rangle. \end{array} \qquad S = \sum_{w\in X^*} \langle S|w\rangle w.$$

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$A\langle\langle X\rangle\rangle := A^{X^*}$ (resp. $A\langle\langle Y\rangle\rangle := A^{Y^*}$). It is the (algebraic) dual of $A\langle X\rangle$ (resp. $A\langle Y\rangle$) and this can be realized through the pairing $\langle -|-\rangle$, *i.e.* $\forall S \in A\langle\langle X\rangle\rangle$ (resp. $A\langle\langle Y\rangle\rangle$), $\forall P \in A\langle X\rangle$ (resp. $A\langle Y\rangle$) :

$$\langle S|P\rangle = \sum_{w\in X^* \text{ (resp. } Y^*)} \langle S|w\rangle\langle P|w\rangle.$$

Density

If A is equipped with the discrete topology then

$$A\langle\langle X\rangle\rangle = \widehat{A\langle X\rangle} \quad (\text{resp.} \quad A\langle\langle Y\rangle\rangle = \widehat{A\langle Y\rangle}).$$

for the pointwise convergence.
A family $(S_i)_{i\in I}$ of formal power series is said summable iff, for all word $w \in X^*$

$$i \mapsto \langle S_i|w\rangle \tag{11}$$

has finite support. The sum is then

$$\sum_{i\in I} S_i = \sum_{w\in X^*} (\sum_{i\in I} \langle S_i|w\rangle) w \tag{12}$$

This criterium adapts perfectly to infinite products and double series.

Encoding the multi-indices by words

X^* and Y^* are generated by the totally ordered alphabets $X = \{x_0, x_1\}$ and $Y = \{y_k\}_{k \geq 1}$ admitting 1_{X^*} and 1_{Y^*}, respectively, as neutral elements.

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$$(\underbrace{1, \ldots, 1}_{r \text{ times}}, s_{k+1}, \ldots, s_r) \leftrightarrow y_1^k y_{s_{k+1}} \ldots y_{s_r} \underset{\pi_Y}{\overset{\pi_X}{\rightleftharpoons}} x_1^k x_0^{s_{k+1}-1} x_1 \ldots x_0^{s_r-1} x_1.$$

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$$\begin{aligned}
\forall w\in Y^*,\quad \mathrm{Li}_\bullet : w &\mapsto \mathrm{Li}_{\pi_X(w)}(z)=\sum_{n_1>\ldots>n_r>0}\frac{z^{n_1}}{n_1^{s_1}\ldots n_r^{s_r}},\\
\mathrm{H}_\bullet : w &\mapsto \mathrm{H}_w(N)=\sum_{N\geq n_1>\ldots>n_r>0}\frac{1}{n_1^{s_1}\ldots n_r^{s_r}},\\
\zeta : w &\mapsto \zeta(w)=\sum_{n_1>\ldots>n_r>0}\frac{1}{n_1^{s_1}\ldots n_r^{s_r}}.
\end{aligned}$$

$(A\langle X\rangle, ., 1_{X^*}, \Delta_{\shuffle}, \epsilon_X)$ and $(A\langle Y\rangle, ., 1_{Y^*}, \Delta_{\stuffle}, \epsilon_Y)$

A : commutative and associative algebra with unit over $\mathbb{Q}$.

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- (**Hoffman**, 97) $$\forall w \in Y^*, \qquad w \uplus 1_{Y^*} = 1_{Y^*} \uplus w = w,$$ $$\forall y_i, y_j \in Y, \forall u, v \in Y^*, \qquad y_i u \uplus y_j v = y_i(u \uplus y_j v) + y_j(y_i u \uplus v) + y_{i+j}(u \uplus v)$$

or by their associated coproduct, $\Delta_{⧢}$ and $\Delta_{\uplus}$, defined as follows

$$\langle u ⧢ v | w\rangle = \langle u \otimes v | \Delta_{⧢}(w)\rangle \quad \text{and} \quad \langle u \uplus v | w\rangle = \langle u \otimes v | \Delta_{\uplus}(w)\rangle$$

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which are morphisms for the concatenation defined, on the letters, by

$$\forall x \in X, \qquad \Delta_{⧢}(x) = 1 \otimes x + x \otimes 1,$$

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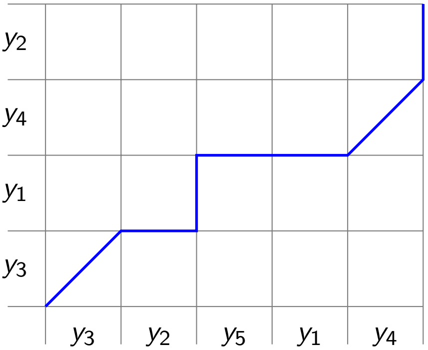
- (**Hoffman**, 97)
$$\forall w \in Y^*, \qquad w \uplus 1_{Y^*} = 1_{Y^*} \uplus w = w,$$
$$\forall y_i, y_j \in Y, \forall u, v \in Y^*, \qquad y_i u \uplus y_j v = y_i(u \uplus y_j v) + y_j(y_i u \uplus v) + y_{i+j}(u \uplus v)$$

or by their associated coproduct, $\Delta_{⧢}$ and $\Delta_{\uplus}$, defined as follows

$$\langle u ⧢ v | w\rangle = \langle u \otimes v | \Delta_{⧢}(w)\rangle \quad \text{and} \quad \langle u \uplus v | w\rangle = \langle u \otimes v | \Delta_{\uplus}(w)\rangle$$

which are morphisms for the concatenation defined, on the letters, by

$$\forall x \in X, \qquad \Delta_{⧢}(x) = 1 \otimes x + x \otimes 1,$$
$$\forall y_k \in Y, \qquad \Delta_{\uplus}(y_k) = 1 \otimes y_k + y_k \otimes 1 + \sum_{i+j=k} y_i \otimes y_j.$$



Path which contributes $y_6 y_2 y_1 y_5 y_1 y_8 y_2$ in the stuffle product $y_3 y_2 y_5 y_1 y_4 \uplus y_3 y_1 y_4 y_2$.

$$u \uplus v = \sum_{\pi \in \mathcal{M}(|u|,|v|)} ev(\pi, u, v)$$

$$\mathcal{M}(p,q) = \{\pi \in \{n,e,d\}^* \mid |\pi|_{e,d} = p,\ |\pi|_{n,d} = q\}$$

Lyndon words as transcendence basis

- A word is a Lyndon word if it is primitive and less than each of its right factor for $\prec_{lex}$ (**Lyndon**, 1954).

Example

$X = \{x_0, x_1\}, x_0 < x_1$. The Lyndon words of length ≤ 5 are

$$x_0, x_0^4x_1, x_0^3x_1, x_0^3x_1^2, x_0^2x_1, x_0^2x_1x_0x_1, x_0^2x_1^2, x_0^2x_1^3, x_0x_1, x_0x_1x_0x_1^2, x_0x_1^2, x_0x_1^3, x_0x_1^4, x_1.$$

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- For any $w \in X^*, w = l_1^{i_1} \ldots l_k^{i_k}, l_1 > \ldots > l_k$ (**Lyndon**, 1954, **Širšov**, 1962).

Example

$x_1x_0x_1^2x_0x_1^2x_0^2x_1 = x_1.x_0x_1^2.x_0x_1^2.x_0^2x_1 = x_1(x_0x_1^2)^2x_0^2x_1.$

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Example

$\mathrm{st}(x_0^2x_1x_0x_1) = (x_0^2x_1, x_0x_1)$, $\mathrm{st}(x_0^2x_1x_0^2x_1x_0x_1) = (x_0^2x_1, x_0^2x_1x_0x_1)$.

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- $(A\langle X\rangle, ⧢, 1_{X^*})$ is a polynomial ring and $\mathcal{L}ynX$ forms a (pure) transcendence basis for it over A (**Radford**, 1956).

Example

$x_0x_1x_0^2x_1 = x_0x_1 ⧢ x_0^2x_1 - 3\,x_0^2x_1x_0x_1 - 6\,x_0^3x_1^2,$

$x_0^3x_1x_0^4x_1 = x_0^3x_1 ⧢ x_0^4x_1 - 5x_0^4x_1x_0^3x_1 - 15x_0^5x_1x_0^2x_1 - 35x_0^6x_1x_0x_1 - 70x_0^7x_1^2.$

Schützenberger's factorization in $(A\langle X\rangle, ., 1_{X^*}, \Delta_{\sqcup\sqcup}, \epsilon_X)$

$\mathcal{L}ynX$ (resp. $\mathcal{L}ynY$) denotes the set of Lyndon words over X (resp. Y).

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- $\{P_l\}_{l\in\mathcal{L}ynX}$: basis of $\mathcal{L}ie_A\langle X\rangle$, where P_l is defined by
  $P_l = l$ if $l \in X$ and $P_l = [P_u, P_v]$ if $l \in \mathcal{L}ynX$ and $\mathrm{st}(l) = (u, v)$.

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- $\{P_w\}_{w\in X^*}$: PBW-L basis of $\mathcal{U}(\mathcal{L}ie_A\langle X\rangle)$ is obtained by putting
  $P_w = P_{l_1}^{i_1} \ldots P_{l_k}^{i_k}$ for $w = l_1^{i_1} \ldots l_k^{i_k}, l_1, \ldots, l_k \in \mathcal{L}ynX, l_1 > \ldots > l_k$.

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  $$\forall u, v \in X^*,\ \langle P_u | S_v\rangle = \delta_{u,v}$$
  can be obtained by putting

$$\begin{aligned} S_l &= xS_u, && \text{for} \quad l = xu \in \mathcal{L}ynX,\\ S_w &= \frac{1}{i_1!\ldots i_k!} S_{l_1}^{\sqcup\!\sqcup i_1} \sqcup\!\sqcup \ldots \sqcup\!\sqcup S_{l_k}^{\sqcup\!\sqcup i_k}, && \text{for} \quad w = l_1^{i_1} \ldots l_k^{i_k}, l_1 > \ldots > l_k. \end{aligned}$$

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$$S_w = \frac{1}{i_1! \ldots i_k!} S_{l_1}^{\sqcup\!\sqcup i_1} \sqcup\!\sqcup \ldots \sqcup\!\sqcup S_{l_k}^{\sqcup\!\sqcup i_k}, \qquad \text{for} \quad w = l_1^{i_1} \ldots l_k^{i_k}, l_1 > \ldots > l_k.$$

Theorem (Schützenberger, 1958, Reutenauer 1988)

$$\mathcal{D}_X := \sum_{w\in X^*} w \otimes w = \sum_{w\in X^*} S_w \otimes P_w = \prod_{l\in\mathcal{L}ynX}^{\searrow} \exp(S_l \otimes P_l).$$

Computational examples

I	P_I	S_I
x_0	x_0	x_0
x_1	x_1	x_1
x_0x_1	$[x_0, x_1]$	x_0x_1
$x_0^2x_1$	$[x_0, [x_0, x_1]]$	$x_0^2x_1$
$x_0x_1^2$	$[[x_0, x_1], x_1]$	$x_0x_1^2$
$x_0^3x_1$	$[x_0, [x_0, [x_0, x_1]]]$	$x_0^3x_1$
$x_0^2x_1^2$	$[x_0, [[x_0, x_1], x_1]]$	$x_0^2x_1^2$
$x_0x_1^3$	$[[[x_0, x_1], x_1], x_1]$	$x_0x_1^3$
$x_0^4x_1$	$[x_0, [x_0, [x_0, [x_0, x_1]]]]$	$x_0^4x_1$
$x_0^3x_1^2$	$[x_0, [x_0, [[x_0, x_1], x_1]]]$	$x_0^3x_1^2$
$x_0^2x_1x_0x_1$	$[[x_0, [x_0, x_1]], [x_0, x_1]]$	$2x_0^3x_1^2 + x_0^2x_1x_0x_1$
$x_0^2x_1^3$	$[x_0, [[[x_0, x_1], x_1], x_1]]$	$x_0^2x_1^3$
$x_0x_1x_0x_1^2$	$[[x_0, x_1], [[x_0, x_1], x_1]]$	$3x_0^2x_1^3 + x_0x_1x_0x_1^2$
$x_0x_1^4$	$[[[[x_0, x_1], x_1], x_1], x_1]$	$x_0x_1^4$
$x_0^5x_1$	$[x_0, [x_0, [x_0, [x_0, [x_0, x_1]]]]]$	$x_0^5x_1$
$x_0^4x_1^2$	$[x_0, [x_0, [x_0, [[x_0, x_1], x_1]]]]$	$x_0^4x_1^2$
$x_0^3x_1x_0x_1$	$[x_0, [[x_0, [x_0, x_1]], [x_0, x_1]]]$	$2x_0^4x_1^2 + x_0^3x_1x_0x_1$
$x_0^3x_1^3$	$[x_0, [x_0, [[[x_0, x_1], x_1], x_1]]]$	$x_0^3x_1^3$
$x_0^2x_1x_0x_1^2$	$[x_0, [[x_0, x_1], [[x_0, x_1], x_1]]]$	$3x_0^3x_1^3 + x_0^2x_1x_0x_1^2$
$x_0^2x_1^2x_0x_1$	$[[x_0, [[x_0, x_1], x_1]], [x_0, x_1]]$	$6x_0^3x_1^3 + 3x_0^2x_1x_0x_1^2 + x_0^2x_1^2x_0x_1$
$x_0^2x_1^4$	$[x_0, [[[[x_0, x_1], x_1], x_1], x_1]]$	$x_0^2x_1^4$
$x_0x_1x_0x_1^3$	$[[x_0, x_1], [[[x_0, x_1], x_1], x_1]]$	$4x_0^2x_1^4 + x_0x_1x_0x_1^3$
$x_0x_1^5$	$[[[[[x_0, x_1], x_1], x_1], x_1], x_1]$	$x_0x_1^5$

Drindfel'd equation and the Hausdorff group

In fact, one can initiate the first steps in the theory of noncommutative differential equations. Let $\mathcal{A} = C^{\omega}(\Omega, \mathbb{C})$. We have the following

Proposition

For a series $S \in \mathcal{A}\langle\langle X \rangle\rangle$, set

$$\Delta_{\sqcup\sqcup}(S) = \sum_{w \in X^*} \langle S|w\rangle \Delta_{\sqcup\sqcup}(w) = \sum_{u,v \in X^*} \langle S|u \sqcup\sqcup v\rangle u \otimes v \qquad (13)$$

For a series, the following are equivalent

1. *for all $u, v \in X^*$ one has $\langle S|u \sqcup\sqcup v\rangle = \langle S|u\rangle\langle S|v\rangle$*
2. $\Delta_{\sqcup\sqcup}(S) = S \hat{\otimes} S$

We will say that such a series is group-like if, moreover $\langle S|1_{X^*}\rangle = 1$ it is not difficult to check that these series form a group (called classically the Hausdorff group).

Drindfel'd equation and the Hausdorff group/2

With the formalism of derivations and coproduct, one gets at hand a true differential (noncommutative) machinery. We can prove that some solutions of $S' = MS$ are group-like and can be considered as a path drawn on the Hausdorff group.
One has the following (S is still a formal power series with functional coefficients over a connected and simply connected domain)

Proposition (D., Minh, Deneufchâtel (1))

Let S be a solution of $S' = MS$ with $\Delta_{\sqcup\sqcup}(M) = M\hat{\otimes}1 + 1\hat{\otimes}M$ (one says that M is primitive). Then

- *If S is once group-like, which means that $\Delta_{\sqcup\sqcup}(S(z_0)) = S(z_0)\hat{\otimes}S(z_0),\ \langle S(z_0)|1_{X^*}\rangle$ for some $z_0 \in \Omega$ (Chen's condition), then S is (always) group-like.*
- *If S is asymptotically group-like (means that it exists a group-like element $G(z)$ such that $lim(S(z)G(z)) = 1$) then S is (always) group-like.*

Unicity and the differential Galois group

If we have two solutions of the equation

$$S' = MS \quad \text{with } \Delta_{\sqcup\!\sqcup}(M) = M\hat{\otimes}1 + 1\hat{\otimes}M \tag{14}$$

they differ by a constant in the following way

Proposition

Let $S_i,\ i = 1, 2$ be two solutions of eq. 14 and suppose that $\langle S_1(z_0)|1_{X^*}\rangle \neq 0$ at some $z_0 \in \Omega$, then

1. $\langle S_1(z)|1_{X^*}\rangle \neq 0$ *everywhere (so S can be inverted)*
2. *It exists $G \in \mathbb{C}\langle\langle X\rangle\rangle$ such that $S_2 = S_1 G$*
3. *If, moreover, S_i are group-like then so is G*

So, one can legitimately call the group-like constant series, the differential Galois group of the group-like solutions of eq. 14.

Condition of independence of the (coordinates of) the solutions.

Theorem (D., Minh, Deneufchâtel (1))

Let $(\mathcal{A}, d)$ be a k-commutative associative differential algebra with unit ($ch(k) = 0$) and $\mathcal{C}$ be a differential subfield of $\mathcal{A}$ (i.e. $d(\mathcal{C}) \subset \mathcal{C}$). We suppose that $S \in \mathcal{A}\langle\langle X \rangle\rangle$ is a solution of the differential equation

$$\mathbf{d}(S) = MS \; ; \; \langle S|1 \rangle = 1 \tag{15}$$

where the multiplier M is a homogeneous series (a polynomial in the case of finite X) of degree 1, i.e.

$$M = \sum_{x \in X} u_x x \in \mathcal{C}\langle\langle X \rangle\rangle \; . \tag{16}$$

Condition of independence of the (coordinates of) the solutions (end of theorem).

Theorem (cont'd)

The following conditions are equivalent :

i) *The family $(\langle S|w\rangle)_{w\in X^*}$ of coefficients of S is free over $\mathcal{C}$.*

ii) *The family of coefficients $(\langle S|y\rangle)_{y\in X\cup\{1_{X^*}\}}$ is free over $\mathcal{C}$.*

iii) *The family $(u_x)_{x\in X}$ is such that, for $f\in\mathcal{C}$ and $\alpha_x\in k$*

$$d(f)=\sum_{x\in X}\alpha_x u_x \Longrightarrow (\forall x\in X)(\alpha_x=0)\ . \tag{17}$$

iv) *The family $(u_x)_{x\in X}$ is free over k and*

$$d(\mathcal{C})\cap span_k\Big((u_x)_{x\in X}\Big)=\{0\}\ . \tag{18}$$

Factorisation of group-like series

If a series T is group-like, the map $T \otimes Id$ is a continuous morphism

$$\mathcal{A}\langle\langle (X^* \otimes X^*)^{(iso)} \rangle\rangle \to \mathcal{A}\langle\langle X \rangle\rangle$$

where $(X^* \otimes X^*)^{(iso)}$ is the monoid of isobaric bi-words (i.e. (u, v)) with $|u|_x = |v|_x$ for all $x \in X$) and then we can apply it to $\mathcal{D}_X = \sum_{w \in X^*} w \otimes w$ and from the infinite product (Schützenberger's factorisation)

$$\mathcal{D}_X := \sum_{w \in X^*} w \otimes w = \sum_{w \in X^*} S_w \otimes P_w = \prod_{l \in \mathcal{L}ynX}^{\searrow} \exp(S_l \otimes P_l) \quad (19)$$

we get

$$T = \sum_{w \in X^*} \langle T|w \rangle w = \prod_{l \in \mathcal{L}ynX}^{\searrow} \exp(\langle T|S_l \rangle \otimes P_l) \ . \qquad (20)$$

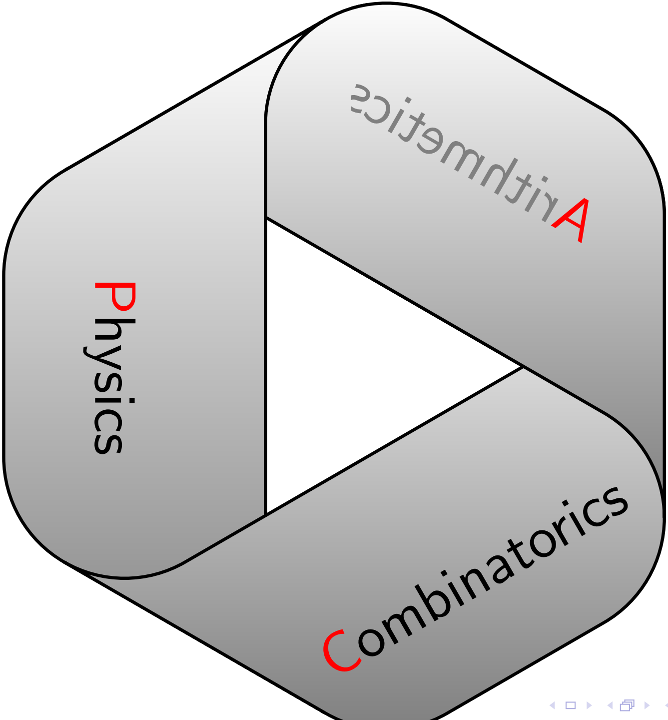
The Lyndon words then constitute the labelling of a local system of coordinates of the Hausdorff group.

Bibliography

M. Deneufchtel, G. H. E. Duchamp, Hoang Ngoc Minh, A. I. Solomon, *Independence of hyperlogarithms over function fields via algebraic combinatorics*, Lecture Notes in Computer Science (2011), Volume 6742 (2011), 127-139.
arXiv:1101.4497v1 [math.CO]

CONCLUSION

- Starting from the classical case of linear differential equations with several singularities, we separated them and replaced the multiplying matrices by noncommuting letters (it is afterwards possible to re-specialise the letters to these matrices). We get a noncommutative linear differential equation with multiplier. Under certain tangency condition (the multiplier be primitive), we get entirely group-like solutions, characterize the (differential) Galois group of the equation and compute local coordinates of them.
- Using special fields of functions, we could also give a necessary and sufficient condition ensuring that the coordinates of the solutions (i.e. the family of functions $(z \to \langle S|w\rangle)_{w\in X^*})$ be linearly independant on enlarged fields of coefficients.
- The hope is to apply this formalism (which is equivalent to that of Dyson, but much more tractable) to arithmetics and physics.


Arithmetics
Physics
Combinatorics

THANK YOU FOR YOUR ATTENTION !