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COMBINATORIAL RULES FOR THREE BASES OF POLYNOMIALS

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In memory of Alain Lascoux, who inspired this paper one night in Osaka

ABSTRACT. We present combinatorial rules (one theorem and two conjectures) concerning three bases of $\mathbb{Z}[x_1, x_2, \ldots]$. First, we prove a "splitting" rule for the basis of Key polynomials [Demazure, *Bull. Sci. Math.* **98** (1974), 163–172], thereby establishing a new positivity theorem about them. Second, we introduce an extension of Kohnert's [*Bayreuth. Math. Schriften* **38** (1990), 1–97] "moves" to conjecture the first combinatorial rule for a certain deformation [Lascoux, in: *Physics and Combinatorics*, World Scientific Publishing, 2001, pp. 164–179] of the Key polynomials. Third, we use the same extension to conjecture a new rule for the Grothendieck polynomials [Lascoux and Schützenberger, *C. R. Acad. Sci. Paris Sér. I Math.* **295** (1982), 629–633].

1. INTRODUCTION

1.1. **Overview.** This paper contributes to the study of certain bases of the ring of polynomials $\mathsf{Pol} = \mathbb{Z}[x_1, x_2, \ldots]$ that are defined by symmetrizing operators. Our two main sources on this subject, and the specific perspective we pursue, are A. Lascoux's books [12, 14].

The Schur basis of the ring Λ of symmetric polynomials is central to algebraic combinatorics in at least two ways. These polynomials have fundamental applications outside of the theory of symmetric functions, specifically to representation theory of the symmetric group and of the general linear group, and to Schubert calculus, see, e.g., [20]. Moreover, understanding combinatorial descriptions of the Schur polynomials has led to a rich theory of Young tableaux. In particular, the problem of how to multiply two Schur polynomials, and expand the result back in the Schur basis, is important in the aforementioned applications. This problem is solved by the Littlewood–Richardson rule.

Now, since the ring of symmetric polynomials is a subring of Pol, one considers the following basic question [14]:

How does one lift properties of Λ (and its Schur basis) to the entirety of Pol*?*

A number of bases, that may be considered natural lifts of the Schur basis, are considered in [12, 14]. These include the Schubert, Grothendieck, Macdonald, and Key polynomials; we will also consider a deformation of the Key polynomials defined by A. Lascoux [11]. These are lifts of the Schur basis in the sense that a certain subset in each of these families is precisely the Schur basis, or otherwise deforms the elements of the Schur basis. In fact, like the Schur polynomials, each of these families have applications to representation theory and geometry.

Now, one would like to find combinatorial descriptions for the polynomials in each of these families. Indeed, such descriptions exist. Yet at present, there is no analogue of the Littlewood–Richardson rule. That is, for each basis, one desires a combinatorial description of how to multiply and expand in the basis so that one recovers a Littlewood–Richardson rule in the special case of Schur polynomials. For instance, for the case of

Schubert polynomials, and more generally, the case of Grothendieck polynomials, this is a longstanding open problem in combinatorial Schubert calculus, cf. [20].

There is a close tie between Littlewood–Richardson rules and the Young tableau description of Schur polynomials. Therefore, by analogy, one would like to find alternative combinatorial descriptions of the aforementioned bases of Pol. The hope is that such alternatives might shed light on finding corresponding generalizations of the Littlewood–Richardson rule.

Our work consists of a new combinatorial description for three of the aforementioned bases of polynomials. We give one theorem and two conjectures, which we summarize as follows:

First, we prove a "splitting" rule for the basis of *Key polynomials* $\{\kappa_\alpha \mid \alpha \in \mathbb{Z}_{\geq 0}^{\infty}\}$, thereby establishing a new positivity theorem about these polynomials. This family was introduced in [5] and first studied combinatorially in [15, 16]. Combinatorial rules for their monomial expansion are known, see, e.g., [15, 16, 21, 8]. Our rule refines the rule of [21, Theorem 5(1)]. Our rule is also analogous to the splitting rule [4, Corollary 3] for the basis of *Schubert polynomials* $\{\mathfrak{S}_w \mid w \in S_\infty\}$.

Second, we investigate the aforementioned basis of polynomials $\{\Omega_\alpha \mid \alpha \in \mathbb{Z}_{\geq 0}^{\infty}\}$ defined by A. Lascoux [11] that deforms the Key basis. By extending the *Kohnert moves* of [10] we conjecturally give the first combinatorial rule for the Ω-polynomials.

Third, in [10], the Kohnert moves were used to conjecture the first combinatorial rule for Schubert polynomials (a proof was later presented in [24]). Similarly, we use the extended Kohnert moves to give a conjecture for the basis of *Grothendieck polynomials* $\{\mathfrak{G}_w \mid w \in S_\infty\}$ [17]. This rule has a significantly different appearance than earlier (proved) rules, such as those in [7, 11, 3, 19].

1.2. **Splitting Key polynomials.** Let S_∞ be the group of permutations of $\mathbb{N}$ with finitely many non-fixed points. This group acts on Pol by permuting the variables. Let s_i be the simple transposition interchanging x_i and x_{i+1}. The **divided difference operator** acts on Pol by

$$\partial_i = \frac{1 - s_i}{x_i - x_{i+1}}.$$

Define the **Demazure operator** by setting

$$\pi_i(f) = \partial_i(x_i \cdot f), \text{ for } f \in \mathsf{Pol}.$$

Let $\alpha = (\alpha_1, \alpha_2, \ldots) \in \mathbb{Z}_{\geq 0}^{\infty}$ and assume throughout that $|\alpha| = \sum_i \alpha_i < \infty$. Define the **Key polynomial** κ_α to be

$$x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots, \quad \text{if } \alpha \text{ is weakly decreasing.}$$

Otherwise, set

$$\kappa_\alpha = \pi_i(\kappa_{\widehat{\alpha}}) \text{ where } \widehat{\alpha} = (\ldots, \alpha_{i+1}, \alpha_i, \ldots) \text{ and } \alpha_{i+1} > \alpha_i.$$

(The π_i's, and ∂_i's also, are well-known to satisfy the braid relations for S_n, and so the κ_α's are independent of the order in which the π_i's are applied.) Since the leading term (under the pure reverse lexicographic order) of κ_α is x^α, the Key polynomials form a $\mathbb{Z}$-basis of Pol.

The Key polynomials lift the Schur polynomials: when

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_t, 0, 0, 0, \ldots), \text{ where } \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_t, \tag{1.1}$$

then

$$\kappa_\alpha = s_{(\alpha_t,\ldots,\alpha_2,\alpha_1)}(x_1,\ldots,x_t). \tag{1.2}$$

A **descent** of α is an index i such that $\alpha_i \geq \alpha_{i+1}$; a **strict descent** is an index i such that $\alpha_i > \alpha_{i+1}$. Fix descents $d_1 < d_2 < \cdots < d_k$ of α containing all strict descents of α. Since π_i is a symmetrizing operator, κ_α is separately symmetric in each collection

$$X_1 = \{x_1, x_2, \ldots, x_{d_1}\},\ X_2 = \{x_{d_1+1}, x_{d_1+2}, \ldots, x_{d_2}\}, \ldots,\ X_k = \{x_{d_{k-1}+1}, x_{d_{k-1}+2}, \ldots, x_{d_k}\}.$$

(The variables $x_{d_k+1}, x_{d_k+2}, \ldots$ do not appear in κ_α.) Therefore, there is a unique expansion

$$\kappa_\alpha(X) = \sum_{\lambda^1,\ldots,\lambda^k} \mathcal{E}^{\alpha}_{\lambda^1,\ldots,\lambda^k}\, s_{\lambda^1}(X_1)\cdots s_{\lambda^k}(X_k), \tag{1.3}$$

where each λ^i is a partition. *A priori*, one only knows that $\mathcal{E}^{\alpha}_{\lambda^1,\ldots,\lambda^k} \in \mathbb{Z}$.

The **Rothe diagram** of a permutation $w \in S_n$ is

$$\texttt{Rothe}(w) = \{(x,y) \mid y < w(x) \text{ and } x < w^{-1}(y)\} \subset [n] \times [n]$$

(indexed so that the southwest corner is labeled $(1,1)$). The **code** of w, denoted by $\texttt{code}(w) \in \mathbb{Z}^n_{\geq 0}$ counts the number of boxes in columns of $\texttt{Rothe}(w)$ (from left to right). Given $\alpha \in \mathbb{Z}^{\infty}_{\geq 0}$, there is a unique $w[\alpha] \in S_\infty$ such that $\texttt{code}(w[\alpha]) = \alpha$ (up to trailing 0's); see, e.g., [20, Proposition 2.1.2]. We will need a special tableau coming from [23, Section 4]:

The tableau $T[\alpha]$: Given $w[\alpha]$, $i_1 < i_2 < \cdots < i_a$ in the first column of $T[\alpha]$ are given by having i_j be the largest descent position smaller than i_{j+1} in the permutation $ws_{i_a}s_{i_{a-1}}\cdots s_{i_{j+1}}$. The next column of $T[\alpha]$ is similarly determined, starting from $ws_{i_a}\cdots s_{i_1}$, etc.

An **increasing tableau** T of shape λ is a filling with strictly increasing rows and columns. (In fact, $T[\alpha]$ is an increasing tableau.) Let $\texttt{row}(T)$ be the reading word of T, obtained by reading the entries of T along rows, from right to left, and from top to bottom. Let $\min(T)$ be the smallest label in T. Finally, given a reduced word $\mathbf{a} = a_1a_2\ldots a_m$, let $\texttt{EGLS}(\mathbf{a})$ be the output of the *Edelman–Greene correspondence* (see Section 2.1).

The following result shows that $\mathcal{E}^{\alpha}_{\lambda^1,\ldots,\lambda^k} \in \mathbb{Z}_{\geq 0}$. It is analogous to one on Schubert polynomials [4, Corollary 3] (which our proof uses).

Theorem 1.1. *The number $\mathcal{E}^{\alpha}_{\lambda^1,\ldots,\lambda^k}$ counts sequences of increasing tableaux $(T_1, T_2, \ldots, T_k)$, where*

- *T_i is of shape λ^i;*
- *$\min T_1 > 0, \min T_2 > d_1, \min T_3 > d_2, \ldots, \min T_k > d_{k-1}$;*
- *$\texttt{row}(T_1)\cdot\texttt{row}(T_2)\cdots\texttt{row}(T_k)$ is a reduced word of $w[\alpha]$ such that $\texttt{EGLS}(\texttt{row}(T_1)\cdot\texttt{row}(T_2)\cdots\texttt{row}(T_k)) = T[\alpha]$.*

When $d_j = j$ for all $j \geq 1$, Theorem 1.1 specializes to an instance of the monomial expansion formula [21, Theorem 5(1)] for κ_α (restated as Theorem 2.5 below). Also, when (1.1) holds, $k = 1$, $d_1 = t$, and thus Theorem 1.1 gives (1.2).

Example 1.2. The (strict) descents of $\alpha = (1,3,0,2,2,1)$ are $d_1 = 2, d_2 = 5$, and

$$\begin{aligned}\kappa_{1,3,0,2,2,1} = {}& s_{3,2}(x_1,x_2)s_{2,1,1}(x_3,x_4,x_5) + s_{3,2}(x_1,x_2)s_{2,1}(x_3,x_4,x_5)s_1(x_6)\\ &+ s_{3,1}(x_1,x_2)s_{2,2}(x_3,x_4,x_5)s_1(x_6) + s_{3,1}(x_1,x_2)s_{2,2,1}(x_3,x_4,x_5)\end{aligned}$$

exhibits the claimed non-negativity of Theorem 1.1.

Also, $w[\alpha] = 2516743$ (one line notation) and

$$T[\alpha] = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2} 4 & 6 \\ \cline{1-2} 5 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array}.$$

Thus,

$$\mathcal{E}^{(1,3,0,2,2,1)}_{(3,2),(2,1,1),\emptyset} = \mathcal{E}^{(1,3,0,2,2,1)}_{(3,2),(2,1),(1)} = \mathcal{E}^{(1,3,0,2,2,1)}_{(3,1),(2,2),(1)} = \mathcal{E}^{(1,3,0,2,2,1)}_{(3,1),(2,2,1),\emptyset} = 1$$

are respectively witnessed by

$$\left(\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}, \begin{array}{|c|c|}\hline 4 & 6 \\ \hline 5 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array}, \emptyset\right), \ \left(\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}, \begin{array}{|c|c|}\hline 4 & 6 \\ \hline 5 \\ \cline{1-1}\end{array}, \begin{array}{|c|}\hline 6 \\ \hline\end{array}\right), \ \left(\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 4 & 5 \\ \hline 5 & 6 \\ \hline\end{array}, \begin{array}{|c|}\hline 6 \\ \hline\end{array}\right), \ \text{and} \ \left(\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 4 & 5 \\ \hline 5 & 6 \\ \hline 6 \\ \cline{1-1}\end{array}, \emptyset\right).$$

For example, for the leftmost sequence, $\mathtt{EGLS}(43152 \cdot 6456 \cdot \emptyset) = T[\alpha]$ holds. $\square$

1.3. **The Ω polynomials.** A. Lascoux [11] defines Ω_α for $\alpha = (\alpha_1, \alpha_2, \ldots) \in \mathbb{Z}^\infty_{\geq 0}$ by replacing π_i in the definition of the Key polynomials with the operator defined by

$$\widetilde{\pi}_i(f) = \partial_i(x_i(1 - x_{i+1})f).$$

(These operators can also be seen to satisfy the braid relations; cf. [14, Chapter 1.4].)

The initial condition is $\Omega_\alpha = x^\alpha (= \kappa_\alpha)$, if α is weakly decreasing. The Ω polynomials deform the Key polynomials. While at present there is no known geometric or representation theoretic interpretation of the Ω polynomials, as is pointed out in *loc. cit.*, many of the known relationships between the Key and Schubert basis extend to ones between the Ω and Grothendieck basis (the latter family is formally recalled in the next subsection).

The **skyline diagram** of α is $\mathtt{Skyline}(\alpha) = \{(i, y) : 1 \leq y \leq \alpha_i\} \subset \mathbb{N}^2$. Graphically, it is a collection of columns, the i-th column having height α_i. For instance,

$$\mathtt{Skyline}(1,3,0,2,2,1) = \begin{pmatrix} . & + & . & . & . & . \\ . & + & . & + & + & . \\ + & + & . & + & + & + \end{pmatrix}$$

Beginning with $\mathtt{Skyline}(\alpha)$, **Kohnert's rule** [10] generates diagrams D by sequentially moving any $+$ at the top of its column to the rightmost open position in its row and to its left. (The result of such a move need not be the skyline of any $\gamma \in \mathbb{Z}^\infty_{\geq 0}$.) Let $x^D = \prod_i x_i^{d_i}$ be the column weight, where d_i is the number of $+$'s in column i of D. If the same D results from a different sequence of moves, it only counts once. Kohnert's theorem states $\kappa_\alpha = \sum x^D$, where the sum is over all such D. Extending this, we introduce:

> **The K-Kohnert rule:** Each $+$ either moves as in Kohnert's rule, or stays in place *and* moves. That is, in the latter case, we mark the original position with a "g", and we place a $+$ in the rightmost open position in its row and to the left of the original position. The g's are unmovable, but a given $+$ treats g in the same way as other $+$'s when deciding if it can move, and to where. Diagrams with the same occupied positions but different arrangements of $+$'s and g's are counted separately.

Example 1.3. Below, we give all K-Kohnert moves one step from D:

$$D = \begin{pmatrix} + & . & g & + & . \\ . & + & + & + & + \end{pmatrix} \mapsto \begin{pmatrix} + & . & g & + & . \\ + & . & + & + & + \end{pmatrix}, \begin{pmatrix} + & . & g & + & . \\ + & g & + & + & + \end{pmatrix}, \begin{pmatrix} + & + & g & . & . \\ . & + & + & + & + \end{pmatrix},$$

$$\begin{pmatrix} + & + & g & g & \cdot \\ \cdot & + & + & + & + \end{pmatrix}, \begin{pmatrix} + & \cdot & g & + & \cdot \\ + & + & + & + & \cdot \end{pmatrix}, \begin{pmatrix} + & \cdot & g & + & \cdot \\ + & + & + & + & g \end{pmatrix}.$$

Let

$$J_\alpha^{(\beta)} = \sum \beta^{\#(g\text{'s appearing in } D)} x^D.$$

Conjecture 1.4. $J_\alpha^{(-1)} = \Omega_\alpha$.

Conjecture 1.4 has been checked by computer, for a wide range of cases up to α being of size 12, leaving us convinced. Clearly, $J_\alpha^{(0)} = \kappa_\alpha$, by Kohnert's theorem.

Example 1.5. Let $\alpha = (1, 0, 2)$. Then the diagrams contributing to $J_{(1,0,2)}$ are:

$$\texttt{Skyline}(1,0,2) = \begin{pmatrix} \cdot & \cdot & + \\ + & \cdot & + \end{pmatrix}, \begin{pmatrix} \cdot & + & \cdot \\ + & \cdot & + \end{pmatrix}, \begin{pmatrix} + & \cdot & \cdot \\ + & \cdot & + \end{pmatrix}, \begin{pmatrix} + & \cdot & \cdot \\ + & + & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & + & \cdot \\ + & + & \cdot \end{pmatrix};$$

$$\begin{pmatrix} + & g & \cdot \\ + & \cdot & + \end{pmatrix}, \begin{pmatrix} + & g & \cdot \\ + & + & \cdot \end{pmatrix}, \begin{pmatrix} + & \cdot & \cdot \\ + & + & g \end{pmatrix}, \begin{pmatrix} \cdot & + & \cdot \\ + & + & g \end{pmatrix}, \begin{pmatrix} \cdot & + & g \\ + & \cdot & + \end{pmatrix}, \begin{pmatrix} + & \cdot & g \\ + & \cdot & + \end{pmatrix}; \begin{pmatrix} + & g & \cdot \\ + & + & g \end{pmatrix}; \begin{pmatrix} + & g & g \\ + & \cdot & + \end{pmatrix}.$$

Thus,

$$\begin{aligned} J_{(1,0,2)} &= (x_1x_3^2 + x_1x_2x_3 + x_1^2x_3 + x_1^2x_2 + x_1x_2^2) \\ &\quad - (x_1^2x_2x_3 + x_1^2x_2^2 + x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2 + x_1^2x_3^2) + (x_1^2x_2^2x_3 + x_1^2x_2x_3^2). \end{aligned}$$

The lowest degree homogeneous component of Ω_α is κ_α. Hence any $f \in \mathsf{Pol}$ is a possibly *infinite* linear combination of the Ω_α's. Finiteness is asserted in [14, Chapter 5]. We show in Section 4.2 that the J_α's also form a (finite) basis.

1.4. **Grothendieck polynomials.** The **Grothendieck polynomial** [17] is defined using the **isobaric divided difference operator** whose action on $f \in \mathsf{Pol}$ is given by

$$\pi_i(f) = \partial_i((1 - x_{i+1})f).$$

(Once again, these operators are known to satisfy the braid relations.) Declare $\mathfrak{G}_{w_0}(X) = x_1^{n-1}x_2^{n-2}\cdots x_{n-1}$ where w_0 is the long element in S_n. Set $\mathfrak{G}_w(X) = \pi_i(\mathfrak{G}_{ws_i})$ if i is an ascent of w. The Grothendieck polynomials are known to lift $\{s_\lambda\}$ to Pol.

One has $\mathfrak{G}_w = \mathfrak{S}_w +$ (higher degree terms). We now state A. Kohnert's conjecture [10] for $\mathfrak{S}_w$. Starting with $\texttt{Rothe}(w)$, Kohnert's rule generates diagrams D by applying the same rules as described for his rule for κ_α. Then $\mathfrak{S}_w = \sum x^D$; the sum is over all such D.

Analogously, we define

$$K_w^{(\beta)} = \sum_D \beta^{\#(g\text{'s appearing in } D)} \mathbf{x}^D$$

where the sum is over all diagrams D generated by the K-Kohnert rule. For example, if $w = 3142$, the diagrams contributing to $K_w^{(\beta)}$ are

$$\texttt{Rothe}(3142) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ + & \cdot & + & \cdot \\ + & \cdot & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ + & + & \cdot & \cdot \\ + & \cdot & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ + & + & g & \cdot \\ + & \cdot & \cdot & \cdot \end{pmatrix}.$$

Hence, correspondingly, $K_{3142}^{(-1)} = (x_1^2x_3 + x_1^2x_2) - (x_1^2x_2x_3)$.

Conjecture 1.6. $K_w^{(-1)} = \mathfrak{G}_w$.

Note that $K_w^{(0)} = \mathfrak{S}_w$ is precisely Kohnert's conjecture. Conjecture 1.6 has been checked by computer for $n \leq 7$, and extensively for larger n. While Kohnert's rule for $\mathfrak{S}_w$ is handy, it remains mysterious, even after [24]. Conjectures 1.4 and 1.6 represent a return to Kohnert's conjecture (albeit after introducing a parameter β).

2. PROOF OF THEOREM 1.1

2.1. **Reduced word combinatorics.** Given $w \in S_n$, let

$$\mathbf{a} = (a_1, a_2, \ldots, a_{\ell(w)}) \text{ and } \mathbf{i} = (i_1, i_2, \ldots, i_{\ell(w)}).$$

In connection to [1], we say the pair $(\mathbf{a}, \mathbf{i})$ is a **stable compatible pair for** w if $s_{a_1} \cdots s_{a_{\ell(w)}}$ is a reduced word for w and the following two conditions on $\mathbf{i}$ hold:

(cs.1) $1 \leq i_1 \leq i_2 \leq \cdots \leq i_{\ell(w)} < n$;
(cs.2) $a_j < a_{j+1} \implies i_j < i_{j+1}$.

We will identify w with $\mathbf{a}$ and the associated reduced word.

The **Edelman–Greene correspondence** [6] (the same basic construction is used in [17]) is a bijection

$$\mathtt{EGLS} : (\mathbf{a}, \mathbf{i}) \mapsto (T, U),$$

where

- T is an increasing tableau such that $\mathtt{row}(T)$ is a reduced word for $\mathbf{a}$;
- U is a semistandard tableau whose multiset of labels contains precisely those in $\mathbf{i}$, and which has the same shape as T.

$\mathtt{EGLS}$ (column) insertion: We insert $\mathbf{a}$ from left to right, starting with a_1. When we reach step j of this process, we initially insert a_j into the leftmost column (of what will be T). If there are no labels strictly larger than a_j, we place a_j at the bottom of that column. If $a_j + t$ for $t > 2$ appears, we bump this $a_j + t$ to the next column to the right, replacing it with a_j. The same holds if $a_j + 1$ appears but not a_j. Finally, if both $a_j + 1$ and a_j already appear, we insert $a_j + 1$ into the next column to the right. Since $\mathbf{a}$ is assumed to be reduced, the above enumerates all possibilities. Finally at step j a new box is created at a corner; in what will be U we place i_j .

Mildly abusing terminology, let $\mathtt{EGLS}(\mathbf{a}) = T$.

We will need another standard notion in the subject. Two reduced words $\mathbf{a}$ and $\mathbf{a}'$ for the same permutation are in the same **Coxeter–Knuth class** if $\mathtt{EGLS}(\mathbf{a}) = \mathtt{EGLS}(\mathbf{a}') = T$. This T **represents** the class. The equivalence relation $\sim$ on reduced words is defined by the symmetric and transitive closure of the relations:

$$\begin{aligned}
\mathbf{A}i(i+1)i\mathbf{B} &\sim \mathbf{A}(i+1)i(i+1)\mathbf{B}, \\
\mathbf{A}acb\mathbf{B} &\sim \mathbf{A}cab\mathbf{B}, \\
\mathbf{A}bac\mathbf{B} &\sim \mathbf{A}bca\mathbf{B},
\end{aligned} \tag{2.1}$$

where $a < b < c$. In particular, it is true that $\mathbf{a} \sim \mathtt{row}(\mathtt{EGLS}(\mathbf{a}))$.

2.2. **Formulas for Schubert polynomials.** A stable compatible pair $(\mathbf{a},\mathbf{i})$ is a **compatible pair for** w if in addition to (cs.1) and (cs.2) the following holds:

(cs.3) $i_j \leq a_j$.

Let $\texttt{Compatible}(w)$ be the set of compatible sequences for w. A rule of [1] states that

$$\mathfrak{S}_w(X) = \sum_{(\mathbf{a},\mathbf{i})\in\texttt{Compatible}(w)} \mathbf{x}^{\mathbf{i}}. \tag{2.2}$$

A **descent** of w is an index j such that $w(j) > w(j+1)$. Let $\texttt{Descents}(w)$ be the set of descents of w. The following is [4, Corollary 3].

Theorem 2.1. *Let $w \in S_n$ and suppose that $\texttt{Descents}(w) \subseteq \{d_1 < d_2 < \cdots < d_k\}$. Then*

$$\mathfrak{S}_w(X) = \sum_{\lambda^1,\ldots,\lambda^k} c^w_{\lambda^1,\ldots,\lambda^k} s_{\lambda^1}(X_1)\cdots s_{\lambda^k}(X_k), \tag{2.3}$$

where $c^w_{\lambda^1,\ldots,\lambda^k}$ counts the number of tuples of increasing tableaux $(T_1,\ldots,T_k)$ satisfying the following properties:

(i) *T_i has shape λ^i;*
(ii) *$\min T_1 > 0$, $\min T_2 > d_1, \ldots, \min T_k > d_{k-1}$;*
(iii) *$\texttt{row}(T_1)\cdots\texttt{row}(T_k)$ is a reduced word of w.*

Assume for the remainder of the proof that

$$\texttt{Descents}(w) \subseteq \{d_1 < d_2 < \cdots < d_k\}. \tag{2.4}$$

Let

$$\texttt{Tuples}(w) = \{[(T_1,U_1),(T_2,U_2),\ldots,(T_k,U_k)]\},$$

where the T_i's satisfy (i), (ii), and (iii) from Theorem 2.1, and each U_i is a semistandard tableau of shape λ^i using the labels $d_{i-1}+1, d_{i-1}+2, \ldots, d_i$ ($d_0 = 0$).

2.3. **"Splitting" the EGLS correspondence.** Assuming (2.4) we define

$$\Phi : \texttt{Compatible}(w) \to \texttt{Tuples}(w).$$

Description of Φ (using EGLS): Uniquely split $(\mathbf{a},\mathbf{i}) \in \texttt{Compatible}$ as follows:

$$\left((\mathbf{a}^{(1)},\mathbf{i}^{(1)}),(\mathbf{a}^{(2)},\mathbf{i}^{(2)}),\ldots,(\mathbf{a}^{(k)},\mathbf{i}^{(k)})\right), \tag{2.5}$$

where

- $\mathbf{a} = \mathbf{a}^{(1)}\cdots\mathbf{a}^{(k)}$ and $\mathbf{i} = \mathbf{i}^{(1)}\cdots\mathbf{i}^{(k)}$ ("$\cdots$" means concatenation);
- the entries of $i^{(j)}$ are contained in the set $\{d_{j-1}+1, d_{j-1}+2, \ldots, d_j\}$.

Now define

$$\Phi((\mathbf{a},\mathbf{i})) := \left(\texttt{EGLS}(\mathbf{a}^{(1)},\mathbf{i}^{(1)}),\ldots,\texttt{EGLS}(\mathbf{a}^{(k)},\mathbf{i}^{(k)})\right).$$

Proposition 2.2. *The map $\Phi : \texttt{Compatible}(w) \to \texttt{Tuples}(w)$ is well-defined and a bijection.*

Proof. Φ is well-defined: The condition (i) just says that T_j and U_j have the same shape, which is true by the description of EGLS. For (ii), the splitting says that each label in $\mathbf{i}^{(j)}$ is strictly bigger than d_{j-1}. Now, by (cs.3), each label in $\mathbf{a}^{(j)}$ is strictly bigger than d_{j-1} as well. By the definition of EGLS, the set of labels appearing in T_j is the same as that of $\mathbf{a}^{(j)}$; hence (ii) holds. Lastly, $\texttt{row}(T_j)$ is a reduced word for $a^{(j)}$. Then (iii) is clear.

Φ is a bijection: Since EGLS is a bijective correspondence, clearly Φ is an injection. Consider the weight function on $\texttt{Compatible}(w)$ that assigns $(\mathbf{a}, \mathbf{i})$ weight $\mathbf{x}^{\mathbf{i}}$ and assigns $[(T_1, U_1), \ldots, (T_k, U_k)]$ the weight $\mathbf{x}^{U_1} \cdots \mathbf{x}^{U_k}$, where $\mathbf{x}^{U_i}$ is the usual monomial associated to the tableau U_i. Then clearly Φ is a weight-preserving map (since EGLS is similarly weight-preserving). Hence the surjectivity of Φ holds by (2.2) and Theorem 2.1. □

See [18, Section 5] for a proof of Theorem 2.1 which is close to the study of the split EGLS correspondence (the argument constructs certain crystal operators).

2.4. **The tableau $T[\alpha]$.** Recall that $w[\alpha] \in S_\infty$ satisfies $\texttt{code}(w[\alpha]) = \alpha$. Let $\prec$ be the pure reverse lexicographic total ordering on monomials. The Schubert polynomial $\mathfrak{S}_{w[\alpha]}$ has leading term $\mathbf{x}^\alpha$ (with respect to $\prec$). The same is true of κ_α (see [21, Corollary 7]), so

$$\mathfrak{S}_{w[\alpha]} = \kappa_\alpha + \text{linear combination of other Key polynomials.} \tag{2.6}$$

Given an increasing tableau U, the **nil left Key** $K_-^0(U)$ is defined by [16] (cf. [21, p. 111–114]). Let $\texttt{sort}(\alpha)$ be the partition obtained by rearranging α into weakly decreasing order. Also let $\texttt{content}(T)$ be the usual content vector of a semistandard tableau T. The following is a result of A. Lascoux and M.-P. Schützenberger (cf. [21, Theorem 4]):

Theorem 2.3.
$$\mathfrak{S}_w(X) = \sum \kappa_{\texttt{content}(K_-^0(U))},$$
where the sum is over all increasing tableaux U of shape $\texttt{sort}(\alpha)$ with $\texttt{row}(U) = w$.

Thus, by (2.6) combined with Theorem 2.3 there exists a unique increasing tableau $U[\alpha]$ of shape $\texttt{sort}(\alpha)$ with $\texttt{row}(U[\alpha]) = w[\alpha]$ and such that $\alpha = \texttt{content}(K_-^0(U[\alpha]))$.

Let $F_w = \lim_{k\to\infty} \mathfrak{S}_{1^k \times w}$ be the **stable Schubert polynomial** associated to w. This is a symmetric polynomial in infinitely many variables. So therefore one has an expansion

$$F_w = \sum_\lambda a_{w,\lambda} s_\lambda, \tag{2.7}$$

where the $a_{w,\lambda} \in \mathbb{Z}_{\geq 0}$ count increasing tableaux A of shape λ with $\texttt{row}(A) = w$. We mention that (2.3) is derived from (2.7) in [4]; thereby, (2.7) may be seen as a specialization of (2.3).

In [23, Theorem 4.1], it is shown that $a_{w,\mu'(w)} = 1$ for a certain explicitly described "maximal" $\mu'(w)$. Moreover a simple description of the witnessing tableau $A[\alpha]$ is given. Straightforwardly, $\mu'(w[\alpha]) = \texttt{sort}(\alpha)$. Then $T[\alpha]$ is precisely the witnessing tableau $A[\alpha]$ for $a_{w[\alpha],\lambda(w[\alpha])}$ (after accounting for the fact that the conventions in [23] use $F_{w[\alpha]}$ for what we call $F_{w[\alpha]^{-1}}$). We leave the details to the reader.

Finally, since the expansion of Theorem 2.3 refines (2.7) (see, e.g., [21]), we have

$$T[\alpha] = A[\alpha] = U[\alpha]. \tag{2.8}$$

So, $T[\alpha]$ is an increasing tableau of shape $\texttt{sort}[\alpha]$ with the properties that $\texttt{row}(T[\alpha]) = w[\alpha]$ and $\texttt{content}(K_-(T[\alpha])) = \alpha$.

2.5. **Conclusion of the proof of Theorem 1.1:** From the definition of $\texttt{Rothe}(w[\alpha])$, we deduce the following fact.

Lemma 2.4. *The descents of $w[\alpha]$ are contained in the set of descents $d_1 < d_2 < \cdots < d_k$ of α.*

By Lemma 2.4 combined with Theorem 2.1, we obtain

$$\mathfrak{S}_{w[\alpha]}(X) = \sum_{(\mathbf{a},\mathbf{i})} \mathbf{x}^{\mathbf{i}} = \sum_{\lambda^1,\ldots,\lambda^k} c^{w[\alpha]}_{\lambda^1,\ldots,\lambda^k} s_{\lambda^1}(X_1)\cdots s_{\lambda^k}(X_k). \tag{2.9}$$

We recall the following formula [21, Theorem 5].

Theorem 2.5. *Fix an increasing tableau T with $\texttt{content}(K_-^0(T)) = \alpha$. Then*

$$\kappa_\alpha = \sum_{(\mathbf{a},\mathbf{i})} \mathbf{x}^{\mathbf{i}},$$

where the sum is over compatible sequences $(\mathbf{a},\mathbf{i})$ satisfying (cs.1), (cs.2), (cs.3), *and* $\texttt{EGLS}(\mathbf{a}) = T$.

In view of (2.8) and the properties about $T[\alpha]$ stated immediately after that equation, we may set $T = T[\alpha]$ in Theorem 2.5 to obtain a monomial expansion formula for κ_α in terms of compatible pairs. Thus, our theorem statement is that κ_α is precisely equal to a prescribed subset of the summands of (2.9).

Thus to complete the proof, restrict Φ to those $(\mathbf{a},\mathbf{i}) \in \texttt{Compatible}(w[\alpha])$ such that $\texttt{EGLS}(\mathbf{a}) = T[\alpha]$. Consider $\Phi(\mathbf{a},\mathbf{i}) = [(T_1,U_1),\ldots,(T_k,U_k)]$. Since $\mathbf{a}^{(i)} \sim \texttt{row}(T_i)$, by (2.1) we see

$$\texttt{row}(T_1)\cdots\texttt{row}(T_k) \sim \mathbf{a}^{(1)}\cdots\mathbf{a}^{(k)} = \mathbf{a}. \tag{2.10}$$

However, since we have assumed $\texttt{EGLS}(\mathbf{a}) = T[\alpha]$, it follows that

$$\texttt{EGLS}(\texttt{row}(T_1)\cdots\texttt{row}(T_k)) = T[\alpha], \tag{2.11}$$

The other two requirements on $(T_1,\ldots,T_k)$ hold since Φ is well-defined (Proposition 2.2). The desired conditions on $(U_1,\ldots,U_k)$ follow from the well-definedness of Φ and Lemma 2.4.

Conversely, suppose $[(T_1,U_1),\ldots,(T_k,U_k)]$ has $(T_1,\ldots,T_k)$ satisfying Theorem 1.1's conditions. Since Φ is a bijection (Proposition 2.2), we have

$$\Phi^{-1}([(T_1,U_1),\ldots,(T_k,U_k)]) = (\mathbf{a},\mathbf{i}) \in \texttt{Compatible}(w[\alpha]).$$

Also, by (2.10), $\mathbf{a} \sim \texttt{row}(T_1)\cdots\texttt{row}(T_k)$. Now, we assumed that (2.11) holds. Hence, $\texttt{EGLS}(\mathbf{a}) = T[\alpha]$ as desired. $\square$

3. Additional remarks

3.1. **Comments on Theorem 1.1.** Since the κ_α's specialize non-symmetric Macdonald polynomials (see, e.g., [8, Section 5.3]), can one extend Theorem 1.1 in that direction?

Theorem 1.1 implies that the Key module of [21, Section 5] should have an action of $GL(d_1) \times GL(d_2 - d_1) \times \cdots \times GL(d_k - d_{k-1})$ such that the character is κ_α.

V. Reiner suggests a variation of Theorem 1.1 using plactic theory. The derivation should be similar, using formulas from [22]. However we are missing the analogue of [4, Corollary 4]; cf. [9, Sections 7, 8]. Theorem 1.1 naturally generalizes to Grothendieck polynomials, using [3, 2]; details may appear elsewhere.

3.2. **J_α's form a (finite) basis of** Pol. Clearly, $J_\alpha(X) = \mathbf{x}^\alpha + \sum_{\beta \prec \alpha} c_\beta \mathbf{x}^\beta$. One decomposes $f \in$ Pol into a possibly infinite sum of J_α's,

$$f = \sum_{\alpha} g_\alpha J_\alpha. \tag{3.1}$$

That is, find the $\prec$ largest monomial $\mathbf{x}^{\theta_0}$ appearing in $f^{(0)} := f$ (say with coefficient c_{θ_0}) and let $f^{(1)} := f - c_{\theta_0} \cdot J_{\theta_t}$. Thus $f^{(1)}$ only contains monomials strictly smaller in the $\prec$ ordering. Now repeat, defining $f^{(t+1)} := f^{(t)} - c_{\theta_t} J_{\theta_t}$ where $\mathbf{x}^{\theta_t}$ is the $\prec$-largest monomial appearing in $f^{(t)}$, etc. Since J_α is not homogeneous, each step t potentially introduces $\prec$-smaller monomials but of higher degree. However, we claim the following.

Proposition 3.1. *The expansion* (3.1) *is finite.*

Proof. By the K-Kohnert rule, each β that appears in J_α is contained in the smallest rectangle R that contains α. So the above procedure only involves the finitely many diagrams contained in R for one of the finitely many initial $\alpha \in \mathbb{Z}_{\geq 0}^{\infty}$ such that $\mathbf{x}^\alpha$ is in f. □

3.3. **More on the interplay of Grothendieck and the Ω polynomials.** M. Shimozono has suggested that the expansion of $\mathfrak{G}_w$ in Ω_α's should alternate in sign, by degree. An explicit rule exhibiting this has been conjectured by V. Reiner and the second author.

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