Combinatorics of some deformed convolution algebras

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Collaborators :

Polylogarithms and harmonic sums :
V.C. Bui, J.-Y. Enjalbert, Hoang Ngoc Minh, Q.H. Ngô

Combinatorial Physics and Dyson's equations :
Katarzyna Gorska, Andrzej Horzela, Karol A. Penson

Séminaire Lotharingien de Combinatoire,
September, 2015, Bertinoro

Pierre-André Picon, PAP or PA as we used to call him

The question/1

Throughout the talk, A stands for a $\mathbb{Q}$-algebra (associative, commutative with unit). In order to make the exposition no heavier than absolutely necessary, details will not always be provided but can, of course, be on request or through references ...

On the 17th of August (2015) ...

The question

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Important formulas in Combinatorics

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Motivation:

The poster for the conference celebrating Noga Alon's 60th birthday, fifteen formulas describing some of Alon's work are presented. (See this post, for the poster, and cash prizes offered for identifying the formulas.) This demonstrates that sometimes (but certainly not always) a major research progress, even areas, can be represented by a single formula. Naturally, following Alon's poster, I thought about representing other people's works through formulas. (My own work, Doron Zeilberger's, etc. Maybe I will pursue this in some future posts.) But I think it will be very useful to collect major formulas representing major research in combinatorics.

The Question

The question collects important formulas representing major progress in combinatorics.

The rules are:

Rules

1) one formula per answer

2) Present the formula *explicitly* (not just by name or by a link or reference), and briefly explain the formula and its importance, again not just link or reference. (But then you may add links and references.)

3) Formulas should represent important research level mathematics. (So, say $\sum \binom{n}{k}^2 = \binom{2n}{n}$ is too elementary.)

4) The formula should be explicit as possible, moving from the formula to the theory it represent should also be explicit, and explaining the formula and its importance at least in rough terms should be feasible.

5) I am a little hesitant if classic formulas like $V - E + F = 2$ are qualified.

co.combinatorics   big-list

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asked Aug 17 at 8:59

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Gil Kalai

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11 One of my favorites is the Cauchy identity (a fundamental product-sum relation):

Exponential formula

25

The exponential formula Can be phrased as

All = exp(Connected)

In a more precise way, if you have a class $\mathcal{C}$ of labelled graphs which is *locally finite i.e.* for every finite set F and $k \in \mathbb{N}$

$$S(n,k) = \mathrm{card}(\mathcal{C}(F,k)) < +\infty$$

where $\mathcal{C}(F,k)$ stands for the subclass of graphs with F as labels and k connected components ($S(n,k)$ is supposed to depend only on $n = \mathrm{card}(F)$). If, moreover, the class $\mathcal{C}$ is closed by

1. relabeling
2. connected components (i.e. $\Gamma \in \mathcal{C}$ iff all connected components of Γ are in $\mathcal{C}$)
3. disjoint union

then

$$\sum_{n,k\geq 0} S(n,k)\frac{x^n}{n!}y^k = e^{y(\sum_{n\geq 1} S(n,1)\frac{x^n}{n!})} \qquad (1).$$

This formula has many **applications** and variants in combinatorics as the computation of the GF of the Bell, Stirling numbers, number of cycles, graphs of endofunctions (with or without constraints), set partitions and the analog for unlabelled graphs to cite only a few.

All the matrices $S(n,k)$ possess the *Sheffer property* i.e. the EGF of the k-th column is (up to a scalar) the k-th power of the EGF of the first (for $k=1$). It is equivalent to formula (1).

Matrices having the *Sheffer property* (not only provided by classes of labelled graphs) form an infinite dimensional Lie group generated by vector fields on the line (see Tom Copeland's answer). Connections of this group can be seen in combinatorial physics, statistics on graphs and over categories.

A usual, useful and (almost) immediate generalisation. In fact, we have

$$S(n,k) = \mathrm{card}(\mathcal{C}(F,k)) = \sum_{\gamma\in\mathcal{C}(F,k)} \mathbf{1}(\gamma)$$

where $\mathbf{1}$ is the constant (equal to 1) function on the class $\mathcal{C}$, and one can, for free (*i.e.* with the same proof), replace $\mathbf{1}$ by any $\mathbb{Q}$-algebra valued *multiplicative statistics*, "c" *i.e.* such that

$$c(\gamma_1 \sqcup \gamma_2) = c(\gamma_1)c(\gamma_2);\ c(\mathcal{C}_\emptyset) = 1$$

(where $\mathcal{C}_\emptyset$ is the empty graph and $\sqcup$ stands for the disjoint union).

Then, with

Shuffles, stuffles and other dual laws

Mother Formula

All what follows is around the same recursive formula/pattern.

$$au * bv = a(u * bv) + b(au * v) + \varphi(a,b)(u * v) \quad (0)$$

The **shuffle product** appears in many contexts (representation theory, iterated integrals, Hecke algebras, symmetric functions, decomposition of polytopes, theory of languages, of codes, of automata).

It turns out that it can be better understood as a law dual to a comultiplication. These co-operations were introduced, in combinatorics, by a seminal paper of Joni and Rota (S.A. Joni and G.-C. Rota, Coalgebras and bialgebras in combinatorics, Stud. Appl. Math. 61 (1979) 93–139.).

Considering two (non empty) words as card decks au, bv the top cards being respectively a, b, the shuffle product of au and bv reads (I do not know how to write the Cyrillic ``Sha", which is the standard sign for the shuffle, in MathJax, so I use $\sqcup$)

$$au \sqcup bv = a(u \sqcup bv) + b(au \sqcup v) \quad (1)$$

which is the sum of all possible shuffles between au and bv (two disjoint cases a or b on top).

Formula (1) together with the initialization making neutral the empty word i.e.

$$w \sqcup 1 = 1 \sqcup w = w$$

defines perfectly the shuffle product.

Now, this law is better understood as "dual". I mean, if you define the natural pairing on the words by $\langle u \mid v\rangle := \delta_{u,v}$ you get

$$\langle u \sqcup v \mid w\rangle = \langle u \otimes v \mid \Delta(w)\rangle$$

with

$$\Delta(w) = \sum_{I+J=[1..|w|]} w[I] \otimes w[J] \quad (2)$$

where $|w|$ stands for the length of w and, for $I = \{i_1, i_2, \cdots i_k\}$ a choice of places (indexed in increasing order $i_1 < i_2 < \cdots < i_k$), $w[I]$ is the subword

$$w[I] = w[i_1]w[i_2]\cdots w[i_k]$$

(therefore $\Delta(w)$ is sometimes called the ``unshuffling" of w).

Shuffles/2

... this reinforces my motivation to advocate in favour of the ease, utility and deepness of shuffle products and (some of) their deformations.

Shuffle products and their deformations appear in many contexts

- representation theory
- iterated integrals
- Dyson series
- Hecke algebras
- symmetric functions
- decomposition of polytopes
- computer science : theory of languages, of codes, of automata

The formula of shuffle product in brief

Considering two (non empty) words as card decks au, bv the top cards being respectively a, b, the shuffle product of au and bv reads

$$au \sqcup\!\sqcup bv = a(u \sqcup\!\sqcup bv) + b(au \sqcup\!\sqcup v) \tag{1}$$

which is the sum of all possible shuffles between au and bv (two disjoint cases a or b on top).
Formula (1) together with the initialization making neutral the empty word i.e.

$$w \sqcup\!\sqcup 1 = 1 \sqcup\!\sqcup w = w \tag{2}$$

defines perfectly the shuffle product.

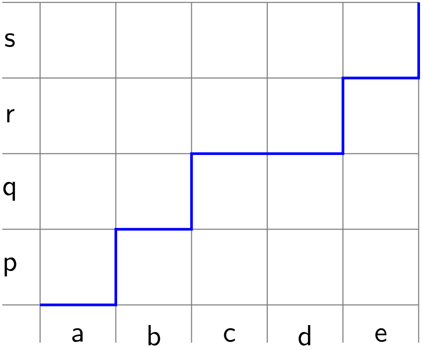
A first deformation : the stuffle product

The stuffle product (also called Hoffman's shuffle, quasi-shuffle, sticky shuffle) appears in many contexts (harmonic sums, lambda rings, quasi-symmetric functions). This time, the set of cards is infinite, more precisely, you have an alphabet $\{y_i\}_{i\in\mathbb{N}_{>0}}$ indexed by non-zero integers. The stuffle law is defined recursively as

$$\begin{aligned} w * 1 &= 1 * w = w \\ y_i u * y_j v &= y_i(u * y_j v) + y_j(y_i u * v) + y_{i+j}(u * v) \end{aligned} \tag{3}$$

the term $y_{i+j}(u * v)$ is the reason why certain physicists call it "sticky shuffle" because, in this case, the cards y_i, y_j stick together.

Shuffle via Dyck paths

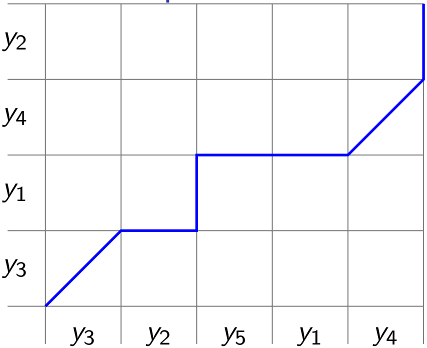


Path which contributes $apbqcdres$ in the shuffle product $abcde ⧢ pqrs$.

$$u ⧢ v = \sum_{\pi \in \mathcal{D}(|u|,|v|)} ev(\pi, u, v)$$

$$\mathcal{D}(p,q) = \{\pi \in \{n,e\}^* \mid |\pi|_e = p,\ |\pi|_n = q\}$$

Stuffle via Motzkin paths



Path which contributes $y_6 y_2 y_1 y_5 y_1 y_8 y_2$ in the stuffle product $y_3 y_2 y_5 y_1 y_4 \uplus y_3 y_1 y_4 y_2$.

$$u \uplus v \quad = \sum_{\pi \in \mathcal{M}(|u|,|v|)} ev(\pi, u, v)$$

$$\mathcal{M}(p, q) \quad = \quad \{\pi \in \{n, e, d\}^* \quad | \quad |\pi|_{e,d} = p, \ |\pi|_{n,d} = q\}$$

Remark The evaluation of the diagonal steps are here $\varphi(y_i, y_j) = y_{i+j}$ but $\varphi : Y \times Y \to AY$ can be arbitrary.

φ-shuffle

In this case the recursion becomes

$$\begin{aligned} w \sqcup\!\sqcup_{\varphi} 1 &= 1 \sqcup\!\sqcup_{\varphi} w = w \\ y_i u \sqcup\!\sqcup_{\varphi} y_j v &= y_i (u \sqcup\!\sqcup_{\varphi} y_j v) + y_j (y_i u \sqcup\!\sqcup_{\varphi} v) \\ &\quad + \varphi(y_i, y_j)(u \sqcup\!\sqcup_{\varphi} v) \end{aligned} \tag{4}$$

Where $Y = \{y_i\}_{i \in I}$ is an indexed alphabet and $\varphi : Y \times Y \to AY$ is defined by its structure constants

$$\varphi(y_i, y_j) = \sum_{k \in I} \gamma_{i,j}^{k} y_k \tag{5}$$

We get the following (not exhaustive) zoology found in the literature.

What can be found in the literature ?

Name	Formula (recursion)	φ
Shuffle	$au \shuffle bv = a(u \shuffle bv) + b(au \shuffle v)$	$\varphi \equiv 0$
Stuffle	$x_i u \stuffle x_j v = x_i(u \stuffle x_j v) + x_j(x_i u \stuffle v)$ $+ x_{i+j}(u \stuffle v)$	$\varphi(x_i, x_j) = x_{i+j}$
Min-stuffle	$x_i u \minstuffle x_j v = x_i(u \minstuffle x_j v) + x_j(x_i u \minstuffle v)$ $- x_{i+j}(u \minstuffle v)$	$\varphi(x_i, x_j) = -x_{i+j}$
Muffle	$x_i u \muffle x_j v = x_i(u \muffle x_j v) + x_j(x_i u \muffle v)$ $+ x_{i\times j}(u \muffle v)$	$\varphi(x_i, x_j) = x_{i\times j}$
q-shuffle	$x_i u \stuffle_q x_j v = x_i(u \stuffle_q x_j v) + x_j(x_i u \stuffle_q v)$ $+ q x_{i+j}(u \stuffle_q v)$	$\varphi(x_i, x_j) = q x_{i+j}$
q-shuffle$_2$	$x_i u \stuffle_q x_j v = x_i(u \stuffle_q x_j v) + x_j(x_i u \stuffle_q v)$ $+ q^{i.j} x_{i+j}(u \stuffle_q v)$	$\varphi(x_i, x_j) = q^{i.j} x_{i+j}$
$\mathtt{LDIAG}(1, q_s)$ (non-crossed, non-shifted)	$au \shuffle bv = a(u \shuffle bv) + b(au \shuffle v)$ $+ q_s^{\lvert a\rvert \lvert b\rvert} a.b(u \shuffle v)$	$\varphi(a, b) = q_s^{\lvert a\rvert \lvert b\rvert}(a.b)$
q-Infiltration	$au \uparrow bv = a(u \uparrow bv) + b(au \uparrow v)$ $+ q\delta_{a,b} a(u \uparrow v)$	$\varphi(a, b) = q\delta_{a,b} a$
AC-stuffle	$au \shuffle_\varphi bv = a(u \shuffle_\varphi bv) + b(au \shuffle_\varphi v)$ $+ \varphi(a, b)(u \shuffle_\varphi v)$	$\varphi(a, b) = \varphi(b, a)$ $\varphi(\varphi(a, b), c) = \varphi(a, \varphi(b, c))$
Semigroup-stuffle	$x_t u \shuffle_\perp x_s v = x_t(u \shuffle_\perp x_s v) + x_s(x_t u \shuffle_\perp v)$ $+ x_{t\perp s}(u \shuffle_\perp v)$	$\varphi(x_t, x_s) = x_{t\perp s}$
φ-shuffle	$au \shuffle_\varphi bv = a(u \shuffle_\varphi bv) + b(au \shuffle_\varphi v)$ $+ \varphi(a, b)(u \shuffle_\varphi v)$	$\varphi(a, b)$ law of AAU

An example of complicated φ

In order to get return of nice properties, φ must be at least associative but, even in the AC (Associative, Commutative) and "natural" cases, its structure constants can be very complicated. As an example, let us invoke the *truncated Hurwitz polyzêta* functions given by :

$$\forall N \in \mathbb{N}_{>0}, \quad \zeta_N(\mathbf{s},\mathbf{t}) = \sum_{N \geqslant n_r > \ldots > n_1 > 0} \frac{1}{(n_1 - t_1)^{s_1} \ldots (n_r - t_r)^{s_r}} \tag{6}$$

In order to obtain the product law, we will use here two alphabets $Y = \{y_i\}_{i \in \mathbb{N}_{>0}}$, $Z = \{z_t\}_{t \in \mathbb{C} \setminus \mathbb{N}_{>0}}$, the (free) submonoid M generated by $Y \times Z$. We now have a product $⧢$ on the indices such that for all indices and $N \in \mathbb{N}$

$$\zeta_N\left((\mathbf{s},\mathbf{t}) ⧢ (\mathbf{s}',\mathbf{t}')\right) = \zeta_N(\mathbf{s},\mathbf{t})\zeta_N(\mathbf{s}',\mathbf{t}') \ . \tag{7}$$

Recursion for $\sqcup\!\sqcup$

Let $Y = \{y_i\}$, $Z = \{z_t\}$ and M as above.
The huffle is defined as a bilinear product over $k[M] = k\langle Y \times Z\rangle$ such that

$$\forall w \in M^*, \qquad w \sqcup\!\sqcup 1_{N^*} = 1_{N^*} \sqcup\!\sqcup w = w,$$

$$\forall y_i, y_j \in Y^2, \forall z_t, z_{t'} \in Z^2, \forall u, v \in N^{*2},$$

$$\begin{aligned}
t = t' \quad \Rightarrow \quad & (y_i, z_t)u \sqcup\!\sqcup (y_j, z_t)v \\
& = (y_i, z_t)(u \sqcup\!\sqcup (y_j, z_t)v) + (y_j, z_t)((y_i, z_t)u \sqcup\!\sqcup v) \\
& +(y_{i+j}, z_t)(u \sqcup\!\sqcup v) \\
t \neq t' \quad \Rightarrow \quad & (y_i, z_t).u \sqcup\!\sqcup (y_j, z_{t'}).v \\
& = (y_i, z_t).\,(u \sqcup\!\sqcup (y_j, z_{t'}).v) + (y_j, z_{t'}).\,((y_i, z_t).u \sqcup\!\sqcup v) \\
& + \sum_{n=0}^{i-1} \binom{j-1+n}{j-1} \frac{(-1)^n}{(t-t')^{j+n}} (y_{i-n}, z_t).\,(u \sqcup\!\sqcup v) \\
& + \sum_{n=0}^{j-1} \binom{i-1+n}{i-1} \frac{(-1)^n}{(t'-t)^{i+n}} (y_{j-n}, z_{t'}).\,(u \sqcup\!\sqcup v) \quad .
\end{aligned}$$

Recursion for $\sqcup\!\sqcup$ /2

The reason of this bizzarely shaped φ stands in the following (exercise) lemma

Lemma

For any integers $s, r \geq 1$, for any complex numbers $a,\ b \neq a$:

$$\forall x \in \mathbb{C} \setminus \{a, b\}, \frac{1}{(x-a)^s (x-b)^r} = \sum_{k=1}^{s} \frac{a_k}{(x-a)^k} + \sum_{k=1}^{r} \frac{b_k}{(x-b)^k} \tag{8}$$

where, for all $k \in \{1, \ldots, s\}$, $a_k = \binom{s+r-k-1}{r-1} \frac{(-1)^{s-k}}{(a-b)^{s+r-k}}$
and, for all $k \in \{1, \ldots, r\}$, $b_k = \binom{s+r-k-1}{s-1} \frac{(-1)^{r-k}}{(b-a)^{s+r-k}}$.

Shuffle and φ-shuffle characters

Series S which satisfy the following equations

$$\begin{cases} \langle S|1\rangle = 1, \\ \langle S|u \sqcup\!\sqcup v\rangle = \langle S|u\rangle\langle S|v\rangle, \ (\forall u, v \in Y^*) \end{cases} \tag{9}$$

can legitimately be called shuffle characters. If you replace $\sqcup\!\sqcup$ by $\sqcup\!\sqcup_\varphi$, then S is a φ-shuffle character (remark that $(u \sqcup\!\sqcup_\varphi v)$ is, in any case, a polynomial). We have two famous examples of such characters :

- Solutions of differential equations (shuffle, linked to special functions and combinatorial physics, see e.g. SLC 74)
- Harmonic sums (stuffle, linked to polyzêtas)

Dual formulation

One can set

$$\Delta_{\sqcup\!\sqcup_\varphi}(S) = \sum_{u,v\in Y^*} \langle S | u \sqcup\!\sqcup_\varphi v\rangle u \otimes v \tag{10}$$

(it is a double series and a linear form on the space of double polynomials). System (9) can be rephrased as

$$\left\{\begin{array}{lcl} \langle S|1\rangle & = & 1, \\ \Delta(S) & = & S\otimes S, \text{ (as linear forms)} \end{array}\right. . \tag{11}$$

These elements are called *group-like* and as it can be checked easily that $\Delta(ST) = \Delta(S)\Delta(T)$, these series form a group (for the concatenation product) called the Hausdorff group (for $\Delta_{\sqcup\!\sqcup_\varphi}$). This group is an infinite-dimensional Lie group, with a nice log-exp correspondence and Lie algebra, the space of series s.t.

$$\Delta(S) = S\otimes 1 + 1\otimes S \tag{12}$$

these elements are called *primitive*.

Dualizability of φ-deformed shuffle products

Definition

Let ш_φ be the product $Y^* \times Y^* \to A\langle Y\rangle$ satisfying the conditions :

i) for any $w \in Y^*$, $1_{Y^*} \text{ш}_\varphi w = w \text{ш}_\varphi 1_{Y^*} = w$,

ii) for any $a, b \in Y$ and $u, v \in Y^*$,

$$(R) \qquad au \text{ш}_\varphi bv = a(u \text{ш}_\varphi bv) + b(au \text{ш}_\varphi v) + \varphi(a,b)(u \text{ш}_\varphi v),$$

where φ is an arbitrary mapping defined by its structure constants

$$\begin{array}{rcl} \varphi : Y \times Y & \longrightarrow & AY, \\ (y_i, y_j) & \longmapsto & \displaystyle\sum_{k \in I \subset \mathbb{N}_+} \gamma_{i,j}^k \, y_k. \end{array}$$

It is said to be dualizable if there exists $\Delta_{\text{ш}_\varphi} : A\langle Y\rangle \to A\langle Y\rangle \otimes A\langle Y\rangle$ such that the dual mapping $\Big(A\langle Y\rangle \otimes A\langle Y\rangle\Big)^* \to A\langle\langle Y\rangle\rangle$ restricts to ш_φ.

Proposition

(R) and i) define a unique mapping $\text{ш}_\varphi : Y^* \times Y^* \to A\langle Y\rangle$ which is at once extended by multilinearity as a law $\text{ш}_\varphi : A\langle Y\rangle \times A\langle Y\rangle \to A\langle Y\rangle$.

What are the dualizable φ-shuffles among our examples

Name	Formula (recursion)	φ
Shuffle	$au \shuffle bv = a(u \shuffle bv) + b(au \shuffle v)$	$\varphi \equiv 0$ (Y)
Stuffle	$x_i u \overset{+}{\sqcup} x_j v = x_i(u \overset{+}{\sqcup} x_j v) + x_j(x_i u \overset{+}{\sqcup} v)$ $+ x_{i+j}(u \overset{+}{\sqcup} v)$	$\varphi(x_i, x_j) = x_{i+j}$ (Y) if $i, j \in \mathbb{N}$
Min-stuffle	$x_i u \overset{-}{\sqcup} x_j v = x_i(u \overset{-}{\sqcup} x_j v) + x_j(x_i u \overset{-}{\sqcup} v)$ $- x_{i+j}(u \overset{-}{\sqcup} v)$	$\varphi(x_i, x_j) = -x_{i+j}$ (Y) if $i, j \in \mathbb{N}_{>0}$
Muffle	$x_i u \overset{\bullet}{\sqcup} x_j v = x_i(u \overset{\bullet}{\sqcup} x_j v) + x_j(x_i u \overset{\bullet}{\sqcup} v)$ $+ x_{i \times j}(u \overset{\bullet}{\sqcup} v)$	$\varphi(x_i, x_j) = x_{i \times j}$ (N) for $i, j \in \mathbb{Q}_{>0}$
q-shuffle	$x_i u \overset{+}{\sqcup}_q x_j v = x_i(u \overset{+}{\sqcup}_q x_j v) + x_j(x_i u \overset{+}{\sqcup}_q v)$ $+ q x_{i+j}(u \overset{+}{\sqcup}_q v)$	$\varphi(x_i, x_j) = q x_{i+j}$ (Y) if $i, j \in \mathbb{N}$
q-shuffle$_2$	$x_i u \overset{+}{\sqcup}_q x_j v = x_i(u \overset{+}{\sqcup}_q x_j v) + x_j(x_i u \overset{+}{\sqcup}_q v)$ $+ q^{i.j} x_{i+j}(u \overset{+}{\sqcup}_q v)$	$\varphi(x_i, x_j) = q^{i.j} x_{i+j}$ (Y) if $i, j \in \mathbb{N}$
$\mathtt{LDIAG}(1, q_s)$ (non-crossed, non-shifted)	$au \shuffle bv = a(u \shuffle bv) + b(au \shuffle v)$ $+ q_s^{\lvert a \rvert \lvert b \rvert} a.b(u \shuffle v)$	$\varphi(a, b) = q_s^{\lvert a \rvert \lvert b \rvert}(a.b)$ (Y)
q-Infiltration	$au \uparrow bv = a(u \uparrow bv) + b(au \uparrow v)$ $+ q\delta_{a,b} a(u \uparrow v)$	$\varphi(a, b) = q\delta_{a,b} a$ (Y)
AC-stuffle	$au \shuffle_\varphi bv = a(u \shuffle_\varphi bv) + b(au \shuffle_\varphi v)$ $+ \varphi(a, b)(u \shuffle_\varphi v)$	depends on φ, AC law
Semigroup- -stuffle	$x_t u \shuffle_\perp x_s v = x_t(u \shuffle_\perp x_s v) + x_s(x_t u \shuffle_\perp v)$ $+ x_{t \perp s}(u \shuffle_\perp v)$	$\varphi(x_t, x_s) = x_{t \perp s}$ depends on the semigroup
φ-shuffle	$au \shuffle_\varphi bv = a(u \shuffle_\varphi bv) + b(au \shuffle_\varphi v)$ $+ \varphi(a, b)(u \shuffle_\varphi v)$	$\varphi(a, b)$ law of AAU depends on φ, ass. law

Properties of φ-deformed shuffle products

Lemma

Let Δ be the morphism $A\langle Y\rangle \longrightarrow A\langle\langle Y^* \otimes Y^*\rangle\rangle$ defined on the letters by

$$\Delta(y_s) = y_s \otimes 1 + 1 \otimes y_s + \sum_{n,m\in I} \gamma^s_{n,m}\, y_n \otimes y_m.$$

Then

i) $\forall u, v, w \in Y^*, \langle u \uplus_\varphi v | w\rangle = \langle u \otimes v | \Delta(w)\rangle^{\otimes 2}$.

ii) $\forall w \in Y^+, \Delta(w) = w \otimes 1 + 1 \otimes w + \sum_{u,v\in Y^+} \langle \Delta(w) | u \otimes v\rangle u \otimes v$.

Theorem

i) *The law $\uplus_\varphi$ is associative (resp. commutative) if and only if the linear extension $\varphi : AY \otimes AY \longrightarrow AY$ is so.*

ii) *Let $\gamma^z_{x,y} := \langle \varphi(x, y) | z\rangle$ be the structure constants of φ, then $\uplus_\varphi$ is dualizable if and only if $(\gamma^z_{x,y})_{x,y,z\in Y}$ has the following property*

$$(\forall z \in Y)(\#\{(x, y) \in Y^2 | \gamma^z_{x,y} \neq 0\} < +\infty).$$

Associative commutative φ-deformed shuffle products

Theorem

Let us suppose that φ is associative and dualizable. We still denote the dual law of $\uplus_\varphi$ by $\Delta_{\uplus_\varphi} : A\langle Y\rangle \longrightarrow A\langle Y\rangle \otimes A\langle Y\rangle$;
$\mathcal{B}_\varphi := (A\langle Y\rangle, \mathrm{conc}, 1_{Y^*}, \Delta_{\uplus_\varphi}, \varepsilon)$ is a bialgebra. The following conditions are equivalent

i) *$\mathcal{B}_\varphi$ is an enveloping bialgebra* (**CQMM** *theorem)*

ii) *(A without ZD) the algebra AY admits an exhaustive filtration $((AY)_n)_{n\in\mathbb{N}}$*
$$(AY)_0 = \{0\} \subset (AY)_1 \subset \cdots \subset (AY)_n \subset (AY)_{n+1} \subset \cdots$$
compatible with comultiplication, i.e.
$$\Delta_\varphi((AY)_n) \quad \subset \quad \sum_{p+q=n} \mathrm{Im}((AY)_p \otimes (AY)_q).$$

iii) *$\mathcal{B}_\varphi$ is isomorphic to $(A\langle Y\rangle, \mathrm{conc}, 1_{Y^*}, \Delta_{\sqcup\!\sqcup}, \epsilon)$ as a bialgebra.*

iv) *For all $y \in Y$, the following series is a polynomial.*
$$\pi_1(y) = y + \sum_{l\geq 2} \frac{(-1)^{l-1}}{l} \sum_{x_1,\dots,x_l\in Y} \langle y|\varphi(x_1\dots x_l)\rangle\, x_1\dots x_l.$$

In the previous equivalent cases, φ will be called moderate.

φ-extended Eulerian project

Theorem (φ-extended Eulerian projector)

Let $\Phi_{\pi_1^\varphi}$ *be the endomorphism of* $(A\langle Y\rangle, conc, 1_{Y^*})$ *defined on the letters by*

$$\forall y \in Y, \pi_1^\varphi(y) = y + \sum_{l\geq 2} \frac{(-1)^{l-1}}{l} \sum_{x_1,\ldots,x_l\in Y} \gamma^y_{x_1,\ldots,x_l}\, x_1\ldots x_l,$$

$$\gamma^y_{x_1,\ldots,x_l} = \sum_{t_1,\ldots,t_{l-2}\in Y} \gamma^y_{x_1,t_1}\gamma^{t_1}_{x_2,t_2}\ldots\gamma^{t_{l-2}}_{x_{l-1},x_l}.$$

Then $\Phi_{\pi_1^\varphi}$ *is an automorphism of* $(A\langle Y\rangle, conc, 1_{Y^*})$ *which is an isomorphism of bialgebras from* $(A\langle Y\rangle, \mathtt{conc}, \Delta_{\sqcup\!\sqcup}, \epsilon_Y)$ *to* $(A\langle Y\rangle, \mathtt{conc}, \Delta_{\uplus_\varphi}, \epsilon_Y)$.

Pair of bases in duality in $(A\langle Y\rangle, ., 1_{Y^*}, \Delta_{⧢_\varphi}, \epsilon_Y)$

$$
\begin{aligned}
\Pi_{y_k} &= \pi_1^{\varphi}(y_k), && \text{for } k \geq 1,\\
\Pi_l &= [\Pi_s, \Pi_r], && \text{for } l \in \mathcal{L}ynY - Y \text{ and } \sigma(l) = (s, r),\\
\Pi_w &= \Pi_{l_1}^{i_1} \ldots \Pi_{l_k}^{i_k}, && \text{for } w = l_1^{i_1} \ldots l_k^{i_k}, l_1 > \ldots > l_k, l_1 \ldots, l_k \in \mathcal{L}ynY.
\end{aligned}
$$

Here, φ is supposed commutative, associative, dualizable and moderate. One can prove (through $\Phi^{\vee}_{\pi_1^{\varphi}}$) that the elements $\{\Sigma_w\}_{w\in Y^*}$ (computed just below) are polynomials.
$\{\Sigma_w\}_{w\in Y^*}$ = dual basis of $\{\Pi_w\}_{w\in Y^*}$: $\forall u, v \in Y^*, \langle \Sigma_v | \Pi_u\rangle = \delta_{u,v}$.
For any $w = l_1^{i_1} \ldots l_k^{i_k}$, with $l_1, \ldots, l_k \in \mathcal{L}ynY$ and $l_1 > \ldots > l_k$,

$$\Sigma_w = \frac{1}{i_1! \ldots i_k!} \Sigma_{l_1}^{⧢_\varphi i_1} ⧢_\varphi \ldots ⧢_\varphi \Sigma_{l_k}^{⧢_\varphi i_k}.$$

Triangularity

Proposition

$\{\Pi_w\}_{w\in Y^*}$ is upper triangular and $\{\Sigma_w\}_{w\in Y^*}$ is lower triangular :

$$\forall w \in Y^*, \quad \Pi_w = P_w + \sum_{|v|>|w|} c_v v, \quad \Sigma_w = S_w + \sum_{|v|<|w|} d_v v.$$

$$\Longrightarrow \forall l \in \mathcal{L}ynY, \quad \Pi_l = P_l + \sum_{|v|>|l|} c_v v, \quad \Sigma_l = S_l + \sum_{|v|<|l|} c_v v.$$

Where the families $(P_w)_{w\in Y^*}$; $(S_w)_{w\in Y^*}$ are computed as in slide (24) but with $\varphi \equiv 0$.

Combinatorial structure of $(A\langle Y\rangle, ., 1_{Y^*}, \Delta_{\uplus_\varphi}, \epsilon_Y)$

Theorem

Let $\mathcal{P}_Y$ be the space of primitive elements in $\mathcal{B}_\varphi$ and $\mathcal{I}_Y$, the space generated by the proper shuffles.

1. *The free associative algebra $A\langle Y\rangle$ is isomorphic to $\mathcal{U}(\mathcal{P}_Y)$.*
2. *$\mathcal{P}_Y$ as a A-module is freely generated by $\{\Pi_l\}_{l\in\mathcal{L}ynY}$.*
3. *The polynomials $\{\Sigma_l\}_{l\in\mathcal{L}ynY}$ and $\{\Sigma_w\}_{w\in Y^*}$ are (pure) transcendence and linear bases, respectively, of $(A\langle Y\rangle, \uplus_\varphi, 1_{Y^*})$.*
4. *The families $\{\Sigma_{l_1}{}^{\uplus_\varphi i_1} \uplus_\varphi \ldots \uplus_\varphi \Sigma_{l_k}{}^{\uplus_\varphi i_k}\}^{i_1,\ldots,i_k\geq 1, k\geq 2}_{l_1,\ldots,l_k\in\mathcal{L}ynY}$ and $\{l_1^{\uplus_\varphi i_1} \uplus_\varphi \ldots \uplus_\varphi l_k^{\uplus_\varphi i_k}\}^{i_1,\ldots,i_k\geq 1, k\geq 2}_{l_1,\ldots,l_k\in\mathcal{L}ynY}$ form bases for $\mathcal{I}_Y$.*
5. *$\{\Pi_l\}_{l\in\mathcal{L}ynY}$ and $\{\Pi_{l_1}{}^{\uplus_\varphi i_1} \uplus_\varphi \ldots \uplus_\varphi \Pi_{l_k}{}^{\uplus_\varphi i_k}\}^{i_1,\ldots,i_k\geq 1}_{l_1,\ldots,l_k\in\mathcal{L}ynY}$ are, respectively, (pure) transcendence and linear bases of $A\langle Y\rangle$.*
6. $\mathcal{I}_Y = \bigoplus_{k\geq 2} \mathcal{P}_Y^{\uplus_\varphi k}$.

Schützenberger's factorization

Theorem (φ-extended Schützenberger's factorization)
Let $\mathcal{D}_Y := \sum_{w \in Y^*} w \otimes w$. *Then*

$$\mathcal{D}_Y = \sum_{w \in Y^*} \Sigma_w \otimes \Pi_w = \prod_{l \in \mathcal{L}ynY}^{\searrow} e^{\Sigma_l \otimes \Pi_l}.$$

Application Let S be a $\sqcup\!\sqcup_\varphi$ character then, applying $S \otimes Id$ on each member one gets

$$S = (S \otimes Id)\Big(\sum_{w \in Y^*} w \otimes w\Big) = \prod_{l \in \mathcal{L}ynY}^{\searrow} e^{\langle S | \Sigma_l \rangle \Pi_l} \tag{13}$$

Which provides a Wei-Norman type system of local coordinates on the Hausdorff group.

Conclusion

We have investigated more deeply the φ-deformed shuffle products (work in progress).

- as soon as φ is associative we get a Hopf algebra

$$\mathcal{B}_{\varphi}^{\vee} = (A\langle X\rangle, ⧢_{\varphi}, 1_{X^*}, \Delta_{conc}, \varepsilon)$$

- if, moreover φ is commutative, we have Radford's theorem
- if, moreover φ is dualizable, we get a dual bialgebra $\mathcal{B}_{\varphi} = (A\langle X\rangle, conc, 1_{X^*}, \Delta_{⧢_{\varphi}}, \varepsilon)$
- if, moreover φ is moderate $(A\langle X\rangle, conc, 1_{X^*}, \Delta_{⧢_{\varphi}}, \varepsilon)$ is the enveloping algebra of its primitive elements and one can compute effectively
  - bases in duality
  - Schützenberger's factorization (which gives a system of local coordinates on the Hausdorff group).