Interactions of holes in two dimensional dimer systems

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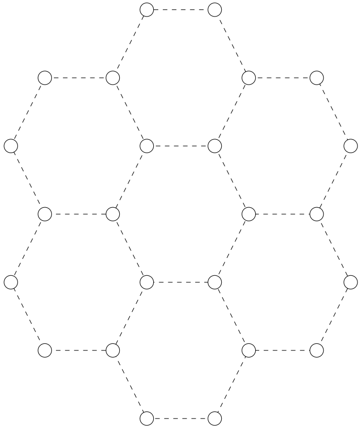
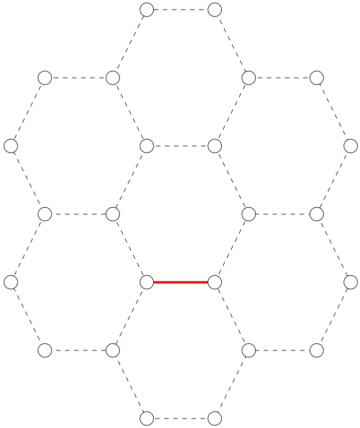
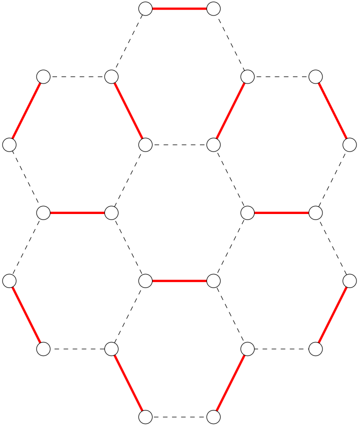
Ellwangen.



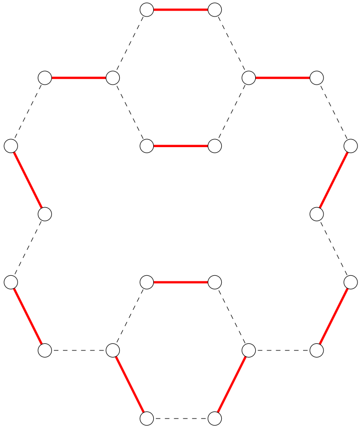
Let L be a subset of the hexagonal lattice.



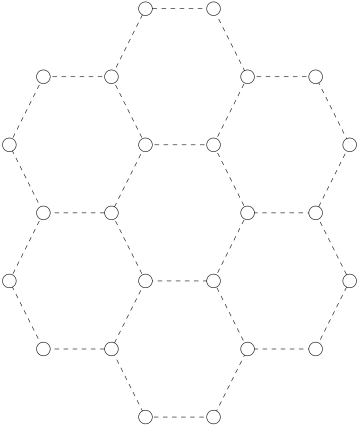
A *dimer* on L is a pair of adjacent vertices, joined by precisely one edge.

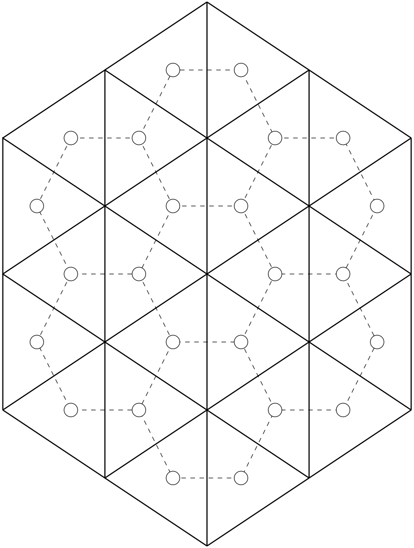


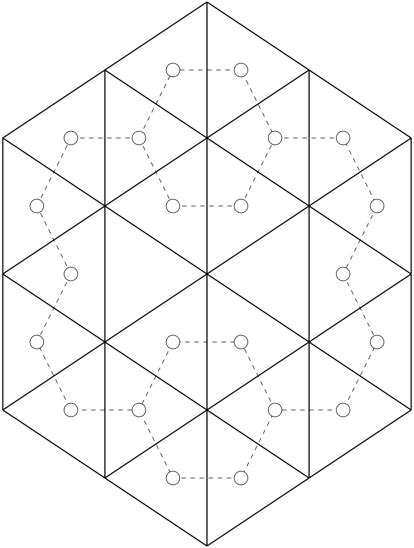
A *dimer covering* on L is a set of dimers that cover every vertex of L exactly once.

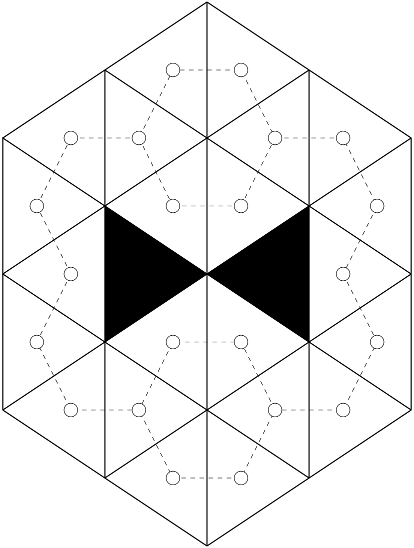


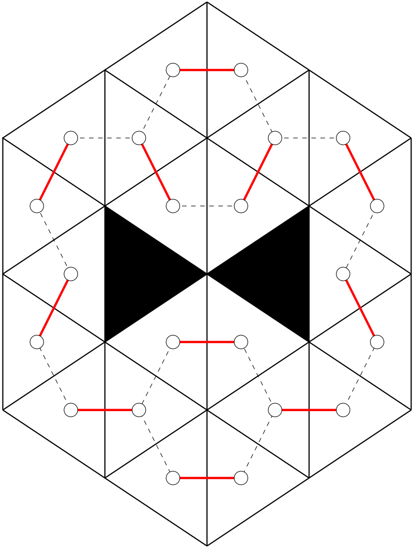
A dimer covering on L containing two holes.

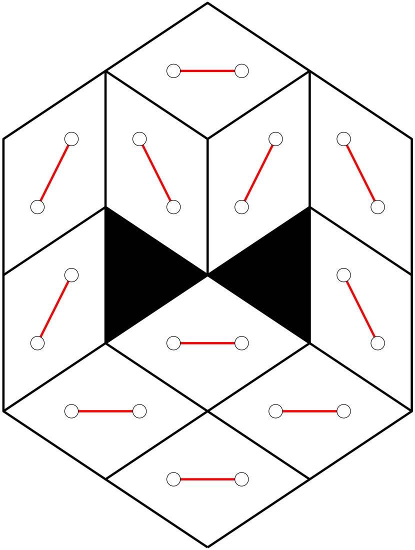


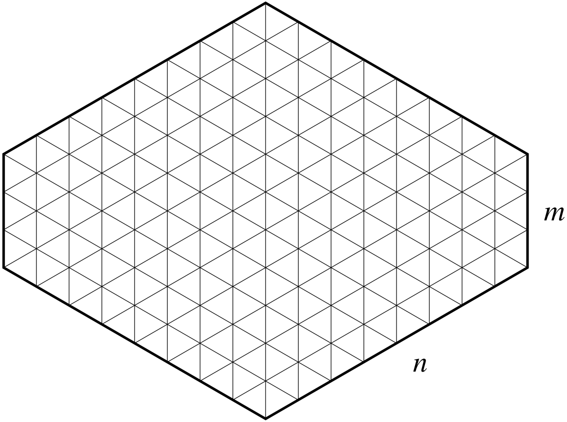




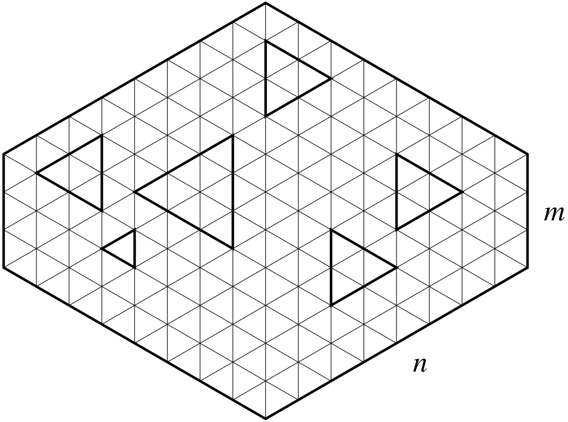




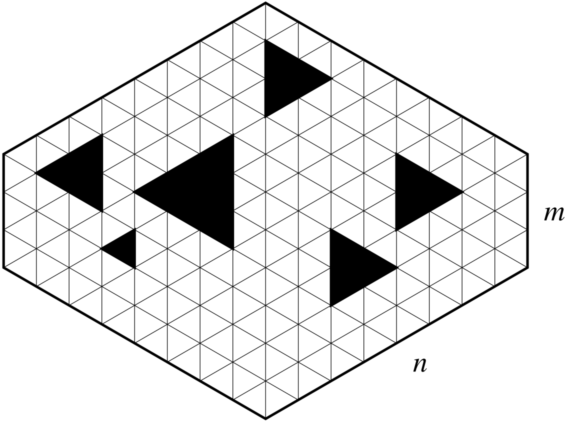




A hexagon, $H_{n,m}$.



A set of triangles, T, contained within $H_{n,m}$.



The holey hexagon $H_{n,m} \setminus T$.

Definition

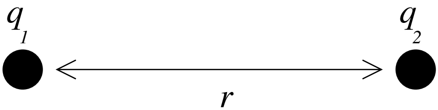
Given a hexagon $H_{n,n}$ and a set of triangles T, the *interaction* (or *correlation function*) of the holes is defined to be

$$\omega(T) = \lim_{n\to\infty} \frac{M(H_{n,n} \setminus T)}{M(H_{n,n})},$$

where $M(R)$ denotes the total number of rhombus tilings of the region R.

Conjecture (M. Ciucu, 2008)

The asymptotic interaction of a set of holes T within a sea of dimers is governed (up to a multiplicative constant) by Coulomb's law for two dimensional electrostatics.

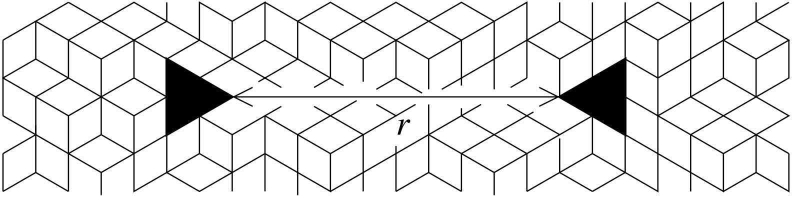


Coulomb's Law

The magnitude of the electrostatic force F between two point charges (q_1 and q_2), each with a signed magnitude, is given by

$$|F| = k_e \frac{|q_1 q_2|}{r^2},$$

where k_e is Coulomb's constant.



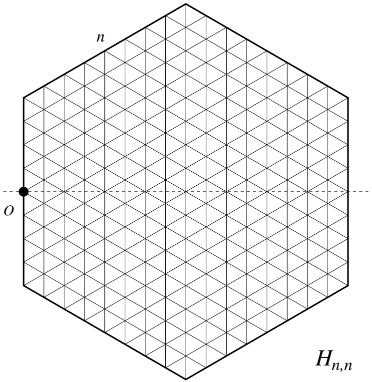
If T denotes the above pair of triangles then according to Ciucu's conjecture

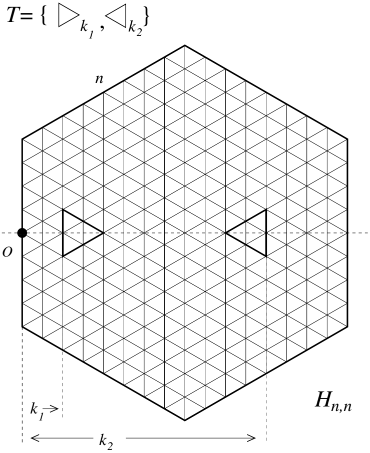
$$\omega(T) \sim C \cdot \frac{1}{r^2}.$$

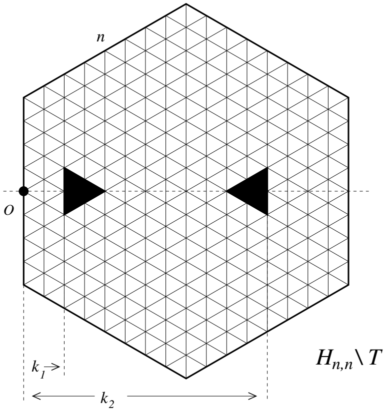
Theorem (TG)

The interaction, $\omega(T)$, between two inward pointing triangular holes of side length two within a sea of dimers is asymptotically

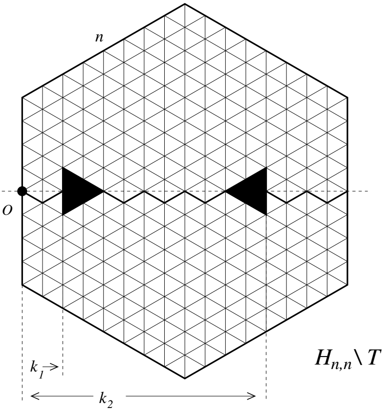
$$\left(\frac{\sqrt{3}}{\pi r}\right)^2.$$


n
O
$H_{n,n}$

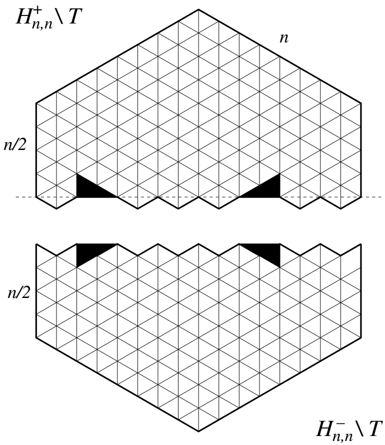

T= { ▷k1 , ◁k2}
n
O
k1
k2
Hn,n

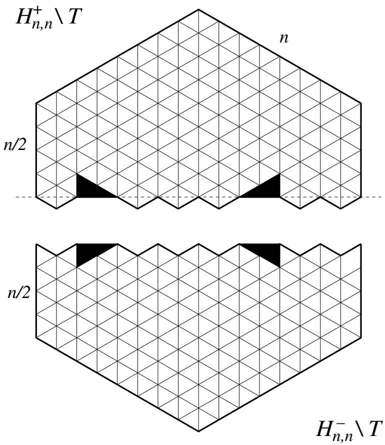


n
O
k_1
k_2
$H_{n,n} \setminus T$



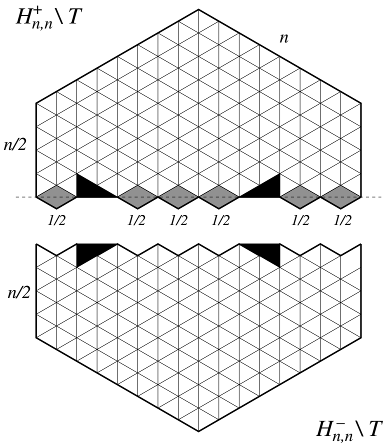
n
O
k_1
k_2
$H_{n,n} \setminus T$


$H^{+}_{n,n} \setminus T$
n
$n/2$
$n/2$
$H^{-}_{n,n} \setminus T$



Matchings Factorisation Theorem (M. Ciucu)

$$M(H_{n,n} \setminus T) = 2^l \cdot M(H^-_{n,n} \setminus T) \cdot M_w(H^+_{n,n} \setminus T).$$



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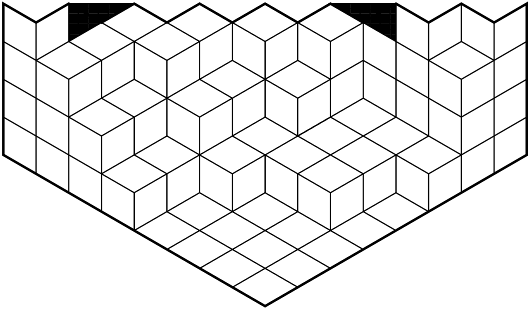
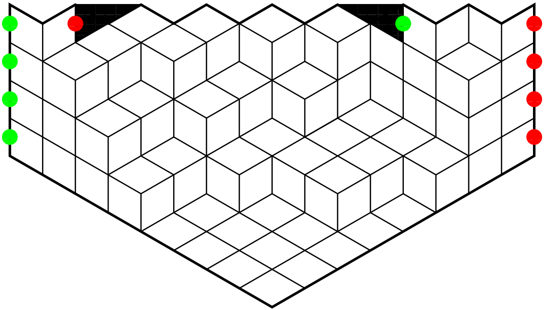
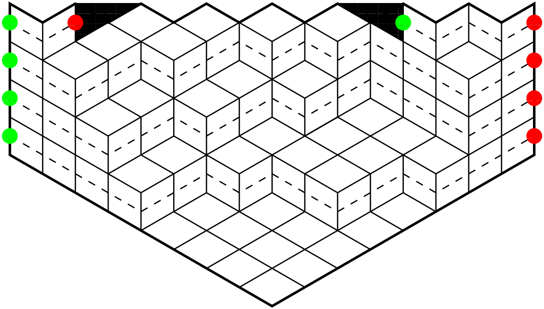
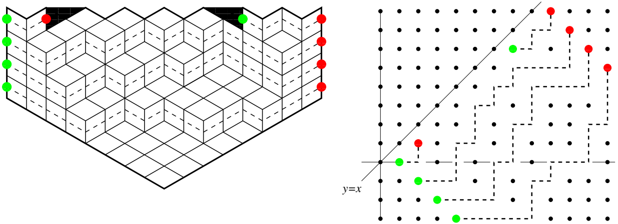








$$M(H^-_{n,n} \setminus T) = \mathscr{P}(V \to W),$$

where $\mathscr{P}(V \to W)$ denotes the set of non-intersecting paths starting at a set of points V and ending at a set of points W where

$$V = \{(i, 1-i) : 1 \leq i \leq \tfrac{n}{2}\} \cup \{(1 + \tfrac{k_2}{2}, \tfrac{k_2}{2})\},$$
$$W = \{(n+j, n+1-j) : 1 \leq j \leq \tfrac{n}{2}\} \cup \{(1 + \tfrac{k_1}{2}, \tfrac{k_1}{2})\}$$

and such that no path crosses the line $y = x$.

Theorem (Lindström-Gessel-Viennot)

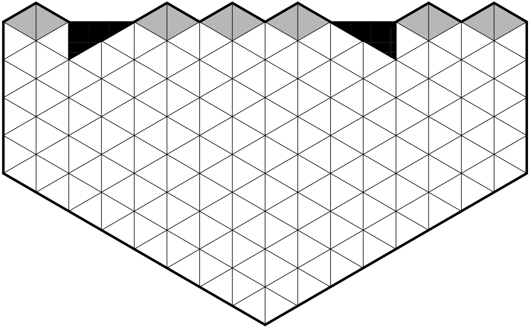
The number of non-intersecting paths that begin at V and end at W is given by $|\det(G)|$, where the matrix $G = (g_{i,j})_{1 \le i,j \le n/2+1}$ has (i,j)-entry $g_{i,j} = \mathscr{P}(V_i \to W_j)$.

Proposition

$$M(H^-_{n,n} \setminus T) = |\det(G)|$$

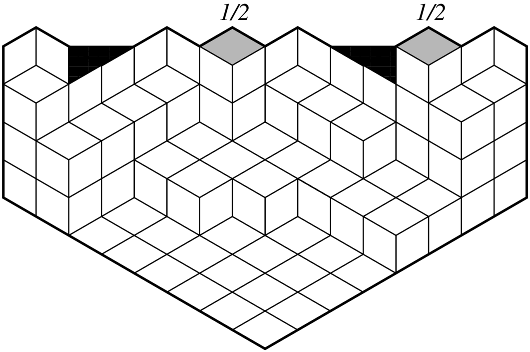
where $G = (g_{i,j})_{1 \le i,j \le n/2+1}$ is the $(n/2+1) \times (n/2+1)$ matrix with (i,j)-entries given by

$$g_{i,j} = \begin{cases} \binom{2n}{n+j-i} - \binom{2n}{n+j-1+i}, & 1 \le i,j \le n/2 \\ \binom{2n-k_2}{n-k_2/2+j-1} - \binom{2n-k_2}{n-k_2/2+j}, & i = n/2+1, 1 \le j \le n/2 \\ \binom{k_1}{k_1/2+i-1} - \binom{k_1}{k_1/2+i}, & j = n/2+1, 1 \le i \le n/2 \\ 0, & \textit{otherwise.} \end{cases}$$



Proposition

$$2^l \cdot M_w(H_{n,n}^+ \setminus T) = M_{w'}(H_{n,n}^+ \setminus T)$$



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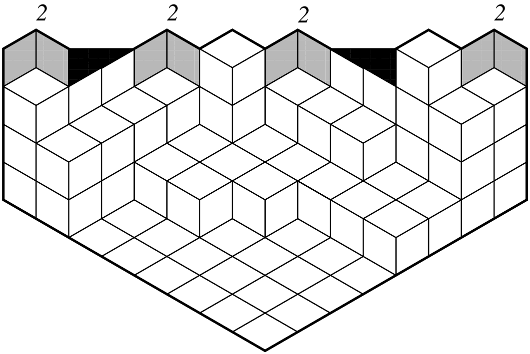

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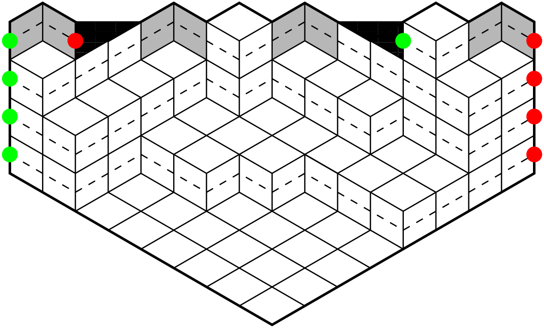

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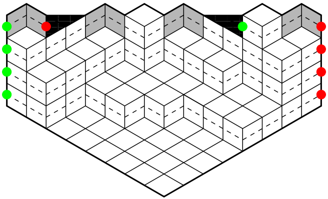
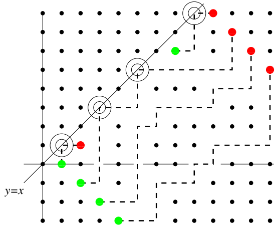
$$2^l \cdot M_w(H^+_{n,n} \setminus T) = M_{w'}(H^+_{n,n} \setminus T)$$





$$M_{w'}(H^{+}_{n,n} \setminus T) = \mathscr{P}'(V \to W),$$

where $\mathscr{P}'(V \to W)$ denotes the set of non-intersecting paths starting at the set of points V and ending at the set of points W such that each path from point V_i to point W_i has weight 2^P, where P denotes the number of times each path touches the line $y = x$.

Proposition

$$M_{w'}(H^{+}_{n,n} \setminus T) = |\det(G^{+})|,$$

where the entries of the matrix $G^{+} = (g^{+}_{i,j})_{1 \le i,j \le n/2+1}$ *are given by*

$$g^{+}_{i,j} = \begin{cases} \binom{2n}{n+j-i} + \binom{2n}{n+j-1+i}, & 1 \le i,j \le n/2 \\ \binom{2n-k_2}{n-k_2/2+j-1} + \binom{2n-k_2}{n-k_2/2+j}, & i = n/2+1, 1 \le j \le n/2 \\ \binom{k_1}{k_1/2+i-1} + \binom{k_1}{k_1/2+i}, & j = n/2+1, 1 \le i \le n/2 \\ 0, & \textit{otherwise.} \end{cases}$$

Consequently...

$$M(H_{n,n} \setminus T) = |\det(G)| \cdot |\det(G^{+})|$$

Theorem (TG)

The positive determinant of the matrix G, which counts rhombus tilings of $H^-_{n,n} \setminus T$, is given by

$$\left(\prod_{i=1}^{n/2} \frac{(2i-1)!(2i+2n-2)!}{(2i+n-2)!(2i+n-1)!} \right) \sum_{s=1}^{n/2} D_{n,k_1}(s) \cdot B_{n,k_2}(s).$$

Theorem (TG)

The positive determinant of the matrix G^+, which counts weighted rhombus tilings of $H^+_{n,n} \setminus T$, is given by

$$\left(\prod_{i=1}^{n/2} \frac{(2i-2)!(2i+2n-1)!}{(2i+n-2)!(2i+n-1)!} \right) \sum_{s=1}^{n/2} B'_{n,k_2}(s) \cdot E_{n,k_1}(s).$$

Theorem (TG)

$$M(H_{n,n} \setminus T) = \left(\prod_{i=1}^{n/2} \frac{(2i-2)!(2i-1)!(2i+2n-1)!(2i+2n-2)!}{(2i+n-2)!^2(2i+n-1)!^2} \right)$$
$$\times \left(\sum_{s=1}^{n/2} B'_{n,k_2}(s) \cdot E_{n,k_1}(s) \right) \times \left(\sum_{t=1}^{n/2} D_{n,k_1}(t) \cdot B_{n,k_2}(t) \right),$$

where

$$B'_{n,k_2}(s) = \frac{(-1)^{s+1}(2n-k_2+1)!(n+s-1)!(n+2s-1)!(k_2/2+s-2)!}{(s-1)!(n-k_2/2)!(k_2/2-1)!(2n+2s-1)!(n-k_2/2+s)!},$$
$$E_{n,k_1}(s) = \frac{(-1)^{s+1}(2s-2)!(k_1+1)!(n+s-1)!(n-k_1/2+s-2)!}{(s-1)!(k_1/2)!(n-k_1/2-1)!(n+2s-2)!(k_1/2+s)!},$$
$$B_{n,k_2}(t) = \frac{(-1)^{t-1}(t+n-2)!(2t+n-1)!(2n-k_2)!(t+k_2/2-2)!}{2(t-1)!(2t+2n-3)!(n-k_2/2)!(k_2/2-1)!(t-k_2/2+n)!},$$
$$D_{n,k_1}(t) = \frac{(-1)^{t+1}(2t)!(t+n-1)!(k_1)!(t-k_1/2+n-2)!}{2(t!)(2t+n-2)!(n-k_1/2-1)!(k_1/2)!(t+k_1/2)!}.$$

Defintion

A *plane partition* is an array of integers that is weakly decreasing along rows (from left to right) and down columns (from top to bottom).

Theorem (MacMahon)

The number of plane partitions that fit inside an $a \times b \times c$ *box is given by*

$$T(a,b,c) = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}.$$

Theorem (TG)

$$M(H_{n,n}\setminus T)=\left(\prod_{i=1}^{n/2}\frac{(2i-2)!(2i-1)!(2i+2n-1)!(2i+2n-2)!}{(2i+n-2)!^2(2i+n-1)!^2}\right)$$

$$\times\left(\sum_{s=1}^{n/2}B'_{n,k_2}(s)\cdot E_{n,k_1}(s)\right)\times\left(\sum_{t=1}^{n/2}D_{n,k_1}(t)\cdot B_{n,k_2}(t)\right),$$

where

$$B'_{n,k_2}(s)=\frac{(-1)^{s+1}(2n-k_2+1)!(n+s-1)!(n+2s-1)!(k_2/2+s-2)!}{(s-1)!(n-k_2/2)!(k_2/2-1)!(2n+2s-1)!(n-k_2/2+s)!},$$

$$E_{n,k_1}(s)=\frac{(-1)^{s+1}(2s-2)!(k_1+1)!(n+s-1)!(n-k_1/2+s-2)!}{(s-1)!(k_1/2)!(n-k_1/2-1)!(n+2s-2)!(k_1/2+s)!},$$

$$B_{n,k_2}(t)=\frac{(-1)^{t-1}(t+n-2)!(2t+n-1)!(2n-k_2)!(t+k_2/2-2)!}{2(t-1)!(2t+2n-3)!(n-k_2/2)!(k_2/2-1)!(t-k_2/2+n)!},$$

$$D_{n,k_1}(t)=\frac{(-1)^{t+1}(2t)!(t+n-1)!(k_1)!(t-k_1/2+n-2)!}{2(t!)(2t+n-2)!(n-k_1/2-1)!(k_1/2)!(t+k_1/2)!}.$$

Theorem (TG)

$$M(H_{n,n}\setminus T)=\left(\prod_{i=1}^{n/2}\frac{(2i-2)!(2i-1)!(2i+2n-1)!(2i+2n-2)!}{(2i+n-2)!^2(2i+n-1)!^2}\right)$$
$$\times\left(\sum_{s=1}^{n/2}B'_{n,k_2}(s)\cdot E_{n,k_1}(s)\right)\times\left(\sum_{t=1}^{n/2}D_{n,k_1}(t)\cdot B_{n,k_2}(t)\right),$$

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$$E_{n,k_1}(s)=\frac{(-1)^{s+1}(2s-2)!(k_1+1)!(n+s-1)!(n-k_1/2+s-2)!}{(s-1)!(k_1/2)!(n-k_1/2-1)!(n+2s-2)!(k_1/2+s)!},$$
$$B_{n,k_2}(t)=\frac{(-1)^{t-1}(t+n-2)!(2t+n-1)!(2n-k_2)!(t+k_2/2-2)!}{2(t-1)!(2t+2n-3)!(n-k_2/2)!(k_2/2-1)!(t-k_2/2+n)!},$$
$$D_{n,k_1}(t)=\frac{(-1)^{t+1}(2t)!(t+n-1)!(k_1)!(t-k_1/2+n-2)!}{2(t!)(2t+n-2)!(n-k_1/2-1)!(k_1/2)!(t+k_1/2)!}.$$

Theorem (TG)

$$M(H_{n,n} \setminus T) = T(n,n,n) \times \left(\sum_{s=1}^{n/2} B'_{n,k_2}(s) \cdot E_{n,k_1}(s) \right) \times \left(\sum_{t=1}^{n/2} D_{n,k_1}(t) \cdot B_{n,k_2}(t) \right),$$

where

$$B'_{n,k_2}(s) = \frac{(-1)^{s+1}(2n-k_2+1)!(n+s-1)!(n+2s-1)!(k_2/2+s-2)!}{(s-1)!(n-k_2/2)!(k_2/2-1)!(2n+2s-1)!(n-k_2/2+s)!},$$

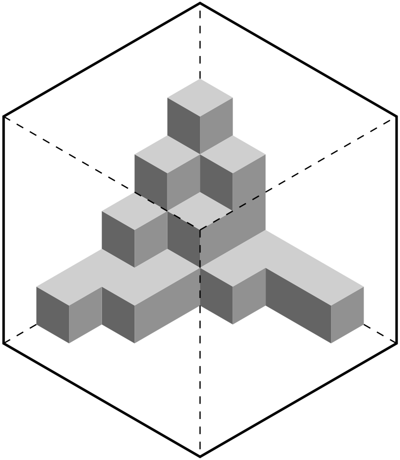
$$E_{n,k_1}(s) = \frac{(-1)^{s+1}(2s-2)!(k_1+1)!(n+s-1)!(n-k_1/2+s-2)!}{(s-1)!(k_1/2)!(n-k_1/2-1)!(n+2s-2)!(k_1/2+s)!},$$

$$B_{n,k_2}(t) = \frac{(-1)^{t-1}(t+n-2)!(2t+n-1)!(2n-k_2)!(t+k_2/2-2)!}{2(t-1)!(2t+2n-3)!(n-k_2/2)!(k_2/2-1)!(t-k_2/2+n)!},$$

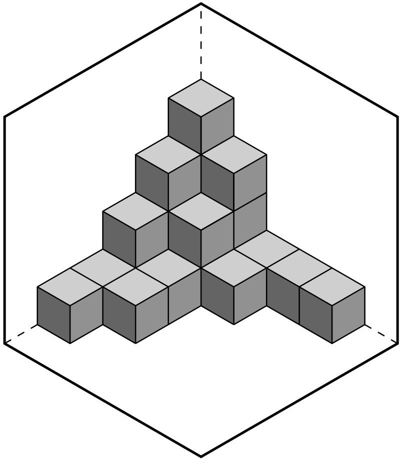
$$D_{n,k_1}(t) = \frac{(-1)^{t+1}(2t)!(t+n-1)!(k_1)!(t-k_1/2+n-2)!}{2(t!)(2t+n-2)!(n-k_1/2-1)!(k_1/2)!(t+k_1/2)!}.$$

4	*3*	*1*	*1*	*1*
3	*2*	*1*		
2	*1*			
1	*1*			
1				

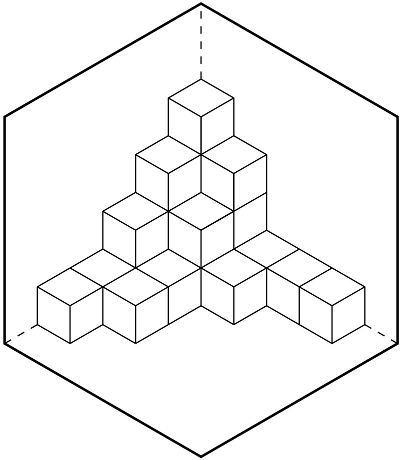
4	3	1	1	1
3	2	1		
2	1			
1	1			
1				



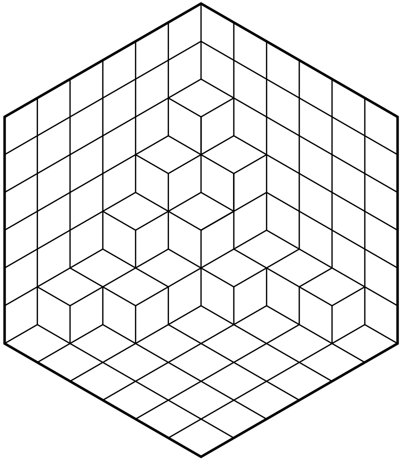
4	*3*	*1*	*1*	*1*
3	*2*	*1*		
2	*1*			
1	*1*			
1				



4	3	1	1	1
3	2	1		
2	1			
1	1			
1				



4	3	1	1	1
3	2	1		
2	1			
1	1			
1				



Every plane partition that fits inside an $n \times n \times n$ box corresponds to a rhombus tiling of $H_{n,n} \setminus T$, so it follows that

$$M(H_{n,n}) = T(n, n, n).$$

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And so...

The formula that gives the number of rhombus tilings of the holey hexagon $H_{n,n} \setminus T$ may be re-written as

$$M(H_{n,n} \setminus T) = \left(\sum_{s=1}^{n/2} B'_{n,k_2}(s) \cdot E_{n,k_1}(s) \right) \times \left(\sum_{t=1}^{n/2} D_{n,k_1}(t) \cdot B_{n,k_2}(t) \right) \times M(H_{n,n})$$

Interaction

The interaction between the holes in T is given by

$$\omega(T) = \lim_{n\to\infty} \frac{M(H_{n,n} \setminus T)}{M(H_{n,n})},$$

that is,

$$\omega(T) = \lim_{n\to\infty} \left(\sum_{s=1}^{n/2} B'_{n,k_2}(s) \cdot E_{n,k_1}(s) \right) \left(\sum_{t=1}^{n/2} D_{n,k_1}(t) \cdot B_{n,k_2}(t) \right).$$

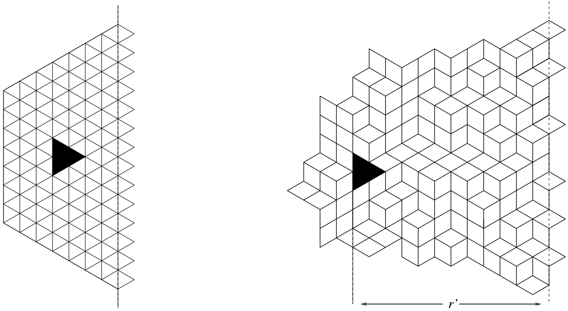
If r denotes the Euclidean distance between the pair holes in T, then it can be shown that as r becomes very large interaction between the holes is asymptotically

$$\omega(T) \sim \left(\frac{\sqrt{3}}{\pi \cdot r} \right)^2.$$

Further Results

Using similar methods it is possible to show that the interaction between a right pointing triangular hole and a free boundary that borders a sea of lozenges on the right is asymptotically

$$\frac{3}{4\pi r'}.$$



Thank you.