Séminaire Lotharingien de Combinatoire **75** (2016), Article B75c

Surprising Relations Between Sums-Of-Squares of Characters of the Symmetric Group Over Two-Rowed Shapes and Over Hook Shapes

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Abstract. In a recent article, we noted (and proved) that the sum of the squares of the characters of the symmetric group, $\chi^\lambda(\mu)$, over all shapes λ with two rows and n cells and $\mu = 31^{n-3}$, equals, surprisingly, to 1/2 of that sum-of-squares taken over all hook shapes with $n+2$ cells and with $\mu = 321^{n-3}$. In the present note, we show that this is only the tip of a huge iceberg! We will prove that, if μ consists of odd parts and (a possibly empty) string of *consecutive* powers of 2, namely $2, 4, \ldots, 2^{t-1}$ for $t \geq 1$, then the sum of $\chi^\lambda(\mu)^2$ over all two-rowed shapes λ with n cells equals exactly $\frac{1}{2}$ times the analogous sum of $\chi^\lambda(\mu')^2$ over all shapes λ of *hook shape* with $n+2$ cells, where μ' is the partition obtained from μ by retaining all odd parts but replacing the string $2, 4, \ldots, 2^{t-1}$ by 2^t.

Recall that the *constant term* of a *Laurent polynomial* in $(x_1, \ldots, x_m)$ is the free term, i.e., the coefficient of $x_1^0 \cdots x_m^0$. For example,

$$\mathrm{CT}_{x_1,x_2}(x_1^{-3}x_2 + x_1x_2^{-2} + 5) = 5.$$

Recall that a *partition* (alias *shape*) of an integer n, with m *parts* (alias *rows*), is a non-increasing sequence of positive integers

$$\lambda = (\lambda_1, \ldots, \lambda_m),$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m > 0$ and $\lambda_1 + \cdots + \lambda_m = n$.

If $\lambda = (\lambda_1, \ldots, \lambda_m)$ and $\mu = (\mu_1, \ldots, \mu_r)$ are partitions of n with m and r parts, respectively, then it easily follows from [M, p. 114, Eq. (7.8)], that the *characters*, $\chi^\lambda(\mu)$, of the *symmetric group*, S_n, may be obtained via the *constant term* expression

$$\chi^\lambda(\mu) \;=\; \mathrm{CT}_{x_1,\ldots,x_m} \frac{\prod_{1\leq i<j\leq m}(1-\frac{x_j}{x_i})\prod_{j=1}^r\left(\sum_{i=1}^m x_i^{\mu_j}\right)}{\prod_{i=1}^m x_i^{\lambda_i}}. \qquad (Chi)$$

As usual, for a partition μ, $|\mu|$ denotes the sum of its parts, in other words, the integer that is being partitioned.

In [RRZ] we considered two quantities. Let μ_0 be a partition with smallest part ≥ 2. The first quantity, that we will call henceforth $A(\mu_0)(n)$, is the following sum-of-squares over two-rowed shapes λ:

$$A(\mu_0)(n) := \sum_{j=0}^{\lfloor n/2\rfloor} \chi^{(n-j,j)}(\mu_0 1^{n-|\mu_0|})^2.$$

(Note that in [RRZ] this quantity was denoted by $\psi^{(2)}(\mu_0 1^{n-|\mu_0|})$.)

The second quantity was the sum-of-squares over *hook-shapes*

$$B(\mu_0)(n) := \sum_{j=1}^{n} \chi^{(j,1^{n-j})}(\mu_0 1^{n-|\mu_0|})^2.$$

(Note that in [RRZ] this quantity was denoted by $\phi^{(2)}(\mu_0 1^{n-|\mu_0|})$.)

In [RRZ] we developed algorithms for discovering (and then proving) closed-form expressions for these quantities, for a *given* (specific) finite partition μ_0 with smallest part larger than one. In fact we proved that each such expression is *always* a multiple of $\binom{2n}{n}$ by a certain rational function of n that depends on μ_0.

Unless μ_0 is very small, these rational functions turn out to be very complicated, but, inspired by the One-Line Encyclopedia of Integer Sequences [S], Alon Regev noted (and then it was proved in [RRZ]) the *remarkable* identity

$$A(3)(n) = \frac{1}{2}B(3,2)(n+2).$$

This led to the following natural question:

> *Are there other partitions, μ_0, such that there exists a partition, μ_0' with $|\mu_0'| = |\mu_0| + 2$, such that the ratio $A(\mu_0)(n)/B(\mu_0')(n+2)$ is a constant?*

This led us to write a new procedure in the Maple package

`http://www.math.rutgers.edu/~zeilberg/tokhniot/Sn.txt`

that accompanies [RRZ], called `SeferNisim(K,N0)`, which searched for such pairs $[\mu_0, \mu_0']$. We then used our *human* ability for *pattern recognition* to notice that all the successful pairs (we went up to $|\mu_0| \le 20$) turned out to be such that μ_0 either consisted of only odd parts, and then μ_0' was μ_0 with 2 appended, or, more generally, μ_0 consisted of odd parts together with a string of *consecutive* powers of 2 (starting with 2), and μ_0' was obtained from μ_0 by retaining all the odd parts but replacing the string of powers of 2 by a single power of 2, one higher than the highest in μ_0. In symbols, we conjectured (and later proved [see below], *alas*, by purely human means) the following theorem.

Theorem. *Let μ_0 be a partition of the form*

$$\mu_0 = \mathrm{Sort}([a_1, \dots, a_s, 2, 2^2, \dots, 2^{t-1}]),$$

where

$$a_1 \ge a_2 \ge \dots \ge a_s \ge 3$$

are all **odd** *and* $t \geq 1$. *(If $t = 1$ then μ_0 only consists of odd parts.) Define*

$$\mu_0' = \mathrm{Sort}([a_1, \ldots, a_s, 2^t]).$$

Then, for every $n \geq |\mu_0|$, we have

$$A(\mu_0)(n) = \frac{1}{2} B(\mu_0')(n+2).$$

(For a sequence of integers S, the symbol $\mathrm{Sort}(S)$ denotes that sequence sorted in non-increasing order.)

In order to prove our theorem, we need to first recall the following **constant-term** expression for $B(\mu_0)(n)$ from [RRZ].

Lemma 1. *If $\mu_0 = (a_1, \ldots, a_r)$, we have*

$$B(\mu_0)(n) = \mathrm{Coeff}_{x^0} \left[\frac{(1+x)^{2n-2-2(a_1+\cdots+a_r)}}{x^{n-1}} \cdot \prod_{i=1}^{r} (x^{a_i} - (-1)^{a_i})(1 - (-1)^{a_i} x^{a_i}) \right].$$

We need an analogous constant-term expression for $A(\mu_0)(n)$. To that end, let us first spell out Equation (Chi) for the two-rowed case, $m = 2$. In that case, we may write $\lambda = (n-j, j)$. With $\mu_0 = (a_1, \ldots, a_r)$, we have

$$\chi^{(n-j,j)}(\mu_0 1^{n-|\mu_0|}) \;=\; \mathrm{CT}_{x_1,x_2} \, \frac{(1-\frac{x_2}{x_1})(x_1+x_2)^{n-a_1-\cdots-a_r} \prod_{i=1}^{r} (x_1^{a_i} + x_2^{a_i})}{x_1^{n-j} x_2^{j}}. \qquad (Chi2)$$

This can be rewritten as

$$\chi^{(n-j,j)}(\mu_0 1^{n-|\mu_0|}) \;=\; \mathrm{CT}_{x_1,x_2} \, \frac{(1-\frac{x_2}{x_1})(1+\frac{x_2}{x_1})^{n-a_1-\cdots-a_r} \prod_{i=1}^{r} \left(1 + (\frac{x_2}{x_1})^{a_j}\right)}{(\frac{x_2}{x_1})^{j}}. \qquad (Chi2')$$

Since the *constant-termand* is of the form $P(\frac{x_2}{x_1})/(\frac{x_2}{x_1})^j$, for some *single-variable* polynomial $P(x)$, the above can be equivalently expressed in the form

$$\chi^{(n-j,j)}(\mu_0 1^{n-|\mu_0|}) \;=\; \mathrm{Coeff}_{x^0} \, \frac{(1-x)(1+x)^{n-a_1-\cdots-a_r} \prod_{i=1}^{r} (1+x^{a_i})}{x^j}. \qquad (Chi2'')$$

Note that the left-hand side is *utter nonsense* if $j > \frac{n}{2}$, but the right-hand side makes perfect sense. It is easy to see that, defining $\chi^{(n-j,j)}(\mu_0 1^{n-|\mu_0|})$ by the right-hand side for $j > \frac{n}{2}$, we get

$$\chi^{(n-j,j)}(\mu_0 1^{n-|\mu_0|}) \;=\; -\chi^{(j,n-j)}(\mu_0 1^{n-|\mu_0|}).$$

Let us denote the numerator of the constant-termand of (Chi''), namely

$$(1-x)(1+x)^{n-a_1-\cdots-a_r}\prod_{i=1}^{r}(1+x^{a_i})\,,$$

by $P(x)$. Then Equation $(Chi2'')$ can be also rewritten as a *generating function*,

$$P(x)=\sum_{j=0}^{n}\chi^{(n-j,j)}(\mu_0 1^{n-|\mu_0|})\,x^j.$$

Since for any polynomial of a single variable, $P(x)=\sum_{j=0}^{n}c_jx^j$, we have

$$\sum_{j=0}^{n}c_j^2=\mathrm{Coeff}_{x^0}\big[P(x)P(x^{-1})\big]\,,$$

we get

$$\begin{aligned}\sum_{j=0}^{n}\chi^{(n-j,j)}(\mu_0 1^{n-|\mu_0|})^2 &= \mathrm{Coeff}_{x^0}\Bigg[\Bigg((1-x)(1+x)^{n-a_1-\cdots-a_r}\prod_{j=1}^{r}(1+x^{a_j})\Bigg)\\ &\qquad\cdot\Bigg((1-x^{-1})(1+x^{-1})^{n-a_1-\cdots-a_r}\prod_{j=1}^{r}\left(1+x^{-a_j}\right)\Bigg)\Bigg]\\ &= -\mathrm{Coeff}_{x^0}\left[\frac{(1-x)^2(1+x)^{2(n-a_1-\cdots-a_r)}\prod_{j=1}^{r}(1+x^{a_j})^2}{x^{n+1}}\right].\end{aligned}$$

But since, by symmetry,

$$\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}\chi^{(n-j,j)}(\mu_0 1^{n-|\mu_0|})^2=\frac{1}{2}\sum_{j=0}^{n}\chi^{(n-j,j)}(\mu_0 1^{n-|\mu_0|})^2,$$

we have the following auxiliary result.

Lemma 2. *Let $\mu_0=(a_1,\ldots,a_r)$ be a partition with smallest part larger than one. Then*

$$A(\mu_0)(n)=-\frac{1}{2}\,\mathrm{Coeff}_{x^0}\left[\frac{(1-x)^2(1+x)^{2(n-a_1-\cdots-a_r)}\prod_{j=1}^{r}(1+x^{a_j})^2}{x^{n+1}}\right].$$

We are now ready to prove the theorem. If $\mu_0=\mathrm{Sort}(a_1,\ldots,a_r,2,\ldots,2^{t-1})$, then

$$\begin{aligned}A(\mu_0)(n)=-\frac{1}{2}\,\mathrm{Coeff}_{x^0}\Bigg[&\frac{(1-x)^2(1+x)^{2(n-a_1-\cdots-a_r-2-2^2-\cdots 2^{t-1})}}{x^{n+1}}\\ &\cdot\prod_{j=1}^{t-1}\left(1+x^{2^j}\right)^2\prod_{j=1}^{r}(1+x^{a_j})^2\Bigg].\end{aligned}$$

But, transferring a factor of $(1+x)^2$ from the second factor to the product, $\prod_{j=1}^{t-1}\left(1+x^{2^j}\right)^2$, we have

$$(1+x)^{2(n-a_1-\cdots-a_r-2-2^2-\cdots 2^{t-1})}\prod_{j=1}^{t-1}\left(1+x^{2^j}\right)^2$$
$$=(1+x)^{2(n-a_1-\cdots-a_r-1-2-2^2-\cdots 2^{t-1})}\prod_{j=0}^{t-1}\left(1+x^{2^j}\right)^2 .$$

Hence,

$$A(\mu_0)(n)=-\frac{1}{2}\,\mathrm{Coeff}_{x^0}\Bigg[\frac{(1-x)^2(1+x)^{2(n-a_1-\cdots-a_r-1-2-2^2-\cdots 2^{t-1})}}{x^{n+1}}$$
$$\cdot\prod_{j=0}^{t-1}\left(1+x^{2^j}\right)^2\prod_{j=1}^{r}(1+x^{a_j})^2\Bigg] .$$

By Euler's good-old $(1-x)\prod_{j=0}^{t-1}(1+x^{2^j})=1-x^{2^t}$, we conclude

$$A(\mu_0)(n)$$
$$=-\frac{1}{2}\,\mathrm{Coeff}_{x^0}\left[\frac{(1-x^{2^t})^2(1+x)^{2(n-a_1-\cdots-a_r-1-2-2^2-\cdots 2^{t-1})}\prod_{j=1}^{r}(1+x^{a_j})^2}{x^{n+1}}\right] .$$

On the other hand, since $\mu_0'=\mathrm{Sort}(a_1,\ldots,a_r,2^t)$, and all the a_i's are odd, we have

$$B(\mu_0')(n+2)=-\mathrm{Coeff}_{x^0}\left[\frac{(1+x)^{2n+2-2(a_1+\cdots+a_r+2^t)}}{x^{n+1}}\cdot(x^{2^t}-1)^2\cdot\prod_{j=1}^{r}(x^{a_j}+1)^2\right] .$$

This completes the proof, since $-(1+2+2^2+\cdots+2^{t-1})=1-2^t$. □

Acknowledgment

The research for this work was done while the second-named author visited the Faculty of Mathematics at the Weizmann Institute of Science, during the week of October 5–9, 2015. He wishes to thank the Weizmann Institute for its hospitality, and its dedicated stuff, most notably Gizel Maimon.

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