Riemann-Roch Theory for Graph Orientations

Spencer Backman

La Sapienza - Università di Roma

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The Riemann-Roch theorem

- The classical Riemann-Roch theorem is one of the cornerstones of modern algebraic geometry.
- It is a certain statement about the dimensions of linear spaces of locally rational functions on Riemann surfaces with prescribed lower bounds for zeros and poles.
- Baker and Norin presented a combinatorial version of this statement for graphs using the language of chip-firing.
- Their formula has been applied to solve problems in algebraic geometry and number theory.

Chip-firing

chip-firing

- A chip configuration is a collection of poker chips sitting at the vertices. In keeping with algebraic geometry we may call chip configurations divisors.
- A vertex fires by sending a chip to each of its neighbors.

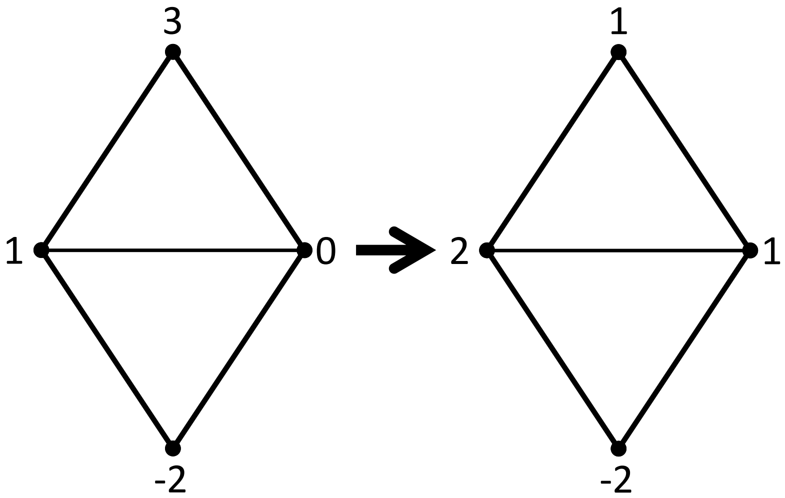

Figure : An example of a chip-firing move

The graph Laplacian and chip-firing

- The graph Laplacian is $Q = D - A$ where D is a diagonal matrix with $D_{i,i} = \deg(v_i)$ and A is the adjacency matrix.
- Chip-firing can be described using the Laplacian.
- If we represent a chip configuration by a vextor $\vec{x}$, then firing v_i gives the new vector $\vec{x} - Qe_i$.

History

Chip-firing was independently introduced in several different communities.

- Poset Theory: '72 Mosesian
- Discrete Probability: '75 Engel
- Statistical Physics: '87 Bak-Tang-Weisenfeld
- Coxeter Theory: '87 Mozes
- Arithmetic Geometry: '70 Raynaud and '90 Lorenzini
- Graph Theory: '91 Björner-Lovász-Shor

A natural question

Baker and Norin:

Given a chip configuration D, when can we bring every vertex out of debt by chip-firing?

- Algorithmic solution:
  1. : Fix a vertex q, and bring every other vertex out of debt.
  2. : Send as many chips back to q as possible by firing sets of vertices simultaneously without sending any vertex back into debt.
- The game is winnable if and only if q is out of debt when the process terminates.

Cool fact: The resulting configurations, called q-reduced divisors or G-parking functions are in bijection with spanning trees.

A nautral refinement

Baker and Norin:

What is the minimum number of chips we need to remove so that we no longer have a winning strategy?

- One less than this quantity is $r(D)$, the rank of a chip-configuration.
- Observation: clearly, $r(D) \leq \#$ of chips in D.

More definitions

- $g = |E| - |V| + 1$ is the genus of a graph.
- $K = \sum_{v \in V(G)} (\deg(v) - 2)(v)$ is the canonical divisor.
- $\deg(D) = \sum_{v \in V(G)} D(v)$ is the degree of D.

The Riemann-Roch theorem for graphs [Baker and Norine 07]

$$r(D) - r(K - D) = \deg(D) - g + 1$$

.

Chip-firing is closely related to graph orientations, particularly acyclic orientations.

History

- Mosesian observed that if you have an acyclic orientation of a graph, you can reverse the edges at a sink to obtain a new acylcic orientation.
- Björner, Lovász, and Shor noted that the indegree sequences of the two acyclic orientations are related by firing the sink in question.
- Mikhalkin-Zharkov and Cori-Le Borgne recognized that divisors associated to acyclic full orientations play a distinguished role in RR theory.
- Gioan generalized this setup to arbitrarily full orientations using cut (cocycle) reversals and dual cycle reversals.

Goal

Describe chip-firing and the Riemann-Roch formula completely in the language of graph orientations.

Immediate obstruction

- Given an orientation, we associate a chip configuration $D_{\mathcal{O}}$ given by the indegree **-1** of each vertex in $\mathcal{O}$.
- Problem: All chip configurations associated to full orientations have $g - 1$ chips and we care about other numbers of chips.
- Solution: Partial graph orientations.

The generalized cycle-cocycle reversal system

- A partial orientation $\mathcal{O}$ of a graph G is an orientation of some edges of G.
- We say that two partial orientations $\mathcal{O}$ and $\mathcal{O}'$ are equivalent in the generalized cycle-cocycle reversal system , written $\mathcal{O} \sim \mathcal{O}'$ if they are related by a sequence of cut reversals, cycle reversals, and edge pivots.

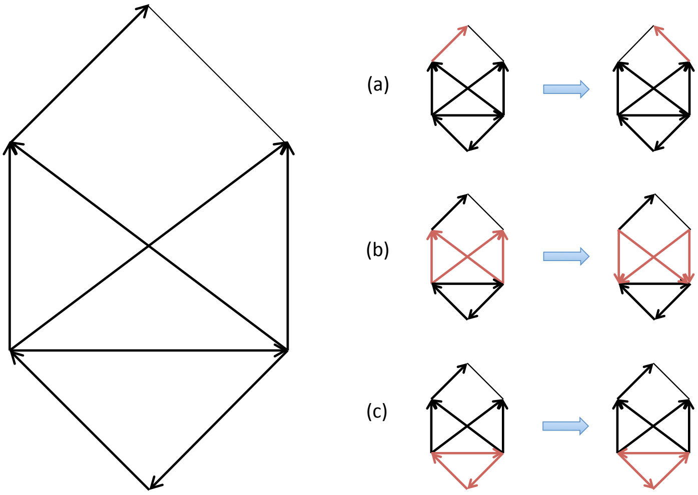

Figure : A partial orientation with (a) an edge pivot, (b) a cocycle reversal, and (c) a cycle reversal.

Theorem [B.]

$\mathcal{O}_1 \sim \mathcal{O}_2$ if and only if $D_{\mathcal{O}_1} \sim D_{\mathcal{O}_2}$.

To prove this theorem, we introduce a nonlocal extension of edge pivots.

Jacob's ladder cascade

Given a directed path terminating at a vertex incident to an unoriented edge, we can perform a sequence of edge pivots to unorient the initial edge of the path.

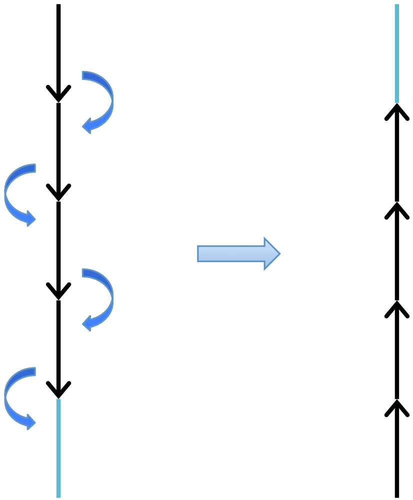

Figure : A Jacob's ladder cascade

Theorem [B.]

Given a partial orientation O, either

1. $\mathcal{O} \sim \mathcal{O}'$ where $\mathcal{O}'$ is sourceless. $(r(D_{\mathcal{O}}) \geq 0)$
2. $\mathcal{O} \sim \mathcal{O}'$ where $\mathcal{O}'$ is acyclic $(r(D_{\mathcal{O}}) = -1)$

We call the algorithm which produces the desired orientation the unfurling algorithm because it unravels directed cycles.

We recover a famous algorithm of Dhar as a shadow of the unfurling algorithm by looking at the associated indegree sequences.

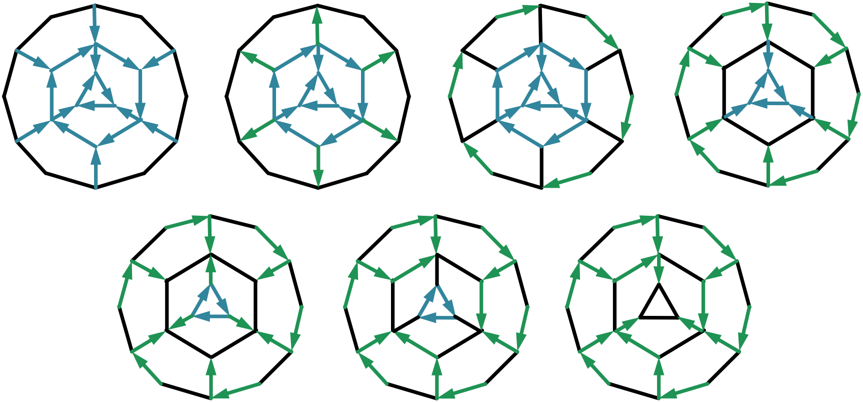

Figure : The unfurling algorithm

Reduction to the study of partial orientations

Theorem [B.]

Let D be a divisor with $\deg(D) \leq g - 1$, then $D \sim D_{\mathcal{O}}$ for some partial orientation $\mathcal{O}$ if and only if $r(D + \vec{1}) \geq 0$.

This is strong enough to reduce the study of ranks of divisors to the study of partial orientations.

Ranks of partial orientations

Theorem [B.]

$r(D_O)$ = the number of directed paths which need to be reversed in the generalized cycle-cocycle reversal system to produce an acyclic partial orientation minus one.

These results are applied to give a new proof of the Riemann-Roch theorem.

Baker and Norin's Key Lemma

Key Lemma

Given a chip configuration D with $r(D) = -1$ then there exists some $\nu \geq D$ with $\deg(\nu) = g - 1$ and $r(\nu) = -1$.

In the language of orientations this says:

1. Every acyclic partial orientation can be extended to a full acyclic orientation.
2. Every acyclic full orientation is equivalent via source reversals to an acyclic full orientation with a unique source.

Remark: Part 1) of this statement can be applied to prove that the number of acyclic partial orientations is $2^g T(3, 1/2)$ where $T(x, y)$ is the Tutte polynomial.

Grazie!

Indiscrete remark: I'm back on the job market.