A new hook formula due to a generalization of Nekrasov-Okounkov identity

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Plan

Partitions

A partition λ of n is a decreasing sequence $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. We represent a partition by its Ferrers diagram.

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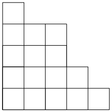

Figure: The Ferrers diagram of λ=(5,4,3,3,1)

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7	5	4	1	
9	7	6	3	1

Figure: The Ferrers diagram of λ=(5,4,3,3,1) and its hook lengths

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$+$	$+$	$+$		
$+$	$+$	$-$		
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Figure: The Ferrers diagram of λ=(5,4,3,3,1) and the sign ε_h of its boxes

Set $\varepsilon_h = \begin{cases} +1 & \text{if } h \text{ is stricly above the diagonal} \\ -1 & \text{else} \end{cases}$

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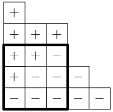

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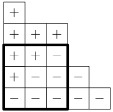

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$\mathcal{H}_t(\lambda)$ the multi-set of hook lengths which are multiple of t

t-core of a partition

Let $t \geq 2$ be an integer. A partition is a *t-core* if its hook lengths set does not contain t, *i.e.* $\mathcal{H}_t(\lambda) = \emptyset$. It is equivalent to the fact that the hook lengths set does not contain any integral multiple of t.

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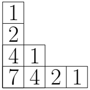

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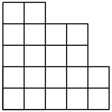

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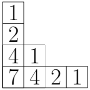

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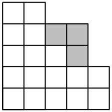

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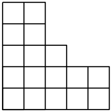

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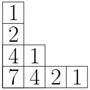

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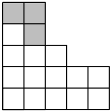

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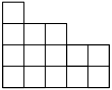

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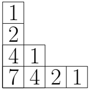

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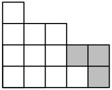

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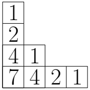

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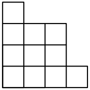

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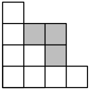

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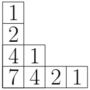

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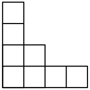

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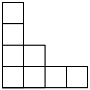

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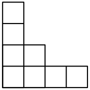

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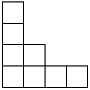

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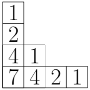

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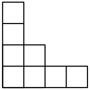

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Han (2009): expansion of η function in terms of hook lengths

Nekrasov-Okounkov formula

Theorem (Nekrasov-Okounkov, 2003; Han, 2009)

For any complex number z we have

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{z}{h^2}\right) = \prod_{k \geq 1} (1 - x^k)^{z-1}$$

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Han's proof uses two tools:

- Macdonald identity (1972) in type $\widetilde{A}$ for t an odd integer

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- a bijection due to Garvan-Kim-Stanton between t-cores and vectors of integers

A generalization of Nekrasov-Okounkov formula

Theorem (Han, 2009)

Let t be a positive integer. For any complex numbers y and z we have

$$\sum_{\lambda\in\mathcal{P}} x^{|\lambda|} \prod_{h\in\mathcal{H}_t(\lambda)} \left(y - \frac{tyz}{h^2}\right) = \prod_{k\geq 1} \frac{(1-x^{tk})^t}{(1-x^k)(1-(yx^t)^k)^{t-z}}$$

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Consequences:

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- Many refinements of the generating function of t-cores
- A reformulation of Lehmer's conjecture in number theory

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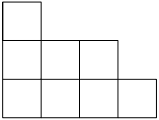

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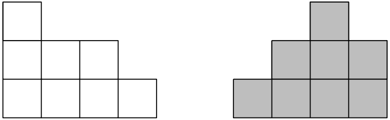

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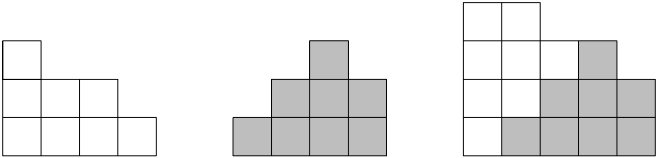

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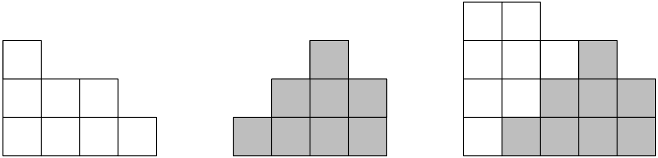

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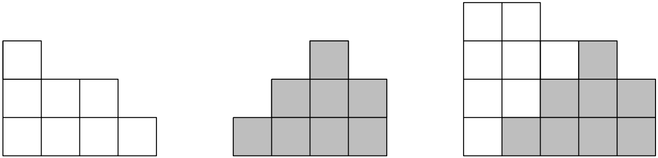

$DD_{(t)}$: set of doubled distinct t-cores

The t-core of a doubled distinct partition is a doubled distinct partition

Nekrasov-Okounkov formula in type $\tilde{C}$ (and $\tilde{B}$ and $\tilde{BC}$)

Theorem (P., 2014)

For any complex number z, the following expansion holds:

$$\sum_{\lambda \in DD} \delta_\lambda x^{|\lambda|/2} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{2z+2}{h\,\varepsilon_h}\right) = \prod_{k \geq 1} (1 - x^k)^{2z^2+z}$$

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Also generalizes Macdonald identity in types $\widetilde{B}$ and $\widetilde{BC}$

A generalization of Nekrasov-Okounkov formula in type $\tilde{C}$

Theorem (P., 2015)

Let $t = 2t' + 1$ be an odd positive integer. For any complex numbers y and z we have

$$\sum_{\lambda \in DD} \delta_\lambda x^{|\lambda|/2} \prod_{h \in \mathcal{H}_t(\lambda)} \left(y - \frac{yt(2z+2)}{\varepsilon_h \, h} \right)$$

$$= \prod_{k \geq 1} (1 - x^k)(1 - x^{kt})^{t'-1} \left(1 - x^{tk} y^{2k}\right)^{(2z+1)(zt+3t')}$$

Littlewood decomposition

Theorem (Littlewood, 1951, probably)

The Littlewood decomposition maps a partition λ *to* $(\tilde{\lambda}, \lambda^0, \lambda^1, \ldots, \lambda^{t-1})$ *such that:*

(i) $\tilde{\lambda}$ *is the t-core of* λ *and* $\lambda^0, \lambda^1, \ldots, \lambda^{t-1}$ *are partitions;*

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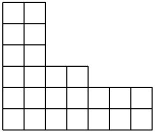

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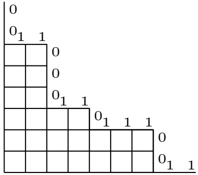

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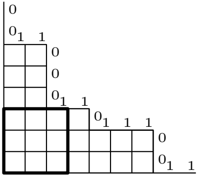

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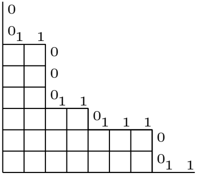

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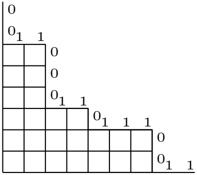

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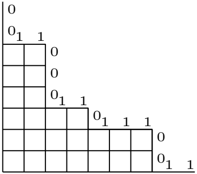

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\begin{aligned}
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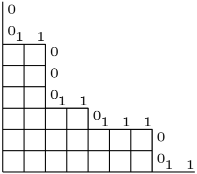

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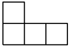
$\lambda^0 =$ 

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New properties of Littlewood decomposition

When $\lambda \in DD$, its Littlewood decomposition $(\tilde{\lambda}, \lambda^0, \lambda^1, \ldots, \lambda^{t-1})$ satisfies:

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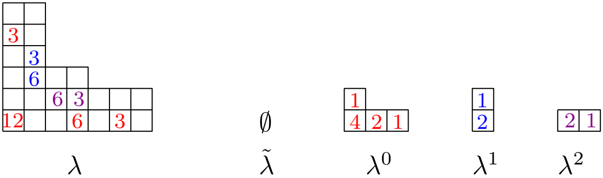

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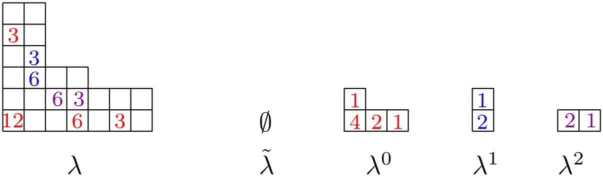

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- And sum over all doubled distinct partitions.

Some consequences

Corollary (P., 2015)

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$$\sum_{\lambda \in DD} \delta_\lambda x^{|\lambda|/2} \prod_{h \in \mathcal{H}_t(\lambda)} \frac{bt}{h\,\varepsilon_h} = \exp(-tb^2 x^t/2) \prod_{k \geq 1} (1 - x^k)(1 - x^{kt})^{t'-1}$$

A new hook formula

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We have:

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Question: can we prove this by using the RSK algorithm?

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- What about other affine types (as $\widetilde{D}$)?

Thank you for your attention