Two classification results on skew Schur Q-functions

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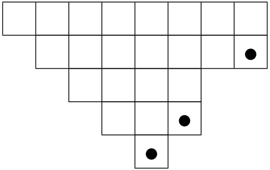
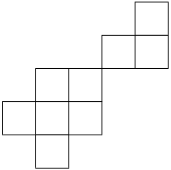
Definition

- A partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ is called strict when $\lambda_1 > \lambda_2 > \dots > \lambda_l > 0$.
- The shifted diagram indexed by $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ is $D_\lambda := \bigcup_{i \in \mathbb{N}} \{(i,i), (i,i+1), \dots (i, i+\lambda_i - 1)\}$.
- For $\lambda, \mu \in DP$, where $D_\mu \subseteq D_\lambda$, the skew shifted diagram is $D_{\lambda/\mu} := D_\lambda \setminus D_\mu$.
- The number of (edgewise) connected components of a diagram D is denoted by $comp(D)$.
  If $comp(D) = 1$ we call the diagram D connected.
- A corner of D is a box $(x, y) \in D$ such that $(x+1, y), (x, y+1) \notin D$.

Example

For $\lambda = (8, 7, 4, 3, 1)$, $\mu = (7, 5, 2)$ we have

$D_\lambda =$  and $D_{\lambda/\mu} =$  .

The diagram D_λ is connected and $comp(D_{\lambda/\mu}) = 2$.
The corners of D_λ are indicated above by $\boxed{\bullet}$.

Definition

- A shifted tableau T of shape $D_{\lambda/\mu}$ is a filling of the boxes of $D_{\lambda/\mu}$ with elements of the alphabet $A = \{1' < 1 < 2' < 2 < \ldots\}$ such that
  - $T(i,j) \leq T(i+1,j)$, $T(i,j) \leq T(i,j+1)$ for all i, j,
  - each column has at most one k ($k = 1, 2, 3, \ldots$),
  - each row has at most one k' ($k' = 1', 2', 3', \ldots$),

  where $T(x,y)$ denotes the entry of the box (x,y) of the skew shifted tableau T.
- The content of the shifted tableau T is $c(T) = (c_1, c_2, \ldots)$, where c_k is the number of k's and ks in the tableau T.
- The reading word $w(T)$ of a shifted tableau T is the word obtained by reading the rows from left to right starting with the lowest row.

Example

$$T = \begin{array}{|c|c|c|c|c|}
\cline{2-5}
\multicolumn{1}{c|}{} & 1' & 1 & 1 & 2 \\
\hline
1' & 2 & 2 & 5' & \multicolumn{1}{c}{} \\
\cline{1-4}
1' & 3' & 3 & 5' & \multicolumn{1}{c}{} \\
\cline{1-4}
\multicolumn{1}{c|}{} & 3 & 6 & \multicolumn{2}{c}{} \\
\cline{2-3}
\end{array}$$

$$c(T) = (5,3,3,0,2,1,0,\ldots) = (5,3,3,0,2,1)$$

$$w(T) = 361'3'35'1'225'1'112$$

The combinatorial definition of skew Schur Q-functions

Definition

For strict partitions λ, μ the skew Schur Q-function in an infinite set of variables $x = (x_1, x_2, \ldots)$ is defined by

$$Q_{\lambda/\mu}(x) := \sum_{\text{shifted tableaux } T \text{ of shape } D_{\lambda/\mu}} x^{c(T)},$$

where $x^{(c_1, c_2, \ldots, c_l)} := x_1^{c_1} x_2^{c_2} \cdots x_l^{c_l}$.
Define $Q_\lambda := Q_{\lambda/\emptyset}$.

Remark

W.l.o.g., we will only consider diagrams with no empty rows or columns.

An analogue of the Littlewood-Richardson rule

Theorem (Stembridge 1989)

For strict partitions λ, μ we have

$$Q_{\lambda/\mu} = \sum_{\text{strict partitions } \nu} f^{\lambda}_{\mu\nu} Q_{\nu},$$

where $f^{\lambda}_{\mu\nu}$ is the number of shifted tableaux T of shape $D_{\lambda/\mu}$ with content ν such that $w(T)$ satisfies a lattice property.

Tableaux with the lattice property

Lemma

Let λ, μ be strict partitions. The word $w(T)$ of a shifted tableau T of shape $D_{\lambda/\mu}$ satisfies the lattice property if T satisfies the following conditions for all $1 \leq k \leq t-1$ where t is the greatest entry in the shifted tableau:

(a) *there is a column with a k but no $k+1$;*

(b) *in each column with a $k+1$ there is also a k;*

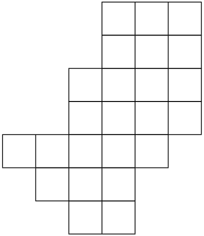
(c) *if $T(x,y) = (k+1)'$ then $T(x-1, y-1) = k'$;*

(d) *the lowest leftmost box filled with an entry in $\{k', k\}$ is filled with a k;*

(e) *the lowest leftmost box filled with an entry in $\{(k+1)', k+1\}$ is filled with a $k+1$.*

Definition (Salmasian 2008)

Let $D_{\lambda/\mu}$ be a skew shifted diagram. The tableau $T_{\lambda/\mu}$ is defined as the tableau obtained as follows:



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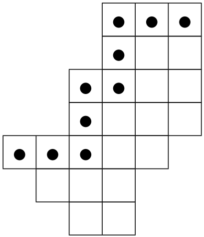

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Definition (Salmasian 2008)

Let $D_{\lambda/\mu}$ be a skew shifted diagram. The tableau $T_{\lambda/\mu}$ is defined as the tableau obtained as follows:

$$\begin{ytableau}
\none & \none & \none & 1' & 1 & 1 \\
\none & \none & \none & 1' & & \\
\none & \none & 1' & 1 & & \\
\none & \none & 1' & & & \\
1 & 1 & 1 & & \\
\none & & & \\
\none & \none & &
\end{ytableau}\,.$$

Definition (Salmasian 2008)

Let $D_{\lambda/\mu}$ be a skew shifted diagram. The tableau $T_{\lambda/\mu}$ is defined as the tableau obtained as follows:

			$1'$	1	1
			$1'$	$2'$	2
		$1'$	1	$2'$	
		$1'$	$2'$	2	
1	1	1	$2'$		
	2	2	2		

.

Definition (Salmasian 2008)

Let $D_{\lambda/\mu}$ be a skew shifted diagram. The tableau $T_{\lambda/\mu}$ is defined as the tableau obtained as follows:

$$
\begin{array}{cccccc}
 & & & 1' & 1 & 1 \\
 & & & 1' & 2' & 2 \\
 & & 1' & 1 & 2' & 3' \\
 & & 1' & 2' & 2 & 3 \\
1 & 1 & 1 & 2' & 3 & \\
 & 2 & 2 & 2 & & \\
 & & 3 & 3 & &
\end{array}.
$$

Let P_k be the set of boxes of $T_{\lambda/\mu}$ filled with entries in $\{k', k\}$.

Remark

The tableau $T_{\lambda/\mu}$ has the lexicographically largest content among all tableaux of $D_{\lambda/\mu}$ satisfying the lattice property.

Proposition

Let $D_{\lambda/\mu}$ be a diagram. Let $\nu = c(T_{\lambda/\mu})$. Then we have

$$f^{\lambda}_{\mu\nu} = \prod_{i=1}^{\ell(\nu)} 2^{comp(P_i)-1}.$$

Definition

A symmetric function f is called Q-homogeneous if $f = k \cdot Q_\nu$ for some k and ν.
A skew shifted diagram $D_{\lambda/\mu}$ is called strange if $Q_{\lambda/\mu} = Q_\nu$ for some ν.

Remark

Salmasian (2008) has classified the strange skew shifted diagrams.

Which skew Schur Q-functions are Q-homogeneous?

Main idea:
For a given property of a diagram of shape $D_{\lambda/\mu}$ find a tableau T with $c(T) \neq c(T_{\lambda/\mu})$ that satisfies the lattice property. Then the diagram of a Q-homogeneous skew Schur Q-function must not have this property. Continue excluding properties for diagrams until the remaining diagrams belong to Q-homogeneous skew Schur Q-functions.

Lemma

For given partitions λ, μ if there is an $i > 1$ such that between two components of P_i there is a column with no box in P_i then $Q_{\lambda/\mu}$ is not Q-homogeneous.

Example

Consider $\lambda/\mu = (9,8,5,3,2)/(6,5,2,1)$:

$$
\begin{array}{ccccc}
 & & 1' & 1 & 1 \\
 & & 1' & 2 & 2 \\
1' & 1 & 1 & & \\
1' & 2' & & & \\
1 & 2 & & &
\end{array}
\mapsto
\begin{array}{ccccc}
 & & 1' & 1 & 1 \\
 & & \mathbf{1} & 2 & 2 \\
1' & 1 & \mathbf{2} & & \\
1' & 2' & & & \\
1 & 2 & & &
\end{array}
.
$$

If $D_{\lambda/\mu}$ is connected then P_1 is connected.
If in a connected diagram $D_{\lambda/\mu}$ there is an $i > 1$ such that $comp(P_i) \geq 2$ then we can find a tableau T with $c(T) \neq c(T_{\lambda/\mu})$, hence $Q_{\lambda/\mu}$ is not Q-homogeneous.
Thus, the Q-homogeneous skew Schur Q-functions have only connected P_is.

Lemma

Let $Q_{\lambda/\mu} = k \cdot Q_\nu$ for some k. If $comp(D_{\lambda/\mu}) = 1$ then $k = 1$, hence $D_{\lambda/\mu}$ is strange.

Theorem

Let λ, μ be such that $D_{\lambda/\mu}$ has no empty rows or columns. We have $Q_{\lambda/\mu} = k \cdot Q_\nu$ for some k and ν if and only if one of the following holds:

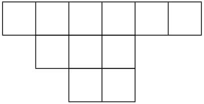
(i) $\mu = \emptyset$ *and* $k = 1$,

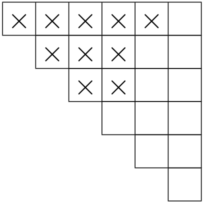
(ii) $\lambda = (m, m-1, \ldots, 1)$ *for some* m, μ *arbitrary and* $k = 1$,

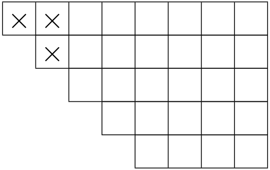
(iii) $\lambda/\mu = (p+q+r, p+q+r-1, p+q+r-2, \ldots, p)/(q, q-1, \ldots, 1)$, *where* $p, q, r \geq 1$ *and* $k = 1$,

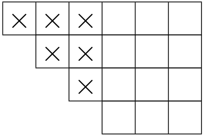
(iv) $\lambda/\mu = (p+q, p+q-1, p+q-2, \ldots, p+1, p)/(q, q-1, \ldots, 1)$, *where* $p, q \geq 1$ *and* $k = 1$,

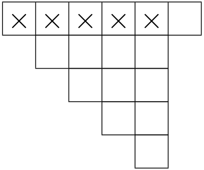
(v) $\lambda/\mu = (r+2, r, r-1, \ldots, 1)/(r+1)$ *and* $\nu = (r+1, r-1, r-2, \ldots, 1)$ *for an* $r \geq 1$ *and* $k = 2$.

case (i):  ,

case (ii):  ,

case (iii):  ,

case (iv):  ,

case (v):  .

Definition

A skew Schur Q-function $Q_{\lambda/\mu}$ is called (Q-)multiplicity-free if $f^{\lambda}_{\mu\nu} \leq 1$ for all ν.

Aim: Classification of the shapes $D_{\lambda/\mu}$ such that $Q_{\lambda/\mu}$ is multiplicity-free.

Main idea: for a given property of a diagram find two tableaux with same content that satisfy the lattice property. Exclude properties until the remaining diagrams belong to multiplicity-free Schur Q-functions.

An important intermediate result

Proposition

If $Q_{\lambda/\mu}$ is multiplicity-free then λ and μ satisfy one of the following properties:

- $\mu = \emptyset$ *or* $\mu = (1)$,
- D_λ *has only one corner and the last part of* λ *is either* 1 *or* 2,
- *both* D_λ *and* D_μ *have at most two corners and if one of the diagrams has two corners then the other diagram has only one corner.*

Theorem

Let $\lambda, \mu \in DP$. Then $Q_{\lambda/\mu}$ is multiplicity-free if and only if λ and μ satisfy one of the following conditions for some a, b, c, d ,w ,x, $y \in \mathbb{N}$:

(i) λ *is arbitrary and* $\mu \in \{\emptyset, (1)\}$,

(ii) $\lambda = (a+b-1, a+b-2, \dots, b)$, *where* $b \in \{1,2\}$ *and* μ *is arbitrary,*

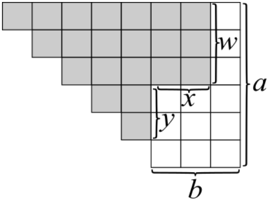
(iii) $\lambda = (a+b-1, a+b-2, \dots, b)$ *and* $\mu = (w+x+y, w+x+y-1, \dots, x+y+2, x+y+1, y, y-1, \dots, 1)$, *where* $w = 1$ *or* $x = 1$ *or* $b \leq 3$ *or* $a+b-w-x-y-1 = 1$,

(iv) $\lambda = (a+b+c+d-1, a+b+c+d-2, \dots, b+c+d+1, b+c+d, c+d-1, c+d-2, \dots, d)$, *where* $d \neq 1$ *and* $\mu = (w, w-1, \dots, 1)$ *with* $1 \in \{a, b, c\}$ *or* $w \leq 2$,

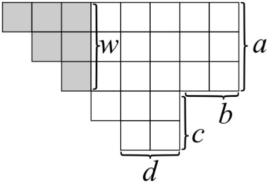
(v) $\lambda = (a+b+c, a+b+c-1, \dots, b+c+2, b+c+1, c, c-1, \dots, 1)$ *and* $\mu = (w, w-1, \dots, 1)$, *where* $a \leq 2$ *or* $b \leq 2$ *or* $c \leq 2$ *or* $w \leq 3$ *or* $w = a+c-1$,

(vi) $\lambda = (a+b-1, a+b-2, \dots, b)$ *and* $\mu = (w+x-1, w+x-2, \dots, x)$, *where* $b \leq 4$ *or* $w \leq 2$ *or* $x \leq 3$ *or* $a = w+1$ *or* $a+b-w-x \leq 2$.

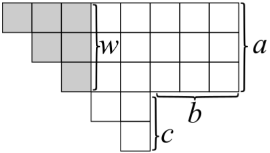
Some of these cases overlap.



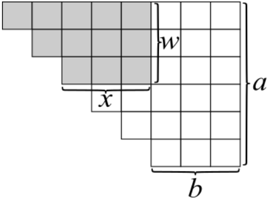
$w = 1$ or $x = 1$ or $b \leq 3$ or $a + b - w - x - y - 1 = 1$.



If $d \geq 2$ then $1 \in \{a, b, c\}$ or $w \leq 2$.



$a \leq 2$ or $b \leq 2$ or $c \leq 2$ or $w \leq 3$ or $w = a + c - 1$.



$b \leq 4$ or $w \leq 2$ or $x \leq 3$ or $a = w + 1$ or $a + b - w - x \leq 2$.

Thank you!