(Affine) generalized associahedra, root systems, and cluster algebras

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Cluster Algebras [Fomin, Zelevinsky]

- Commutative algebra $\mathcal{A}$ defined recursively by generators (*cluster variables*) and relations (*exchange relations*).

- The generators are grouped into overlapping n-elements sets (*clusters*)

$$X = \{x_1, \ldots, x_n\}.$$

  Each cluster, together with a skew-symmetrizable matrix (*exchange matrix*) form a *seed*

$$(B, X).$$

- The recursion is driven by seed *mutation* from a fixed *initial seed*.

Finite type classification

Cartan companion

$$A(B)_{ij} := \begin{cases} 2 & \text{if } i = j \\ -|B_{ij}| & \text{otherwise} \end{cases}$$

Theorem [Fomin, Zelevinsky]

A cluster algebra $\mathcal{A}$ is of *finite type* (i.e. it has only finitely many cluster variables) if and only if it has a seed (X, B) with $A(B)$ finite type Cartan matrix.

A structural result

Laurent Phenomenon

Any cluster variable x of $\mathcal{A}$ is a Laurent polynomial in the cluster variables $X = \{x_1, \ldots, x_n\}$ of the initial seed; i.e.

$$x = \frac{p(x_1, \ldots, x_n)}{x_1^{d_1} \cdots x_n^{d_n}}$$

Definition

The integer vector $\mathbf{d}_x := (d_1, \ldots, d_n)$ is called the *denominator vector* of x with respect to the initial seed (B, X).

Key observation in the classification theorem

Definition

A seed (B, X) is said to be *acyclic* if, up to simultaneous rows and columns permutations, the matrix B is such that

$$B_{ij} \geq 0 \quad \text{if } i < j \qquad\qquad (\text{equivalently} \quad B_{ij} \leq 0 \quad \text{if } i > j).$$

Proposition

When the initial seed is acyclic and $\mathcal{A}$ is of finite type, the cluster variables of $\mathcal{A}$ are in bijection with the *almost positive roots* $\Phi_{\geq -1}$ in the root system Φ associated to $A(B)$.

$$x = \frac{p(x_1, \ldots, x_n)}{x_1^{d_1} \cdots x_n^{d_n}} \longmapsto \sum d_i \alpha_i$$

Compatibility degree (in finite type)

Euler matrices

$$E^-_{ij} := \begin{cases} 1 & \text{if } i = j \\ \min\{B_{ij}, 0\} & \text{otherwise} \end{cases} \qquad E^+_{ij} := \begin{cases} 1 & \text{if } i = j \\ \min\{-B_{ij}, 0\} & \text{otherwise} \end{cases}$$

Compatibility degree on $\Phi_{\geq -1}$

$$(\alpha||\beta) := \begin{cases} \alpha \cdot \beta & \text{if either one of } \alpha \text{ and } \beta \text{ is a negative root} \\ -\min\left\{ \alpha E^- \beta,\ \alpha E^+ \beta \right\} & \text{otherwise} \end{cases}$$

Proposition

Two cluster variables x and y belong to the same cluster of $\mathcal{A}$ if and only if

$$(\mathbf{d}_x||\mathbf{d}_y) = 0.$$

The roots $\mathbf{d}_x$ and $\mathbf{d}_y$ are said to be *compatible*.

Cluster fans and generalized associahedra

Cluster fan

Let $\mathcal{F}$ denote the collection of cones

$$\mathcal{F} := \bigcup_C \mathbb{R}_+ C$$

where C runs over all possible sets of pairwise compatible almost positive roots.

Theorem

- Each maximal (by inclusion) set of pairwaise compatible almost positive roots is a $\mathbb{Z}$-basis of the root lattice.
- $\mathcal{F}$ is a *complete simplicial fan* normal to a polytope (the *generalized associahedron*).

In particular each point in the root lattice admits a unique *cluster expansion* and exchange relations have an interpretation in terms of linear dependencies.

Few names

Ceballos, Chapoton, Fomin, Hohlweg, Lange, Pilaud, Reading, Speyer, S., Thomas, Zelevinsky ...

Affine cluster algebras

Definition

A cluster algebra $\mathcal{A}$ is said to be of *affine type* if it is not of finite type and it has a seed (B, X) such that $A(B)$ is an affine type cartan matrix.

Goal

Extend the cluster fan construction to the affine case. We need to take care of two main questions before proving theorems:

- Which is the right notion of almost positive roots?
- How does one define compatibility degree in the affine case?

c-orbits: the finite type case

Let $c = s_1 \dots s_n$ be a *Coxeter element* in a finite type Weyl group with root system Φ.

Theorem

- Under the action of c the root system Φ decomposes into n distinct orbits Ω_i each containing h roots.
- There are precisely n positive roots ψ_i^+ such that $c^{-1}\left(\psi_i^+\right)$ is a negative root and precisely n positive roots ψ_i^- such that $c\left(\psi_i^-\right)$ is a negative root. Namely

$$\psi_i^+ := s_1 \dots s_{i-1}\alpha_i \qquad \psi_i^- := s_n \dots s_{i+1}\alpha_i$$

- The orbit Ω_i is of the form

$$\psi_i^+ \xrightarrow{c} \cdots \xrightarrow{c} \psi_{i^*}^- \xrightarrow{c} -\psi_{i^*}^+ \xrightarrow{c} \cdots \xrightarrow{c} -\psi_i^- \xrightarrow{c} \psi_i^+$$

c-orbits: the affine type case

Let $c = s_1 \dots s_n$ be a *Coxeter element* in an affine Weyl group with root system Φ.

Theorem [Reading, S.]

Under the action of c, the root system Φ decomposes into $2n$ infinite orbits and, if $n > 2$, infinitely many c-orbits of finite length.

- There are n infinite orbits Ω_i^+ of the form

  $$\cdots \xrightarrow{c} -\psi_i^- \xrightarrow{c} \psi_i^+ \xrightarrow{c} \dots$$

  and n infinite orbits Ω_i^- of the form

  $$\cdots \xrightarrow{c} \psi_i^- \xrightarrow{c} -\psi_i^+ \xrightarrow{c} \dots$$

- There are $n-2$ positive roots β_i (with $\beta - \delta$ negative) such that the roots

  $$\psi_j^0 := s_{\beta_1} \dots s_{\beta_{j-1}} \beta_j$$

  are all in distinct finite orbits Ω_j^0 and any other finite orbit is of the form

  $$\Omega_j^k := \left\{ \gamma + k\delta \mid \gamma \in \Omega_j^0 \right\}$$

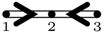
  for some $k \in \mathbb{Z}$.

Example $C_2^{(1)}$

Pick the exchange matrix

$$B = \begin{pmatrix} 0 & 1 & 0 \\ -2 & 0 & 2 \\ 0 & -1 & 0 \end{pmatrix}$$

So that the labeling on the corresponding Dynkin diagram is



The representatives of the infinite orbits are

$$\begin{aligned} \psi_1^- &= \alpha_1 + 2\alpha_2 + 2\alpha_3 \\ \psi_2^- &= \alpha_2 + \alpha_3 \\ \psi_3^- &= \alpha_3 \end{aligned} \qquad \begin{aligned} \psi_1^+ &= \alpha_1 \\ \psi_2^+ &= \alpha_1 + \alpha_2 \\ \psi_3^+ &= 2\alpha_1 + 2\alpha_2 + \alpha_3 \end{aligned}$$

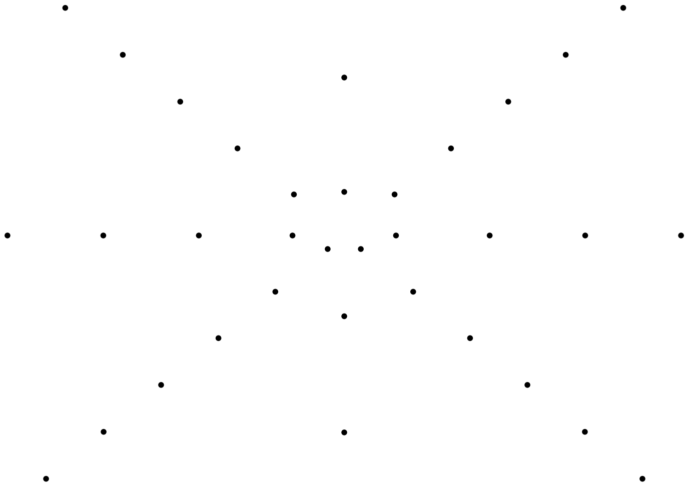
For the finite orbits set

$$\psi_1^0 = \alpha_2$$

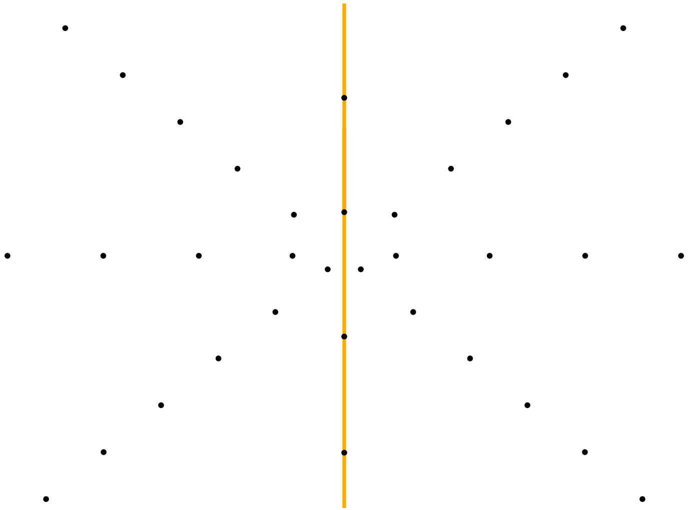
so that Ω_1^0 is

$$\alpha_2 \xrightarrow{c} \alpha_1 + \alpha_2 + \alpha_3 \xrightarrow{c} \alpha_2$$

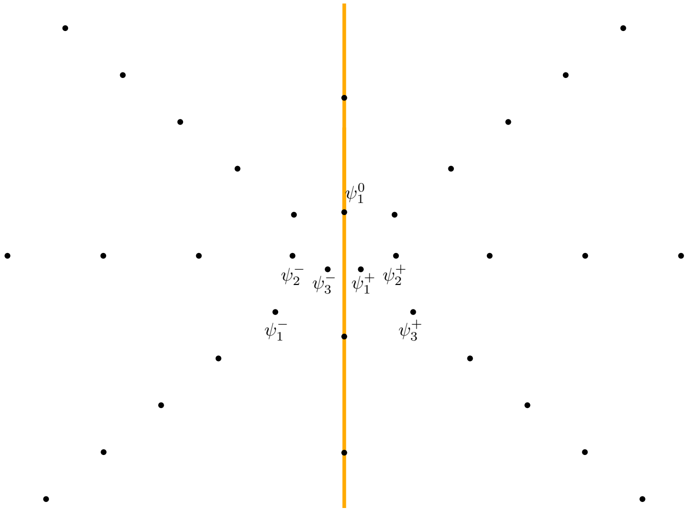
Example $C_2^{(1)}$



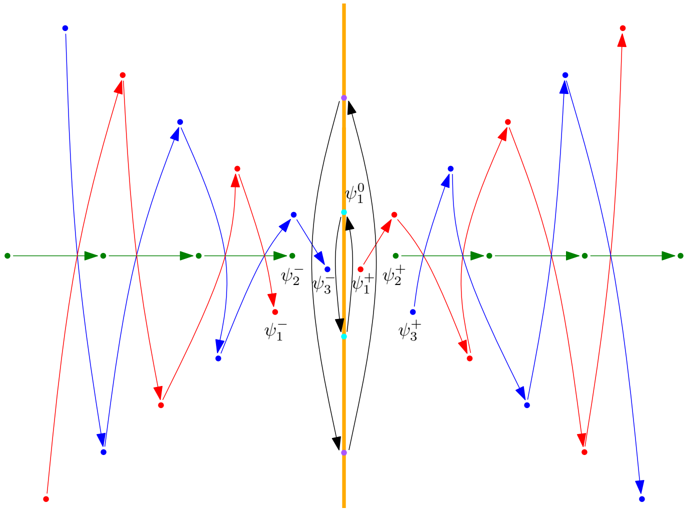
Example $C_2^{(1)}$



Example $C_2^{(1)}$



Example $C_2^{(1)}$



Almost positive Schur roots in affine type

Definition

Let B be any acyclic affine exchange matrix. The *almost positive Schur roots* associated to B are

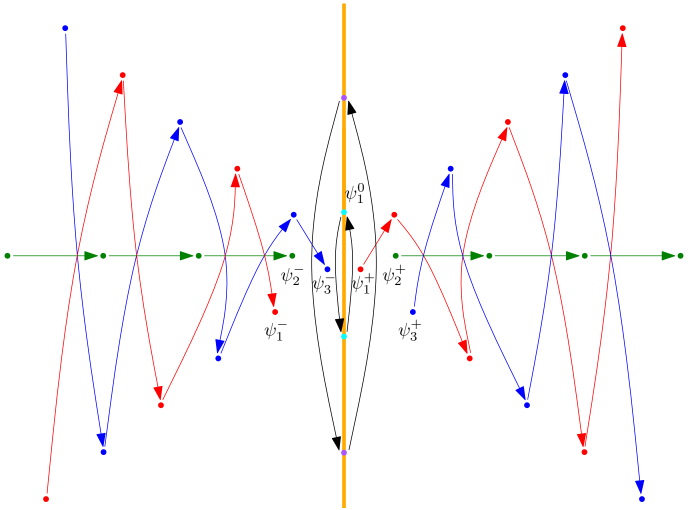
$$\Phi_B := \left(\Phi_+ \setminus \bigcup_{k \neq 0} \Omega_j^k \right) \cup \{ -\alpha_i \mid 1 \leq i \leq n \}$$

Proposition

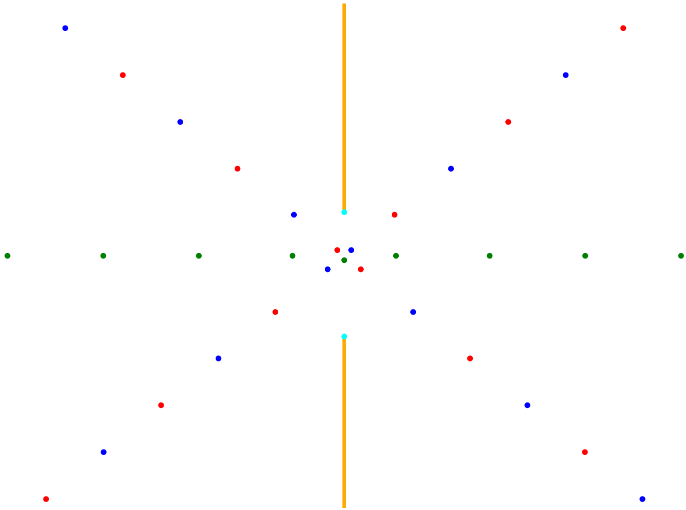
The cluster variables of the cluster algebra having (B, X) as initial seed are in bijection with the roots in Φ_B. The bijection associates to each cluster variable its $\mathbf{d}$-vector.

$$x = \frac{p(x_1, \dots, x_n)}{x_1^{d_1} \cdots x_n^{d_n}} \longmapsto \sum d_i \alpha_i$$

Example $C_2^{(1)}$



Example $C_2^{(1)}$



Compatibility degree in affine type

Let U be the cone

$$U := \mathbb{R}_+ \left(\bigcup_{j=1}^{n-2} \Omega_j^0 \right).$$

Note that U has dimension $n-1$ and is not necessarily a simplicial cone; moreover $\Phi_B \cap \mathring{U} = \emptyset$.

Compatibility degree on Φ_B

$$(\alpha||\beta) := \begin{cases} \alpha \cdot \beta & \text{if either one of } \alpha \text{ and } \beta \text{ is a negative root} \\ \mathrm{adj}(\alpha, \beta) & \text{if } \{\alpha, \beta\} \subset U \text{ and } \alpha + \beta \in \mathring{U} \\ -\min\left\{\, \alpha E^- \beta,\ \alpha E^+ \beta \,\right\} & \text{otherwise} \end{cases}$$

As before, two almost positive Shur roots α, and β are said to be *compatible* if

$$(\alpha||\beta) = 0$$

Cluster fans in affine type

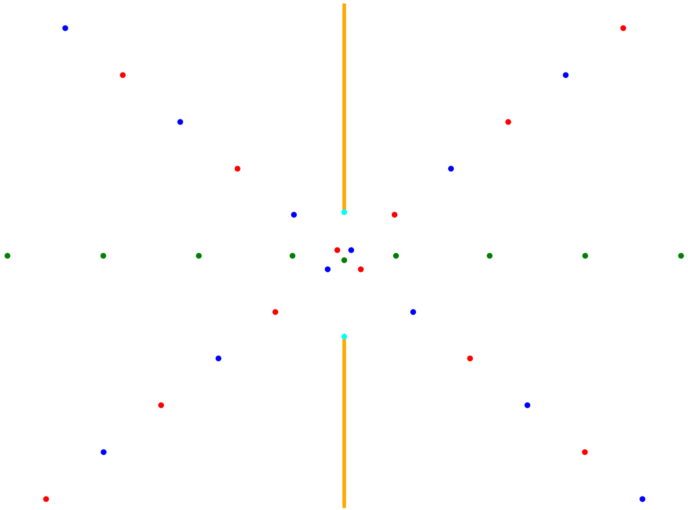
Let B be any acyclic exchange matrix of affine type and define $\mathcal{F}$ as before.

Theorem [Reading, S.]

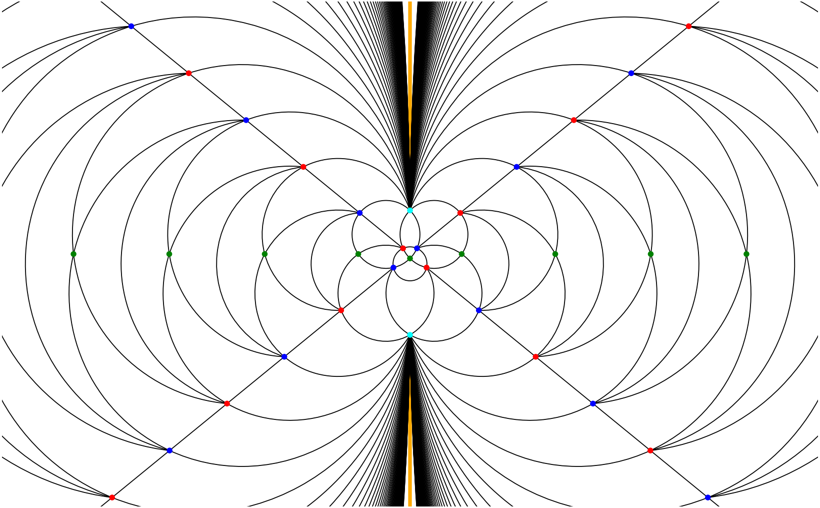
- Each maximal (by inclusion) set of pairwise compatible almost positive Schur roots is a $\mathbb{Z}$-basis of the root lattice.
- $\mathcal{F}$ is a simplicial fan with infinitely many cones filling the all the space except for $\mathring{U}$

In particular each point in the root lattice that does not live in the interior of U admits a unique *cluster expansion*.

Example $C_2^{(1)}$



Example $C_2^{(1)}$



Dangling edges and current work

- It is possible to "fill the hole" in $\mathcal{F}$ by declaring δ to be in Φ_B. This creates *imaginary clusters* each containing $n-1$ roots. The corresponding cones are contained in U and are arranged according to the *tubings* of 1, 2, or 3 cycles (depending only on the type of B).

- We have a conjectural formula for exchange relations similar to the finite type formula.

- In finite type [Ceballos, Pilaud] and for surfaces [Fomin, Shapiro, Thurston] the compatibility degree actually computes the coefficients of $\mathbf{d}$-vectors. Is this true also in the affine types not coming from surfaces?

- A related construction in finite type informs some structural property on the associated cluster algebra that allowed [Yang, Zelevinsky] to realize such algebras as the ring of coordinates of a specific subvariety of the associated Lie group. Together with Rupel and Williams I am currently persuing the same construction for affine types cluster algebras and the corresponding Kac-Moody groups.

Thank you