The combinatorics of web worlds and web diagrams

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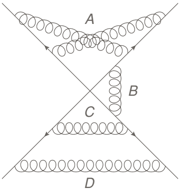
- M. Dukes, E. Gardi, E. Steingrímsson, C. White.
  Web worlds, web-colouring matrices, and web-mixing matrices.
  Journal of Combinatorial Theory Series A 120 (2013), no. 5, 1012-1037.
- M. Dukes, E. Gardi, H. McAslan, D.J. Scott, C. White.
  Webs and posets.
  Journal of High Energy Physics 2014 (2014), no. 1, 1-43.
- M.Dukes and C. White.
  Web matrices: structural properties and generating combinatorial identities.
  Preprint (2015).

1. Background and motivation

QCD - a theory for describing quarks and gluons - the constituents of protons, neutrons and related particles.

Particle colliders smashing such particles together (perhaps to discover new particles) will be acompnied by a lot of quark and gluon radiation.

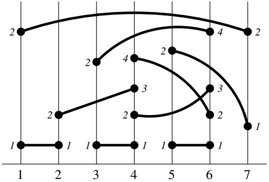


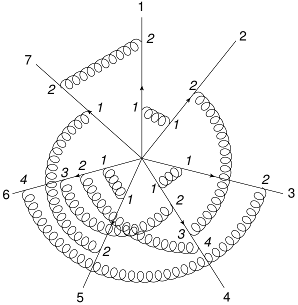
Standard framework for comparing QCD to data is to calculate scattering amplitudes – related to interaction probabilities.

Leaving aside many further comments/assumptions the general goal is to study the scattering amplitudes expressed as $\mathcal{S} = \exp(\mathcal{F}^T \mathcal{R} \mathcal{C})$

2. Web diagrams

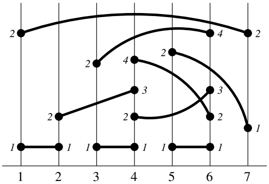
A web diagram consists of a sequence of pegs and a set of edges, each connecting two pegs, as illustrated here:





- In the left diagram the indices of the pegs are shown at the bottom.
- The heights of the endpoints of the edges are shown in italics at each endpoint.
- The unique edge between pegs 3 and 6 is represented by the 4-tuple $(3, 6, 2, 4)$ since the left endpoint of the edge (on peg 3) has height 2 and the right endpoint of the edge (on peg 6) has height 4.
- The diagram on the right is the Feynman diagram illustration of the web diagram.

2.1 Web diagrams

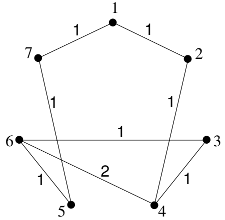


Every web diagram is uniquely represented by listing the 4-tuples that specify its edge set:

$$D = \{(1,2,1,1),(1,7,2,2),(2,4,2,3),(3,4,1,1),(3,6,2,4),\\ (4,6,2,3),(4,6,4,2),(5,6,1,1),(5,7,2,1)\}$$

The web graph $G(D)$ of a web diagram D is the graph whose vertices represent the pegs of D, and whose labelled edges state the number of edges between pegs in D.

$G(D) =$



2.2 Web worlds

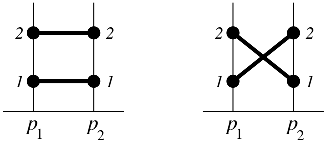
Definition 1

A web world W is a set of web diagrams such that every web diagram D of W can be transformed into another web diagram D' of W by permuting the vertices on pegs.

Equivalently, a set of web diagrams is called a web world if $G(D) = G(D')$ for all $D, D' \in W$. Since all diagrams in a web world have the same web graph, we can write this as $G(W)$.

Example 2

$W = \{D_1 = \{(1,2,1,1),(1,2,2,2)\},\ D_2 = \{(1,2,1,2),(1,2,2,1)\}\}$ is a web world.

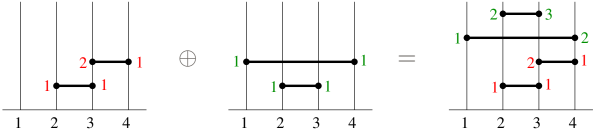


D_1 D_2

2.3 The sum of two web diagrams

Suppose that we have two web diagrams D and D' on the same peg set. We define the sum $D \oplus D'$ to be the web diagram that results from placing D' on top of D and relabelling.

Example 3



Here $D = \{(2,3,1,1),(3,4,2,1)\}$, $D' = \{(1,4,1,1),(2,3,1,1)\}$, and

$$D \oplus D' \; = \; \{(2,3,1,1),(3,4,2,1),(1,4,1,2),(2,3,2,3)\}.$$

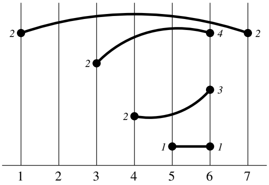
2.4 Subweb diagrams

Given a web diagram D, suppose we select a collection of edges $X \subseteq D$.

In order for X to be a web diagram, we must relabel the 3rd and 4th parts of the edges 4-tuples.

Let D be our Example web diagram. Choose

$$X = \{(1,7,2,2),(3,6,2,4),(4,6,2,3),(5,6,1,1)\}.$$

$X =$ 

Then $\mathrm{rel}(X) = \{(1,7,1,1),(3,6,1,3),(4,6,1,2),(5,6,1,1)\}$.

Suppose that $D = \{e_i = (x_i, y_i, a_i, b_i) : 1 \leq i \leq L\}$ is a web diagram on n pegs, and $\ell \leq L$ a positive integer.

Definition 4 (Colouring and reconstruction)

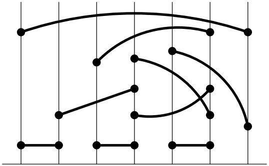
A colouring c of D is a surjective function $c : \{1, \ldots, L\} \to \{1, \ldots, \ell\}$.

Let $D_c(j) = \{e_i \in D : c(i) = j\}$ for all $1 \leq j \leq \ell$, the subweb diagram of D whose edges have colour j.

The reconstruction $\mathrm{Recon}(D, c) \in W(D)$ of D according to the colouring c is the web diagram

$$\mathrm{Recon}(D, c) = \mathrm{rel}(D_c(1)) \oplus \mathrm{rel}(D_c(2)) \oplus \cdots \oplus \mathrm{rel}(D_c(\ell)).$$

2.5 A colouring and reconstruction example



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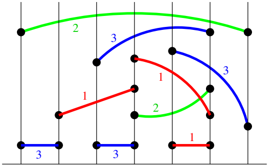

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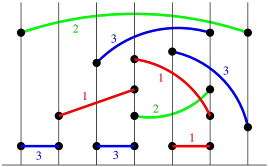

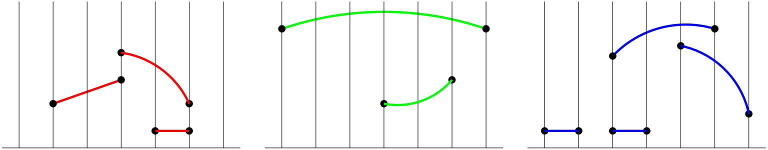

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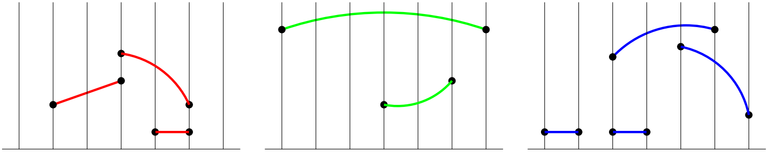

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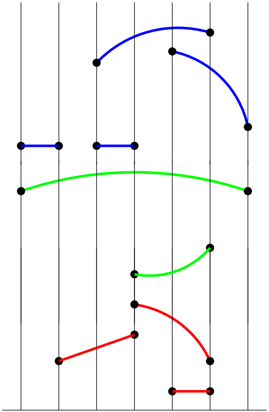

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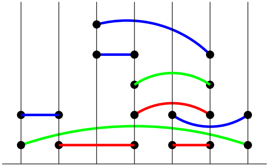

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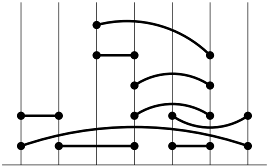

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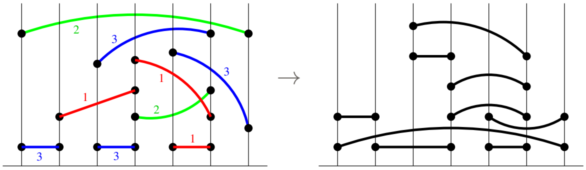

Definition 5 (Self-reconstructing colourings)

Let D be a web diagram and let c be an ℓ-colouring of D. The colouring c is said to be self-reconstructing if $\mathrm{Recon}(D, c) = D$.

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Definition 6 (Colourings that produce D_2 from D_1)

Given a web world W and $D_1, D_2 \in W$, let

$$F(D_1, D_2, \ell) = \{\ell\text{-colourings } c \text{ of } D_1 \ : \ \mathrm{Recon}(D_1, c) = D_2\}$$

and $f(D_1, D_2, \ell) = |F(D_1, D_2, \ell)|$.

2.6 Web-colouring and web-mixing matrices

The following two matrices have rows and columns that are indexed by the diagrams in a given web world.

The web-colouring matrix $\mathfrak{M}^{(W)}(x)$ has (D_1, D_2) entry:

$$\mathfrak{M}^{(W)}_{D_1,D_2}(x) = \sum_{\ell \geq 1} x^{\ell} f(D_1, D_2, \ell).$$

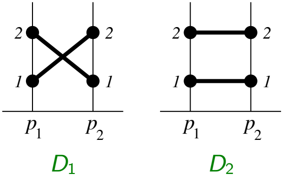
The web-mixing matrix $\mathfrak{R}^{(W)}$ has (D_1, D_2) entry:

$$\mathfrak{R}^{(W)}_{D_1,D_2} = \sum_{\ell \geq 1} \frac{(-1)^{\ell-1}}{\ell} f(D_1, D_2, \ell),$$

The two are related via:

$$\mathfrak{R}^{(W)}_{D_1,D_2} = \int_{-1}^{0} \frac{\mathfrak{M}^{(W)}_{D_1,D_2}(x)}{x} dx.$$

Let W be the web world in Example 2:



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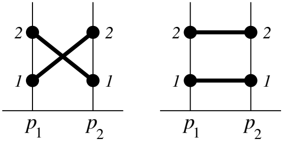

D_1 D_2

There are three different colourings of D_1:

$$
\begin{array}{llcl}
c(e_1) = 1 & c(e_2) = 1 & \Rightarrow & \mathrm{Recon}(D_1, c) = D_1 \\
c(e_1) = 1 & c(e_2) = 2 & \Rightarrow & \mathrm{Recon}(D_1, c) = D_2 \\
c(e_1) = 2 & c(e_2) = 1 & \Rightarrow & \mathrm{Recon}(D_1, c) = D_2
\end{array}
$$

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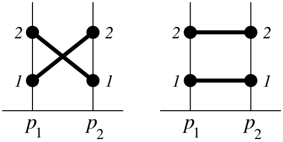

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Consequently $\mathfrak{M}^{(W)}_{D_1,D_1}(x) = x^1$ and $\mathfrak{M}^{(W)}_{D_1,D_2}(x) = 2x^2$.

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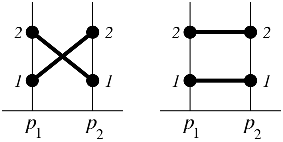

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Consequently $\mathfrak{M}^{(W)}_{D_1,D_1}(x)=x^1$ and $\mathfrak{M}^{(W)}_{D_1,D_2}(x)=2x^2$.

Likewise there are three different colourings of D_2:

$$\begin{array}{llcl} c(e_1')=1 & c(e_2')=1 & \Rightarrow & \mathrm{Recon}(D_2,c)=D_2 \\ c(e_1')=1 & c(e_2')=2 & \Rightarrow & \mathrm{Recon}(D_2,c)=D_2 \\ c(e_1')=2 & c(e_2')=1 & \Rightarrow & \mathrm{Recon}(D_2,c)=D_2 \end{array}$$

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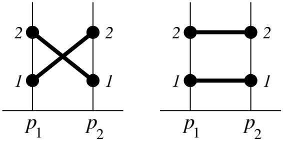

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Consequently $\mathfrak{M}^{(W)}_{D_2,D_1}(x)=0$ and $\mathfrak{M}^{(W)}_{D_2,D_2}(x)=x^1+2x^2$. This gives

$$\mathfrak{M}^{(W)}(x)=\begin{pmatrix} x & 2x^2 \\ 0 & x+2x^2 \end{pmatrix} \quad \text{and} \quad \mathfrak{R}^{(W)}=\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}.$$

Let W be the web world whose web graph is $G(W)$ = 

$$\mathfrak{R}^{(W)} = \frac{1}{24}\begin{bmatrix} 6 & -6 & -6 & 6 & -6 & 6 & 6 & -6 \\ -6 & 6 & 6 & -6 & 6 & -6 & -6 & 6 \\ -2 & 2 & 2 & -2 & 2 & -2 & -2 & 2 \\ 2 & -2 & -2 & 2 & -2 & 2 & 2 & -2 \\ -2 & 2 & 2 & -2 & 2 & -2 & -2 & 2 \\ 2 & -2 & -2 & 2 & -2 & 2 & 2 & -2 \\ 2 & -2 & -2 & 2 & -2 & 2 & 2 & -2 \\ -2 & 2 & 2 & -2 & 2 & -2 & -2 & 2 \end{bmatrix}$$

2.8 Web-colouring and web-mixing properties

The basic problems we consider are as follows: Given a web world W,

- What can we say about the matrices $\mathfrak{M}^{(W)}(x)$ and $\mathfrak{R}^{(W)}$, their entries, trace and rank?
- Can we determine the entries of $\mathfrak{M}^{(W)}(x)$ and $\mathfrak{R}^{(W)}$ for special cases?

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Theorem 7 (Gardi & White 2011)

Let W be a web world.

(i) *The row sums of $\mathfrak{R}^{(W)}$ are all zero.*

(ii) *$\mathfrak{R}^{(W)}$ is idempotent.*

3. Self-reconstruction, diagonal entries, and order-preserving maps

Self-reconstructing colourings $\iff$ Diagonal entries of $\mathfrak{M}^{(W)}(x)$

Note: $\mathfrak{R}^{(W)}$ idempotent $\Rightarrow$ trace$(\mathfrak{R}^{(W)})$ $=$rank$(\mathfrak{R}^{(W)})$

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$$\boxed{\text{Self-reconstructing colourings} \quad \Longleftrightarrow \quad \text{Diagonal entries of } \mathfrak{M}^{(W)}(x)}$$

Note: $\mathfrak{R}^{(W)}$ idempotent $\Rightarrow$ $\text{trace}(\mathfrak{R}^{(W)}) = \text{rank}(\mathfrak{R}^{(W)})$

Definition 8 (Decomposition poset)

Let W be a web world and $D \in W$. Suppose that

$$D = E_1 \oplus E_2 \oplus \cdots \oplus E_k$$

where every E_i is an indecomposable web diagram.

Define the partial order $P = (P, \preceq)$ as follows: $P = (E_1, \ldots, E_k)$ and $E_i \preceq E_j$ if

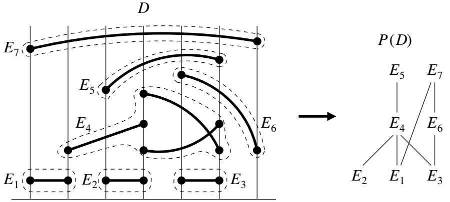
(a) $i < j$, and

(b) there is an edge $e = (x, y, a, b)$ in E_i and an edge $e' = (x', y', a', b')$ in E_j such that an endpoint of e is below an endpoint of e' on some peg.

We call $P(D)$ the decomposition poset of D.

Example 9

The decomposition poset $P(D)$ we get from a web diagram D:



Note that $D = E_1 \oplus E_2 \oplus \cdots \oplus E_7$ where

$$
\begin{array}{lllllllll}
E_1 & = & \{(1,2,1,1)\} & E_2 & = & \{(3,4,1,1)\} & E_3 & = & \{(5,6,1,1)\} \\
E_4 & = & \multicolumn{7}{l}{\{(2,4,1,2),(4,6,1,2),(4,6,3,1)\}} \\
E_5 & = & \{(3,6,1,1\} & E_6 & = & \{(5,7,1,1)\} & E_7 & = & \{(1,7,1,1)\}
\end{array}
$$

Theorem 10

Let D be a web diagram with

$$D \ = \ E_1 \oplus \ldots \oplus E_k$$

where the entries of the sum are all indecomposable web diagrams.

Let $P = P(D)$ and $p = |P(D)|$.

If every member of the sequence $(E_1, \ldots, E_k)$ is distinct then

$$\mathfrak{M}^{(W)}_{D,D}(x) = \sum_{\pi \in \mathcal{L}(P)} x^{1+\mathrm{des}(\pi)}(1+x)^{p-1-\mathrm{des}(\pi)}$$

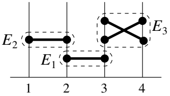
and

$$\mathfrak{R}^{(W)}_{D,D} = \sum_{\pi \in \mathcal{L}(P)} \frac{(-1)^{\mathrm{des}(\pi)}}{p\binom{p-1}{\mathrm{des}(\pi)}},$$

where $\mathcal{L}(P)$ is the Jordan-Hölder set of P (the set of all linear extensions).

Example 11

Let D be the following web diagram:



Since each of the web diagrams (E_1, E_2, E_3) are distinct, Theorem 10 applies:

The poset $P = P(D)$ is the poset on $\{E_1, E_2, E_3\}$ with relations $E_1 < E_2, E_3$.

We find that $\mathcal{L}(P) = \{E_1E_2E_3, E_1E_3E_2\}$, with $\mathrm{des}(E_1E_2E_3) = 0$ and $\mathrm{des}(E_1E_3E_2) = 1$.

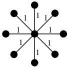
Consequently we have

$$\mathfrak{M}_{D,D}^{(W(n))}(x) = x(1+x)^2 + x^2(1+x) = x + 3x^2 + 2x^3$$

and $\mathfrak{R}_{D,D}^{(W)} = (-1)^0/3 + (-1)^1/3\binom{2}{1} = 1/6$.

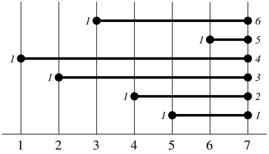
4. Web worlds having a star web graph with unitary edge weights

Consider web worlds $W(n)$ whose web graph $G(W) =$ star graph S_n



Example 12

This web diagram D may be represented by D_π where $\pi = (5, 4, 2, 1, 6, 3)$.



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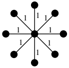

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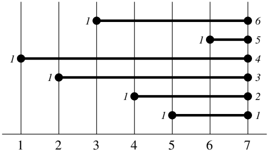

Is it possible to describe the actions of the colourings in terms of the permutations representing the diagrams?

Yes, the number of ways to colour one diagram to get another depends on the number of ways one can colour the corresponding permutation and read from it the new permutation *with respect to a particular order.*

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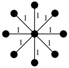

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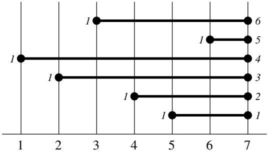
This web diagram D may be represented by D_π where $\pi = (5, 4, 2, 1, 6, 3)$.



Is it possible to describe the actions of the colourings in terms of the permutations representing the diagrams?

Yes, the number of ways to colour one diagram to get another depends on the number of ways one can colour the corresponding permutation and read from it the new permutation *with respect to a particular order.*

If $\pi = (2, 8, 5, 4, 1, 3, 7, 6)$ and $\sigma = (8, 5, 1, 4, 3, 7, 2, 6)$ then we have $\text{Minimal}(\pi, \sigma) = ((8, 5, 1), (4, 3, 7), (2, 6))$. This means $\text{minimal}(\pi, \sigma) = 3$ and the unique colouring having fewest colours that transforms D_π into D_σ is $c = (1, 3, 2, 2, 1, 3, 2, 1)$.

Theorem 13

Suppose that $D_\pi, D_\sigma \in W(n)$ *with* $m = \mathrm{minimal}(\pi, \sigma)$. *Then*

$$\mathfrak{M}^{(W(n))}_{D_\pi, D_\sigma}(x) = x^m(1+x)^{n-m} \quad \text{and} \quad \mathfrak{R}^{(W(n))}_{D_\pi, D_\sigma} = \frac{(-1)^{m-1}}{n\binom{n-1}{m-1}}.$$

Consequently,

$$\begin{aligned} \mathfrak{R}^{(W(n))}_{D_\pi, D_\pi} &= 1/n, & \mathrm{trace}\left(\mathfrak{R}^{(W(n))}\right) &= (n-1)! \\ \mathfrak{M}^{(W(n))}_{D_\pi, D_\pi}(x) &= x(1+x)^{n-1}, & \mathrm{trace}\left(\mathfrak{M}^{(W(n))}(x)\right) &= n!x(1+x)^{n-1}. \end{aligned}$$

5. Disconnected web graphs and their connected components

Let W_1 and W_2 be two web worlds on disjoint peg sets S_1 and S_2 and having web graphs G_1 and G_2, respectively.

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Suppose that $D_1, D_1' \in W_1$ and $D_2, D_2' \in W_2$.

Let $W_3 = W_1 + W_2$ be a new web world which is the disjoint union of W_1 and W_2.

The diagram $D_3 = D_1 + D_2$ is a web diagram in W_3 and the same for D_3'.

Question 14

Suppose W_3 is the disjoint union of the two web worlds W_1 and W_2.

How can we express $\mathfrak{M}^{W_3}_{D_3,D_3'}(x)$ in terms of $\mathfrak{M}^{W_1}_{D_1,D_1'}(x)$ and $\mathfrak{M}^{W_2}_{D_2,D_2'}(x)$?

Given $A(x) = a_0 + a_1 x + \ldots + a_n x^n$ and $B(x) = b_0 + b_1 x + \ldots + b_m x^m$ both in $\mathbb{C}[[x]]$ we define the black diamond product of $A(x)$ and $B(x)$ as

$$A(x) \blacklozenge B(x) = \sum_{k \geq 0} x^k \sum_{i_1, i_2 \geq 0} a_{i_1} b_{i_2} \left(\!\!\left(\!\!\left(\begin{matrix} k \\ i_1, i_2 \end{matrix}\right)\!\!\right)\!\!\right)^{\star}$$

where

$$\left(\!\!\left(\!\!\left(\begin{matrix} k \\ i_1, i_2 \end{matrix}\right)\!\!\right)\!\!\right)^{\star} = \binom{k}{k - i_1, k - i_2, i_1 + i_2 - k} = [u_1^{i_1} u_2^{i_2}]((1+u_1)(1+u_2)-1)^k.$$

Definition 15

Given $A^{(1)}(x), \ldots, A^{(m)}(x) \in \mathbb{C}[[x]]$ where $A^{(k)}(x) = \sum_{n\geq 0} a_n^{(k)} x^n$, we define the black diamond product of $A^{(1)}(x), \ldots, A^{(m)}(x)$ as:

$$A^{(1)}(x) \blacklozenge \ldots \blacklozenge A^{(m)}(x) = \sum_{k\geq 0} x^k \sum_{i_1,\ldots,i_m\geq 0} a_{i_1}^{(1)} \cdots a_{i_m}^{(m)} \left(\!\!\left(\!\!\left({k \atop i_1, \ldots, i_m} \right)\!\!\right)\!\!\right)^{\star}$$

where

$$\left(\!\!\left(\!\!\left({k \atop i_1, \ldots, i_m} \right)\!\!\right)\!\!\right)^{\star} = [u_1^{i_1} \cdots u_m^{i_m}]((1+u_1)\cdots(1+u_m)-1)^k.$$

Theorem 16

Let $W_1, \ldots, W_m$ *be web worlds on pairwise disjoint peg sets.*

Suppose that $D_i, D_i' \in W_i$ *for all* $i \in [1, m]$.

Let $W = W_1 \cup W_2 \cup \ldots \cup W_m$ *be a new web world which is the disjoint union of* $W_1, \ldots, W_m$.

The diagrams $D = D_1 \oplus \ldots \oplus D_m$ *and* $D' = D_1' \oplus \ldots \oplus D_m'$ *are web diagrams in* W *and*

$$\mathfrak{M}_{D,D'}^{(W)}(x) \;=\; \mathfrak{M}_{D_1,D_1'}^{(W_1)}(x) \blacklozenge \ldots \blacklozenge \mathfrak{M}_{D_m,D_m'}^{(W_m)}(x).$$

5.3 Disconnected web worlds and generating combinatorial identities

Proposition 17

Let W be a web world that is the disjoint union of at least two web worlds. Then all entries of the web-mixing matrix $\mathfrak{R}^{(W)}$ are zero, and consequently $\operatorname{trace} \mathfrak{R}^{(W)} = 0$.

5.3 Disconnected web worlds and generating combinatorial identities

Proposition 17

Let W be a web world that is the disjoint union of at least two web worlds. Then all entries of the web-mixing matrix $\mathfrak{R}^{(W)}$ are zero, and consequently $\operatorname{trace}\mathfrak{R}^{(W)} = 0$.

Theorem 18

Let W be a web world whose web-colouring matrix has s different diagonal entries $(H_1(x), \ldots, H_s(x))$ that appear with multiplicities $(h_1, \ldots, h_s)$. Then for all positive integers m, we have

$$\sum_{\substack{a_1, \ldots, a_s \geq 0 \\ a_1 + \ldots + a_s = m}} h_1^{a_1} \cdots h_s^{a_s} \binom{m}{a_1, \ldots, a_s} \int_{-1}^{0} H_1(x)^{\blacklozenge\, a_1} \blacklozenge \cdots \blacklozenge H_s(x)^{\blacklozenge\, a_s} \frac{dx}{x} = 0.$$

The expression for the $s = 2$ case is:

$$\sum_{a=0}^{m} h_1^a h_2^{m-a} \binom{m}{a} \int_{-1}^{0} H_1(x)^{\blacklozenge\, a} \blacklozenge H_2(x)^{\blacklozenge\, m-a} \frac{dx}{x} = 0.$$

Let W be the web world we considered earlier that has web-colouring matrix

$$\mathfrak{M}^{(W)}(x) = \begin{pmatrix} x & 2x^2 \\ 0 & x+2x^2 \end{pmatrix}.$$

Then $H_1(x) = x$, $H_2(x) = x + 2x^2$, $h_1 = h_2 = 1$ and

$$\sum_{a=0}^{m} \sum_{k=1}^{2m-a} \sum_{i_1, i_2} \binom{m}{a} \frac{(-1)^{k+1}}{k} i_1! i_2! \left\{ \begin{matrix} a \\ i_1 \end{matrix} \right\} \left\{ \begin{matrix} 2m-2a \\ i_2 \end{matrix} \right\} \binom{k}{k-i_1, k-i_2, i_1+i_2-k} = 0.$$

Let W be the web world of our running example. Then

$$\mathrm{Represent}(W) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Theorem 19

Let W be a web world on n pegs and $A = \mathrm{Represent}(W)$. The number of different diagrams $D \in W$ is

$$|W| \; = \; \prod_{i=1}^{n} (a_{i*} + a_{*i})! \Big/ \prod_{1 \le i < j \le n} a_{ij}!$$

where a_{i*} (resp. a_{*i}) is the sum of entries in column (resp. row) i of A.

7. What can we say about the square of $\mathfrak{M}^{(W)}(x)$?

Theorem 20

Let W be a web world whose diagrams each have n edges. Let $D, D' \in W$ and suppose that $\mathfrak{M}^{(W)}_{D,D'}(x) = \beta_1 x + \ldots + \beta_n x^n$. Then

$$\left(\mathfrak{M}^{(W)}(x)\right)^2_{D,D'} = \sum_{i=1}^{n} \beta_i L_i(x)$$

where $L_i(x) = \sum_{j,k\geq 1} x^{j+k} \sum_{b=0}^{j} \sum_{a=0}^{k} (-1)^{j+k-(b+a)} \binom{j}{b}\binom{k}{a}\binom{ab}{i}$.

i.e. $\mathfrak{M}^{(W)}(x)^2$ is the image of $\mathfrak{M}^{(W)}(x)$ under the operator $T : \mathbb{C}[[x]] \to \mathbb{C}[[x]]$ which takes the basis $T : (x^i)_{i\geq 0} \ \to \ (L_i(x))_{i\geq 0}$.

n	$L_n(x)$
1	x^2
2	$2x^3 + 2x^4$
3	$6x^4 + 12x^5 + 6x^6$
4	$x^4 + 26x^5 + 73x^6 + 72x^7 + 24x^8$
5	$12x^5 + 156x^6 + 516x^7 + 732x^8 + 480x^9 + 120x^{10}$
6	$2x^5 + 126x^6 + 1206x^7 + 4322x^8 + 7680x^9 + 7320x^{10} + 3600x^{11} + 720x^{12}$

Given a poset $P = (P, \prec)$, its *comparability graph* $\mathrm{comp}(P)$ is the graph whose vertices are the elements of P, with $x, y \in P$ adjacent if $x \prec y$ or $y \prec x$.

Theorem 21

Let D and D' be web diagrams in a web world W with

$$D = E_1 \oplus \cdots \oplus E_k \quad \textit{and} \quad D' = E'_1 \oplus \cdots \oplus E'_{k'},$$

where each of the constituent diagrams E_i and E'_i are indecomposable. Suppose that every member of the sequence $(E_1, \ldots, E_k)$ is distinct and every member of the sequence $(E'_1, \ldots, E'_{k'})$ is also distinct. Then

$$\mathrm{comp}(P(D)) = \mathrm{comp}(P(D')) \implies \mathfrak{M}^{(W)}_{D,D}(x) = \mathfrak{M}^{(W)}_{D',D'}(x).$$

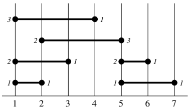
Example 22

Let $D = \{(1,2,1,1),(1,3,2,1),(1,4,3,1),(3,5,2,3),(5,6,2,1),(5,7,1,1)\}$ and $D' = \{(1,2,1,1),(1,3,2,1),(1,4,3,1),(2,7,2,3),(6,7,1,2),(5,7,1,1)\}$.

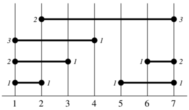
The Hasse diagrams for $P(D)$ and $P(D')$ are illustrated in the following diagram.

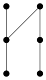
Although the Hasse diagrams are clearly different, since $\mathrm{comp}(P(D)) = \mathrm{comp}(P(D')) = G$ we have $\mathfrak{M}^{(W)}_{D,D}(x) = \mathfrak{M}^{(W)}_{D',D'}(x)$.

D

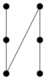


D'

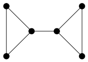




$P(D)$



$P(D')$



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