# Gamma-Nonnegativity in Combinatorics and Geometry

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# Outline

# Introduction

# Symmetry and unimodality

## Definition

A polynomial $f(x) \in \mathbb{R}[x]$ is

- symmetric (or palindromic) and
- unimodal

if for some $n \in \mathbb{N}$,

$$f(x) \;=\; p_0 + p_1 x + p_2 x^2 + \cdots + p_n x^n$$

with

- $p_k = p_{n-k}$ for $0 \leq k \leq n$ and
- $p_0 \leq p_1 \leq \cdots \leq p_{\lfloor n/2 \rfloor}$.

The number $n/2$ is called the center of symmetry.

# Example: Eulerian polynomial

We let

- $\mathfrak{S}_n$ be the group of permutations of $[n] := \{1, 2, \ldots, n\}$

and for $w \in \mathfrak{S}_n$

- $\mathrm{des}(w) \; := \; \# \{ i \in [n-1] : \; w(i) > w(i+1) \}$
- $\mathrm{exc}(w) \; := \; \# \{ i \in [n-1] : \; w(i) > i \}$

be the number of descents and excedances of $w$, respectively. The polynomial

$$A_n(x) \; := \; \sum_{w \in \mathfrak{S}_n} x^{\mathrm{des}(w)} \; = \; \sum_{w \in \mathfrak{S}_n} x^{\mathrm{exc}(w)}$$

is the $n$th Eulerian polynomial.

## Example

$$A_n(x) = \begin{cases} 1, & \text{if } n = 1 \\ 1 + x, & \text{if } n = 2 \\ 1 + 4x + x^2, & \text{if } n = 3 \\ 1 + 11x + 11x^2 + x^3, & \text{if } n = 4 \\ 1 + 26x + 66x^2 + 26x^3 + x^4, & \text{if } n = 5 \\ 1 + 57x + 302x^2 + 302x^3 + 57x^4 + x^5, & \text{if } n = 6. \end{cases}$$

Note: The Eulerian polynomial $A_n(x)$ is well known to be symmetric and unimodal. Is there a simple combinatorial proof?

# Gamma-nonnegativity

**Proposition** (Bränden, 2004, Gal, 2005)

*Suppose* $f(x) \in \mathbb{R}[x]$ *has nonnegative coefficients and only real roots and that it is symmetric, with center of symmetry* $n/2$. *Then*

$$f(x) \;=\; \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i \, x^i (1+x)^{n-2i}$$

*for some nonnegative real numbers* $\gamma_0, \gamma_1, \ldots, \gamma_{\lfloor n/2 \rfloor}$.

**Definition**

The polynomial $f(x)$ is called $\gamma$-nonnegative if there exist nonnegative real numbers $\gamma_0, \gamma_1, \ldots, \gamma_{\lfloor n/2 \rfloor}$ as above, for some $n \in \mathbb{N}$.

## Example

$$A_n(x) = \begin{cases} 1, & \text{if} \quad n = 1 \\ 1 + x, & \text{if} \quad n = 2 \\ (1+x)^2 + 2x, & \text{if} \quad n = 3 \\ (1+x)^3 + 8x(1+x), & \text{if} \quad n = 4 \\ (1+x)^4 + 22x(1+x)^2 + 16x^2, & \text{if} \quad n = 5 \\ (1+x)^5 + 52x(1+x)^3 + 186x^2(1+x), & \text{if} \quad n = 6. \end{cases}$$

Note: Every $\gamma$-nonnegative polynomial (even if it has nonreal roots) is symmetric and unimodal.

An index $i \in [n]$ is called a double descent of a permutation $w \in \mathfrak{S}_n$ if

$$w(i-1) \;>\; w(i) \;>\; w(i+1),$$

where $w(0) = w(n+1) = n+1$.

**Theorem** (Foata–Schützenberger, 1970)

*We have*

$$A_n(x) \;=\; \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \gamma_{n,i}\, x^i (1+x)^{n-1-2i},$$

*where $\gamma_{n,i}$ is the number of $w \in \mathfrak{S}_n$ which have no double descent and $\mathrm{des}(w) = i$. In particular, $A_n(x)$ is symmetric and unimodal.*

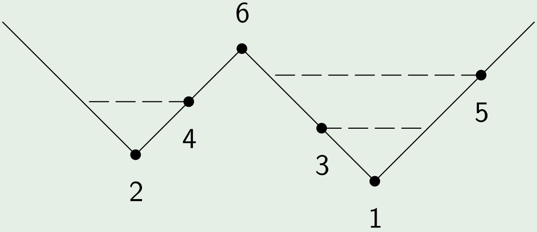
Elegant proof by Foata–Schützenberger (1970) and Foata–Strehl (1974): They partition $\mathfrak{S}_n$ into equivalence classes, so that for each class $\mathcal{K}$,

$$\sum_{w \in \mathcal{K}} x^{\mathrm{des}(w)} \;=\; x^i (1+x)^{n-1-2i}$$

for some $i$. The permutations within each class have the same peaks and valleys.

## Example

For the class of $w = (2, 4, 6, 3, 1, 5) \in \mathfrak{S}_6$ we have $n = 6$ and $i = 1$,




so

$$\sum_{w \in \mathcal{K}} x^{\mathrm{des}(w)} \;=\; x(1+x)^3.$$

Recall that a permutation $w \in \mathfrak{S}_n$ is said to be up-down if

$$w(1) < w(2) > w(3) < \cdots .$$

## Corollary

*We have*

$$A_n(-1) = \begin{cases} 0, & \text{if } n \text{ is even}, \\ (-1)^{(n-1)/2}\,\gamma_{n,(n-1)/2}, & \text{if } n \text{ is odd}, \end{cases}$$

*where $\gamma_{n,(n-1)/2}$ is the number of up-down permutations in $\mathfrak{S}_n$.*

Recently, gamma-nonnegativity attracted attention after the work of

- Bränden (2004, 2008) on $P$-Eulerian polynomials,
- Gal (2005) on flag triangulations of spheres.

A book exposition can be found in:

- T.Kyle Petersen, *Eulerian Numbers*, Birkhaüser, 2015.

# I. Gamma-nonnegativity in combinatorics

# $P$-Eulerian polynomials

We let

- $P$ be a poset with $n$ elements,
- $\omega : P \to [n]$ be an order preserving bijection.

**Definition (Stanley, 1972)**

*The P-Eulerian polynomial is defined as*

$$W_P(x) \;=\; \sum_{w \in \mathcal{L}(P,\omega)} x^{\mathrm{des}(w)},$$

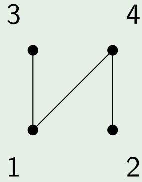
*where* $\mathcal{L}(P,\omega)$ *consists of all permutations* $(a_1, a_2, \ldots, a_n) \in \mathfrak{S}_n$ *with the property*

$$\omega^{-1}(a_i) \;<_P\; \omega^{-1}(a_j) \;\Rightarrow\; i < j.$$

## Example

For

$$(P,\omega) \quad =$$




we have

$$\mathcal{L}(P,\omega) \;=\; \{(1,2,3,4),\, (1,2,4,3),\, (2,1,3,4),\, (2,1,4,3),\, (1,3,2,4)\}$$

and

$$W_P(x) \;=\; 1 + 3x + x^2.$$

## Example

For an $n$-element antichain $P$ (no two elements are comparable)

$$(P,\omega) \quad = \quad \begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ 1 & 2 & 3 & 4 \end{array}$$

we have

$$\mathcal{L}(P,\omega) \;=\; \mathfrak{S}_n$$

and hence

$$W_P(x) \;=\; A_n(x).$$

Note: The polynomial $W_P(x)$:

- plays a role in Stanley's theory of $P$-partitions,
- does not depend on $\omega$,
- is symmetric, provided $P$ is graded,
- can have non-real roots, as shown by Bränden and Stembridge.

**Theorem** (Reiner–Welker, 2005)

*The polynomial $W_P(x)$ is unimodal for every graded poset $P$.*

Their proof uses deep results from geometric combinatorics. Bränden gave two elementary proofs of the following:

**Theorem** (Bränden, 2004, 2008)

*The polynomial $W_P(x)$ is $\gamma$-nonnegative for every graded poset $P$.*

# Derangement polynomials

We let $\mathcal{D}_n$ be the set of derangements in $\mathfrak{S}_n$. The polynomial

$$d_n(x) := \sum_{w \in \mathcal{D}_n} x^{\mathrm{exc}(w)}$$

is the $n$th derangement polynomial.

## Example

$$d_n(x) = \begin{cases} 0, & \text{if } n = 1 \\ x, & \text{if } n = 2 \\ x + x^2, & \text{if } n = 3 \\ x + 7x^2 + x^3, & \text{if } n = 4 \\ x + 21x^2 + 21x^3 + x^4, & \text{if } n = 5 \\ x + 51x^2 + 161x^3 + 51x^4 + x^5, & \text{if } n = 6, \\ x + 113x^2 + 813x^3 + 813x^4 + 113x^5 + x^6, & \text{if } n = 7. \end{cases}$$

Note: The unimodality of $d_n(x)$ follows from deep results of Stanley on local $h$-polynomials of triangulations of simplices. Other proofs of unimodality were given by:

- Brenti (1990),
- Stembridge (1992),
- Zhang (1995).

Note:

$$d_n(x) = \begin{cases} 0, & \text{if } \ n = 1 \\ x, & \text{if } \ n = 2 \\ x(1+x), & \text{if } \ n = 3 \\ x(1+x)^2 + 5x^2, & \text{if } \ n = 4 \\ x(1+x)^3 + 18x^2(1+x), & \text{if } \ n = 5 \\ x(1+x)^4 + 47x^2(1+x)^2 + 61x^3, & \text{if } \ n = 6 \\ x(1+x)^5 + 108x^2(1+x)^3 + 479x^3(1+x), & \text{if } \ n = 7. \end{cases}$$

A descending run of a permutation $w \in \mathfrak{S}_n$ is a maximal string of indices $\{a, a+1, \ldots, b\}$ such that $w(a) > w(a+1) > \cdots > w(b)$. An index $i \in [n-1]$ is a double excedance of $w$ if $w(i) > i > w^{-1}(i)$.

**Theorem**

*We have*

$$d_n(x) \;=\; \sum_{i=0}^{\lfloor n/2 \rfloor} \xi_{n,i}\, x^i (1+x)^{n-2i},$$

*where $\xi_{n,i}$ equals the number of:*

- *permutations $w \in \mathfrak{S}_n$ with $i$ runs and no run of size one,*
- *derangements $w \in \mathcal{D}_n$ with $i$ excedances and no double excedance.*

## Example

For $n = 4$ the permutations

$$(4,3,2,1) \quad (4,2,3,1) \quad (4,1,3,2) \quad (3,2,4,1)$$
$$(3,1,4,2) \quad (2,1,4,3)$$

have no run of size one and the derangements

$$(2,1,4,3) \quad (3,4,1,2) \quad (4,3,2,1) \quad (3,4,2,1)$$
$$(4,3,1,2) \quad (4,1,2,3)$$

have no double excedance, in agreement with

$$d_4(x) \;=\; x(1+x)^2 + 5x^2.$$

This statement, along with several $q$-analogues and generalizations, was discovered independently (using different methods) by:

- A–Savvidou (2012),
- Shareshian–Wachs (2010),
- Linusson–Shareshian–Wachs (2012),
- Shin–Zeng (2012),
- Sun–Wang (2014).

For instance:

We denote by $c(w)$ the number of cycles of $w \in \mathfrak{S}_n$.

**Theorem** (Shin–Zeng, 2012)

*We have*

$$\sum_{w \in \mathcal{D}_n} q^{c(w)} x^{\mathrm{exc}(w)} \;=\; \sum_{i=0}^{\lfloor n/2 \rfloor} \xi_{n,i}(q)\, x^i (1+x)^{n-2i},$$

*where*

$$\xi_{n,i}(q) \;=\; \sum_{w \in \mathcal{D}_n(i)} q^{c(w)}$$

*and $\mathcal{D}_n(i)$ consists of all elements of $\mathcal{D}_n$ with exactly $i$ excedances and no double excedance.*

Recall that

$$\mathrm{maj}(w) \;=\; \sum_{i \in \mathrm{Des}(w)} i$$

is the major index of $w \in \mathfrak{S}_n$.

**Theorem (Shareshian–Wachs, 2010)**

*We have*

$$\sum_{w \in \mathcal{D}_n} p^{\mathrm{des}(w)} q^{\mathrm{maj}(w)-\mathrm{exc}(w)} x^{\mathrm{exc}(w)} \;=\; \sum_{i=0}^{\lfloor n/2 \rfloor} \xi_{n,i}(p,q)\, x^i (1+x)^{n-2i}$$

*for some polynomials $\xi_{n,i}(p,q)$ in $p,q$ with nonnegative coefficients.*

Note: A combinatorial interpretation for $\xi_{n,i}(p,q)$ will be given the day a-fter tomorrow.

## Corollary

*We have*

$$d_n(-1) \;=\; \begin{cases} 0, & \textit{if } n \textit{ is odd,} \\ (-1)^{n/2}\,\xi_{n,n/2}, & \textit{if } n \textit{ is even,} \end{cases}$$

*where* $\xi_{n,n/2}$ *is the number of up-down permutations in* $\mathfrak{S}_n$*.*

# Involutions

We let $\mathcal{I}_n$ be the set of permutations $w \in \mathfrak{S}_n$ with $w = w^{-1}$ and let

$$\mathcal{I}_n(x) := \sum_{w \in \mathcal{I}_n} x^{\mathrm{des}(w)}.$$

## Example

$$\mathcal{I}_n(x) = \begin{cases} 1, & \text{if } n = 1 \\ 1 + x, & \text{if } n = 2 \\ 1 + 2x + x^2, & \text{if } n = 3 \\ 1 + 4x + 4x^2 + x^3, & \text{if } n = 4 \\ 1 + 6x + 12x^2 + 6x^3 + x^4, & \text{if } n = 5 \\ 1 + 9x + 28x^2 + 28x^3 + 9x^4 + x^5, & \text{if } n = 6, \\ 1 + 12x + 57x^2 + 92x^3 + 57x^4 + 12x^5 + x^6, & \text{if } n = 7. \end{cases}$$

Note: The polynomial $\mathcal{I}_n(x)$ was first considered by Strehl (1980).

**Theorem** (Guo–Zeng, 2006)

*The polynomial $\mathcal{I}_n(x)$ is symmetric and unimodal for every $n$.*

The proof uses generating functions and recursions.

**Conjecture** (Guo–Zeng, 2006)

*The polynomial $\mathcal{I}_n(x)$ is $\gamma$-nonnegative for every $n$.*

## Example

$$\mathcal{I}_n(x) = \begin{cases} 1, & \text{if} \quad n = 1 \\ 1 + x, & \text{if} \quad n = 2 \\ (1+x)^2, & \text{if} \quad n = 3 \\ (1+x)^3 + x(1+x), & \text{if} \quad n = 4 \\ (1+x)^4 + 2x(1+x)^2 + 2x^2, & \text{if} \quad n = 5 \\ (1+x)^5 + 4x(1+x)^3 + 6x^2(1+x), & \text{if} \quad n = 6 \\ (1+x)^6 + 6x(1+x)^4 + 18x^2(1+x)^2, & \text{if} \quad n = 7. \end{cases}$$

Note: The symmetry of $\mathcal{I}_n(x)$ is evident from the following statements; it was also shown in a more general context by Hultman.

**Proposition (Strehl, 1980)**

*Let $\mathrm{SYT}(n)$ denote the set of standard Young tableaux of size $n$. Then*

$$\mathcal{I}_n(x) \;=\; \sum_{Q \in \mathrm{SYT}(n)} x^{\mathrm{des}(Q)},$$

*where $\mathrm{des}(Q)$ is the number of entries $i \in [n-1]$ for which $i+1$ lies in a row in $Q$ lower than $i$ does.*

## Example

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}\,, \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}\,, \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}\,, \quad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$$

$$\mathcal{I}_2(x) \;=\; 1 + 2x + x^2$$

## Proposition (A, 2015)

*For $n \geq 1$,*

$$\mathcal{I}_n(x) \;=\; \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} A_{c(w^2)}(x)\,(1-x)^{n-c(w^2)},$$

*where $c(w)$ is the number of cycles of $w \in \mathfrak{S}_n$.*

# $W$-Eulerian polynomials

We let

- $(W, S)$ be a Coxeter system
- $\ell(w)$ be the Coxeter length of $w \in W$,

so that $W = \langle S : (st)^{m(s,t)} = e \rangle$ for some positive integers $m(s,t)$ with $m(s,t) = m(t,s)$ and $m(s,t) = 1 \Leftrightarrow s = t$ for $s, t \in S$, and for $w \in W$

- $\mathrm{des}(w) \ := \ \#\{s \in S : \ell(ws) < \ell(w)\}$.

**Definition**

The $W$-Eulerian polynomial is defined as

$$W(x) \ = \ \sum_{w \in W} x^{\mathrm{des}(w)}$$

for every finite Coxeter group $W$.

Note: Finite Coxeter groups include $\mathfrak{S}_n$, as well as the group of signed permutations $B_n = \{w = (w(1), w(2), \ldots, w(n)) : |w| \in \mathfrak{S}_n\}$. Then

$$B_n(x) \;=\; \sum_{w \in B_n} x^{\mathrm{des}_B(w)}$$

where

- $\mathrm{des}_B(w) \;:=\; \#\{i \in \{0, 1, \ldots, n-1\} : \; w(i) > w(i+1)\}$

for $w \in B_n$ as above, with $w(0) := 0$.

## Example

$$B_n(x) = \begin{cases} 1 + x, & \text{if} \quad n = 1 \\ 1 + 6x + x^2, & \text{if} \quad n = 2 \\ 1 + 23x + 23x^2 + x^3, & \text{if} \quad n = 3 \\ 1 + 76x + 230x^2 + 76x^3 + x^4, & \text{if} \quad n = 4 \\ 1 + 237x + 1682x^2 + 1682x^3 + 237x^4 + x^5, & \text{if} \quad n = 5. \end{cases}$$

Note:

$$B_n(x) = \begin{cases} 1+x, & \text{if } \; n=1 \\ (1+x)^2+4x, & \text{if } \; n=2 \\ (1+x)^3+20x(1+x), & \text{if } \; n=3 \\ (1+x)^4+72x(1+x)^2, & \text{if } \; n=4 \\ (1+x)^5+232x(1+x)^3+976x^2(1+x), & \text{if } \; n=5 \\ (1+x)^6+716x(1+x)^4+7664x^2(1+x)^2, & \text{if } \; n=6. \end{cases}$$

Note: The unimodality of $W(x)$ follows from a deep result of Stanley on $h$-polynomials of simplicial convex polytopes.

**Theorem (Stembridge, 2007)**

*The polynomial $W(x)$ is $\gamma$-nonnegative for every finite Coxeter group $W$.*

Problem: Find a proof which does not depend on the classification of finite Coxeter groups.

Note: More information about the $\gamma$-coefficients can be given:

For $w \in \mathfrak{S}_n$ let

$$\mathrm{pk}(w) \ := \ \#\{i \in [n-1] : \ w(i-1) < w(i) > w(i+1)\}$$

be the number of left peaks of $w$, where $w(0) := 0$.

**Theorem** (Petersen, 2007)

*We have*

$$B_n(x) \ = \ \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma^B_{n,i} \, x^i (1+x)^{n-2i},$$

*where*

$$\gamma^B_{n,i} \ = \ 4^i \cdot \#\{w \in \mathfrak{S}_n : \ \mathrm{pk}(w) = i\}.$$

Note: There is a similar result for $D_n(x)$.

# Narayana polynomials

The Catalan number

$$C_n := \frac{1}{n+1}\binom{2n}{n}$$

has the interesting $q$-analogue

$$C_n(q) = \sum_{i=0}^{n-1} \frac{1}{i+1}\binom{n}{i}\binom{n-1}{i} q^i,$$

known as the $n$th Narayana polynomial, in the sense that $C_n(1) = C_n$. The coefficients of $C_n(q)$ count

- Dyck paths of length $2n$, by the number of peaks,
- noncrossing partitions of $[n]$, by the number of blocks,

among many other families of combinatorial objects.

Note: The polynomial $C_n(q)$ is $\gamma$-nonnegative; in fact, as it follows, for instance, from work of Simion–Ullman,

$$C_n(q) \;=\; \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} C_k \binom{n-1}{2k} q^k (1+q)^{n-1-2k}.$$

Note: There is an interesting Coxeter group analogue of $C_n(q)$:

We let

- $W$ be a finite Coxeter group
- $T$ be the set of reflections,
- $\ell_T(w)$ be the length of $w \in W$ with respect to $T$,
- $c$ be a Coxeter element.

**Definition (Bessis, Brady–Watt, 2001)**

*The set of W-noncrossing partitions is defined as*

$$\mathrm{NC}_W \;=\; \{w \in W : \ell_T(w) + \ell_T(w^{-1}c) \leq \ell_T(c)\}.$$

We let

$$C_W(q) \;=\; \sum_{w \in \mathrm{NC}_W} q^{\ell_T(w)}.$$

Note: We have

$$C_W(1) \;=\; \prod_{i=1}^{\ell} \frac{e_i + h + 1}{e_i + 1}$$

for every irreducible Coxeter group $W$, where $e_1, e_2, \ldots, e_\ell$ are the exponents of $W$ and $h$ is the Coxeter number.

We have

$$C_W(q) \;=\; \begin{cases} \displaystyle\sum_{i=0}^{n} \frac{1}{i+1}\binom{n}{i}\binom{n-1}{i} q^i, & \text{if } \; W = \mathfrak{S}_n \\[2em] \displaystyle\sum_{i=0}^{n} \binom{n}{i}^2 q^i, & \text{if } \; W = B_n \\[2em] \displaystyle\sum_{i=0}^{n} \binom{n}{i}\left(\binom{n-1}{i} + \binom{n-2}{i-2}\right) q^i, & \text{if } \; W = D_n. \end{cases}$$

Note that $C_{\mathfrak{S}_n}(q) = C_n(q)$, as expected.

**Theorem**

*The polynomial $C_W(q)$ is $\gamma$-nonnegative for every finite Coxeter group $W$.*

Problem: Find a proof which does not depend on the classification of finite Coxeter groups.

Note: The theorem was extended to all well-generated complex reflection groups by Mühle.

# More examples

There is an endless list of generalizations and similar results, including:

- $q$-analogues,
- various refinements,
- analogues for colored permutations,
- results for other interesting classes of permutations.

# An example from symmetric functions

We define polynomials $T_\lambda(t)$ by

$$\sum_\lambda T_\lambda(t)\, s_\lambda(x) \;=\; \frac{\displaystyle\sum_{k\geq 1}(1+t+\cdots+t^{k-1})\, s_k(x)}{\displaystyle 1-\sum_{k\geq 2}(t+t^2+\cdots+t^{k-1})\, s_k(x)},$$

where the sum on the left ranges over all integer partitions $\lambda$ and $s_\lambda(x)$ is a Schur function.

Note: The $T_\lambda(t)$ are symmetric, with nonnegative coefficients, and satisfy

$$\sum_{\lambda\vdash n} f^\lambda T_\lambda(t) \;=\; A_n(t),$$

where $f^\lambda$ is the number of standard Young tableaux of shape $\lambda$.

**Theorem** (Brenti, 1989)

*The polynomial $T_\lambda(t)$ is real-rooted for every $\lambda$.*

Note: As a result, the $T_\lambda(t)$ are $\gamma$-nonnegative and their $\gamma$-nonnegativity refines that of the Eulerian polynomials. We will see the day after tomorrow what the $\gamma$-coefficients count.

# Two-sided Eulerian polynomials

Let $W$ be a finite Coxeter group. The two-sided $W$-Eulerian polynomial is defined as

$$W(x, y) = \sum_{w \in W} x^{\mathrm{des}(w)} y^{\mathrm{des}(w^{-1})}.$$

**Conjecture (Gessel, 2005, Petersen)**

*There exist nonnegative integers $\gamma_{i,j} = \gamma_{i,j}^W$ such that*

$$W(x, y) = \sum_{2i+j \leq n} \gamma_{i,j} (xy)^i (x+y)^j (1+xy)^{n-2i-j},$$

*where $n$ is the rank of $W$.*

Note: This has been proved for the symmetric and hyperoctahedral groups by Zhicong Lin. It is an open probelm to find a combinatorial interpretation to the $\gamma$-coefficients.

More examples tomorrow...

## Part II

II. Gamma-nonnegativity in geometry

# Face enumeration of simplicial complexes

We let

- $\Delta$ be a simplicial complex of dimension $n-1$,
- $f_i(\Delta)$ be the number of $i$-dimensional faces.

**Definition**

The *h*-polynomial of $\Delta$ is defined as

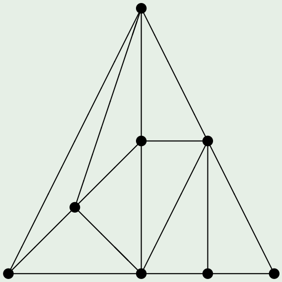
$$h(\Delta, x) = \sum_{i=0}^{n} f_{i-1}(\Delta)\, x^i (1-x)^{n-i} = \sum_{i=0}^{n} h_i(\Delta)\, x^i.$$

The sequence $h(\Delta) = (h_0(\Delta), h_1(\Delta), \dots, h_n(\Delta))$ is the *h*-vector of $\Delta$.

Note: $h(\Delta, 1) = f_{n-1}(\Delta)$.

## Example

For the 2-dimensional complex

$\Delta \quad =$ 

we have $f_0(\Delta) = 8$, $f_1(\Delta) = 15$ and $f_2(\Delta) = 8$ and hence

$$\begin{aligned} h(\Delta, x) &= (1-x)^3 + 8x(1-x)^2 + 15x^2(1-x) + 8x^3 \\ &= 1 + 5x + 2x^2. \end{aligned}$$

**Theorem** (Klee, Reisner, Stanley)

*The polynomial* $h(\Delta, x)$*:*

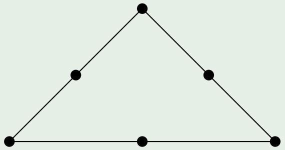
- *has nonnegative coefficients if* $\Delta$ *triangulates a ball or a sphere,*
- *is symmetric if* $\Delta$ *triangulates a sphere,*
- *is unimodal if* $\Delta$ *is the boundary complex of a simplicial polytope.*

## Example

We let

- $V$ be an $n$-element set,
- $2^V$ be the simplex on the vertex set $V$,
- $\Gamma$ be the first barycentric subdivision of the boundary complex of $2^V$.

Then $h(\Gamma, x) = A_n(x)$. For $n = 3$

$\Gamma \quad =$ 

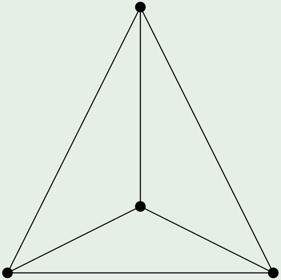
$$h(\Delta, x) \;=\; (1-x)^2 + 6x(1-x) + 6x^2 \;=\; 1 + 4x + x^2.$$

# Flag complexes and Gal's conjecture

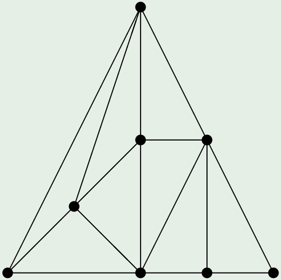
## Definition

A simplicial complex $\Delta$ is called flag if it contains every simplex whose 1-skeleton is a subcomplex of $\Delta$.

## Example



not flag



flag

## Example

For a 1-dimensional sphere $\Delta$ with $m$ vertices we have

$$h(\Delta, x) \;=\; 1 + (m-2)x + x^2.$$

Note that $h(\Delta, x)$ is $\gamma$-nonnegative $\;\Leftrightarrow\; m \geq 4 \;\Leftrightarrow\; \Delta$ is flag.

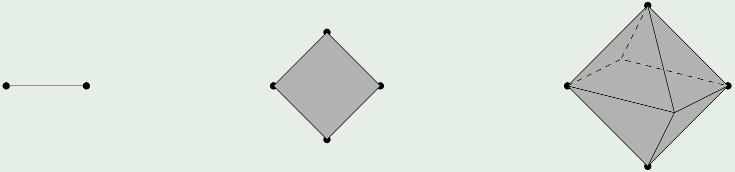
## Conjecture (Gal, 2005)

*The polynomial $h(\Delta, x)$ is $\gamma$-nonnegative for every flag triangulation $\Delta$ of the sphere.*

Note: This extends a conjecture of Charney–Davis (1995).

## Example

The boundary complex $\Sigma_n$ of the $n$-dimensional cross-polytope is a flag triangulation of the $(n-1)$-dimensional sphere:



We have

$$h(\Sigma_n, x) \; = \; (1+x)^n$$

for every $n \geq 1$.

Note: Let us write

$$h(\Delta, x) \;=\; \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i(\Delta)\, x^i (1+x)^{n-2i}.$$

Then Gal's conjecture asserts that $\gamma_i(\Delta) \geq \gamma_i(\Sigma_n)$ for every $i$ and implies that $h_2(\Delta)$ is bounded below by the coefficient of $x^2$ in

$$(1+x)^n \;+\; \gamma_1(\Delta) x (1+x)^{n-2},$$

which means the following:

**Conjecture**

*Among all flag triangulations of the $(n-1)$-dimensional sphere with given number $m$ of vertices, the $(n-2)$-fold double suspension over the boundary complex of an $(m-2n+4)$-gon has the smallest possible number of edges.*

Note: By a result of Karu (2006), Gal's conjecture holds for barycentric subdivisions of regular CW-spheres.

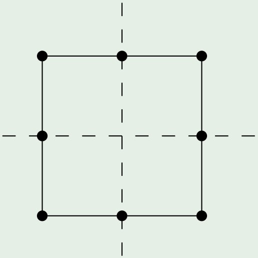
# The Coxeter complex

For every finite Coxeter group $W$ there exists a flag triangulation $\mathrm{Cox}(W)$ of the sphere, known as the Coxeter complex, such that

$$h(\mathrm{Cox}(W), x) \;=\; W(x) \;:=\; \sum_{w \in W} x^{\mathrm{des}(w)}.$$

Note: The Coxeter complex $\mathrm{Cox}(\mathfrak{S}_n)$ is isomorphic to the first barycentric subdivision of the boundary complex of the simplex with $n$ vertices.

## Example



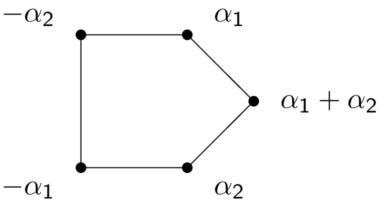
The Coxeter complex for $B_2$

Note: As a result, the $\gamma$-nonnegativity of the $W$-Eulerian polynomial is an instance of Gal's conjecture.

# The cluster complex

For every finite Coxeter group $W$ there exists a flag triangulation $\Delta_W$ of the sphere, namely the cluster complex of Fomin–Zelevinsky, such that

$$h(\Delta_W, x) \;=\; C_W(x) \;:=\; \sum_{w \in \mathrm{NC}_W} x^{\ell_T(w)}.$$




The cluster complex for $\mathfrak{S}_2$

Note: As a result, the $\gamma$-nonnegativity of the $C_W(x)$ is an instance of Gal's conjecture as well.

# The local $h$-polynomial

We let

- $V$ be an $n$-element set,
- $\Gamma$ be a triangulation of the simplex $2^V$ on the vertex set $V$.

**Definition (Stanley, 1992)**

*The local h-polynomial of $\Gamma$ (with respect to $V$) is defined as*

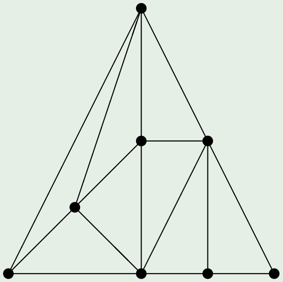
$$\ell_V(\Gamma, x) \; = \; \sum_{F \subseteq V} (-1)^{n-|F|} \, h(\Gamma_F, x),$$

*where $\Gamma_F$ is the restriction of $\Gamma$ to the face $F$ of the simplex $2^V$.*

Note: This polynomial plays a major role in Stanley's theory of subdivisions of simplicial (and more general) complexes.

## Example

For the 2-dimensional triangulation

$\Gamma \quad =$ 

we have

$$\ell_V(\Gamma, x) \;=\; (1+5x+2x^2) - (1+2x) - (1+x) - 1$$
$$+1+1+1-1 \;=\; 2x+2x^2.$$

**Theorem** (Stanley, 1992)

*The polynomial* $\ell_V(\Gamma, x)$

- *is symmetric,*
- *has nonnegative coefficients,*
- *is unimodal for every regular triangulation* $\Gamma$ *of* $2^V$.

**Conjecture** (A, 2012)

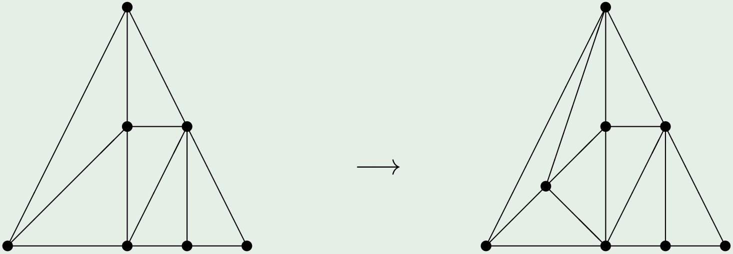
*The polynomial* $\ell_V(\Gamma, x)$ *is* $\gamma$*-nonnegative, if* $\Gamma$ *is a flag triangulation of* $2^V$.

Note: This is stronger than Gal's conjecture. There is considerable evidence for both conjectures. For instance:

### Proposition

- (Gal, 2005) *$h(\Delta, x)$ is $\gamma$-nonnegative for every (necessarily flag) triangulation $\Delta$ of the sphere which can be obtained from $\Sigma_n$ by successive edge subdivisions,*
- (A, 2012) *$\ell_V(\Gamma, x)$ is $\gamma$-nonnegative for every (necessarily flag) triangulation $\Gamma$ which can be obtained from the trivial triangulation of $2^V$ by successive edge subdivisions.*

## Example



An edge subdivision

Recall that we denote by $\Sigma_n$ the boundary complex of the $n$-dimensional cross-polytope.

**Theorem (A, 2012)**

*Every flag triangulation $\Delta$ of the $(n-1)$-dimensional sphere is a flag, vertex-induced homology subdivision $\Gamma$ of $\Sigma_n$. Moreover,*

$$h(\Delta, x) \;=\; \sum_{F \in \Sigma_n} \ell_F(\Gamma_F, x)\,(1+x)^{n-|F|},$$

*hence the $\gamma$-nonnegativity of $h(\Delta, x)$ is implied by that of the $\ell_F(\Gamma_F, x)$.*

**Corollary**

*For every flag triangulation $\Delta$ of the $(n-1)$-dimensional sphere,*

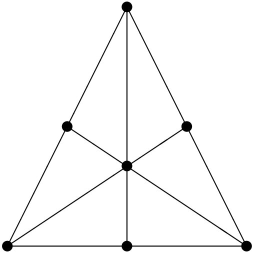
$$h(\Delta, x) \;\geq\; (1+x)^n$$

*holds coefficientwise.*

Note: This holds, more generally, for doubly Cohen–Macaulay flag complexes of dimension $n-1$.

# Barycentric subdivision

For the barycentric subdivision $\Gamma$ of the simplex $2^V$ on the vertex set $V$



Stanley showed that

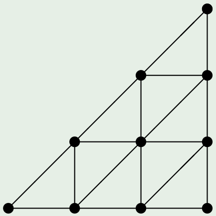
$$\ell_V(\Gamma, x) \;=\; \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} A_k(x) \;=\; \sum_{w \in \mathcal{D}_n} x^{\mathrm{exc}(w)} \;=\; d_n(x),$$

whose $\gamma$-nonnegativity has already been discussed.

# Edgewise subdivision

The *r-fold edgewise subdivision* $\mathrm{esd}_r(2^V)$ is a standard way to triangulate a simplex $2^V$ so that each face $F \in 2^V$ is subdivided into $r^{\dim(F)}$ simplices of the same dimension.

## Example



The 3-fold edgewise subdivision of a 2-simplex

To be more precisely, we let

- $e_1, e_2, \ldots, e_d$ be the unit coordinate vectors in $\mathbb{R}^d$,
- $V = \{0, re_1, r(e_1 + e_2), \ldots, r(e_1 + e_2 + \cdots + e_d)\}$.

Then $\mathrm{esd}_r(2^V)$ is realized as the triangulation of the geometric simplex with vertex set $V$ whose maximal faces are the $d$-dimensional simplices into which that simplex is dissected by the hyperplanes of the form

- $x_i = k$,
- $x_i - x_j = k$,

with $k \in \mathbb{Z}$.

Note: The triangulation $\mathrm{esd}_r(2^V)$ is flag.

Note: The edgewise subdivision has appeared in several mathematical contexts, including:

- algebraic topology (Freudenthal, 1942)
- toric geometry (Kempf–Knudsen–Mumford–Saint-Donat, 1972)
- algebraic $K$-theory (Grayson, 1989)
- topological cyclic homology (Bökstedt–Hsiang–Madsen, 1993)
- combinatorial commutative algebra (Brun–Römer, 2005)
- combinatorial commutative algebra (Brenti–Welker, 2009)
- discrete geometry (Haase–Paffenholz–Piechnik–Santos, 2014).

We let

- $\mathcal{S}(n, r)$ denote the set of sequences

$$w = (w_0, w_1, \ldots, w_n) \in \{0, 1, \ldots, r-1\}^{n+1}$$

  having no two consecutive entries equal and satisfying $w_0 = w_n = 0$

and for such $w \in \mathcal{S}(n, r)$ we set

- $\mathrm{asc}(w) := \#\{i \in \{0, 1, \ldots, n-1\} : w_i < w_{i+1}\}.$

We say that an index $i \in [n-1]$ is a

- double ascent of $w$ if $w_{i-1} < w_i < w_{i+1}$ and
- double descent of $w$ if $w_{i-1} > w_i > w_{i+1}$.

**Theorem (A, 2014)**

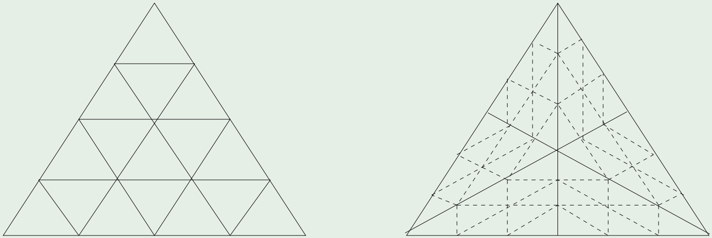
*For every $n$-element set $V$,*

$$\ell_V(\mathrm{esd}_r(2^V), x) = \sum_{w \in \mathcal{S}(n,r)} x^{\mathrm{asc}(w)} = \sum_{i=0}^{\lfloor n/2 \rfloor} \xi_{n,r,i}\, x^i (1+x)^{n-2i},$$

*where $\xi_{n,r,i}$ is the number of $(w_0, w_1, \ldots, w_n) \in \mathcal{S}(n, r)$ which have exactly $i$ ascents and satisfy the following: for every double ascent $k$ there exists a double descent $\ell > k$ such that $w_k = w_\ell$ and $w_k \leq w_j$ for $k < j < \ell$.*

Note: One can define the $r$-fold edgewise subdivision for any simplicial complex.

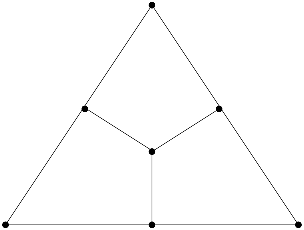
## Example

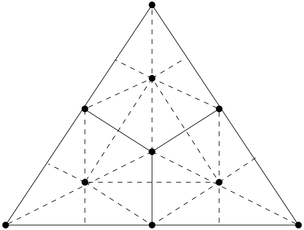


The 4-fold edgewise subdivision of the 2-simplex and the 3-fold edgewise subdivision of its barycentric subdivision

# More barycentric subdivisions

Consider the barycentric subdivision $K$ of the cubical barycentric subdivision of the simplex $2^V$.





Note: The sum of the coefficients of $\ell_V(K, x)$ is equal to the number of

- even derangements in $B_n$,
- derangements in $D_n$,

where $B_n$ is the group of signed permutations of $[n]$ and $D_n$ is the subgroup of even signed permutations.

## Example

$$\ell_V(K,x) = \begin{cases} 0, & \text{if} \quad n=1 \\ 3x, & \text{if} \quad n=2 \\ 7x+7x^2, & \text{if} \quad n=3 \\ 15x+87x^2+15x^3, & \text{if} \quad n=4 \\ 31x+551x^2+551x^3+31x^4, & \text{if} \quad n=5 \\ 63x+2803x^2+8243x^3+2803x^4+63x^5, & \text{if} \quad n=6. \end{cases}$$

## Conjecture

*The polynomial $\ell_V(K,x)$ has only real roots.*

For $w = (w_1, w_2, \ldots, w_n) \in B_n$ we let

- $\mathrm{exc}_A(w) \ := \ \#\{i \in [n-1] : \ w(i) > i\}$,
- $\mathrm{neg}(w) \ := \ \#\{i \in [n] : \ w(i) < 0\}$.

**Definition (Bagno–Garber, 2006)**

*The flag-excedance number of $w \in B_n$ is defined as*

$$\mathrm{fex}(w) \ = \ 2 \cdot \mathrm{exc}_A(w) + \mathrm{neg}(w).$$

Example: For

- $w = (3, -5, 1, 4, -2)$

we have $\mathrm{exc}_A(w) = 1$ and $\mathrm{neg}(w) = 2$, so $\mathrm{fex}(w) = 4$.

**Theorem (A, 2014)**

*We have*

$$\begin{aligned}\ell_V(K,x) &= \sum_{w} x^{\mathrm{fex}(w)/2} \\ &= \sum_{i=0}^{\lfloor n/2 \rfloor} \xi^+_{n,i}\, x^i (1+x)^{n-2i},\end{aligned}$$

*where the first sum runs over all derangements $w \in D_n$ and $\xi^+_{n,i}$ is the number of elements of $B_n$ with $i$ descending runs, none of size one, and positive last coordinate.*

Problem: Find a simple combinatorial proof of the second expression.

Consider the second barycentric subdivision $\Gamma^2$ of the simplex $2^V$.

Note: The sum of the coefficients of $\ell_V(\Gamma^2, x)$ equals the number of pairs $(u, v) \in \mathfrak{S}_n \times \mathfrak{S}_n$ of permutations with no common fixed point.

Problem: Find a combinatorial interpretation for:

- the coefficients of $\ell_V(\Gamma^2, x)$,
- the coefficients in the expansion

$$\ell_V(\Gamma^2, x) \;=\; \sum \gamma_i \, x^i (1 + x)^{n-2i}.$$

Problem: Study the barycentric subdivision of more general polyhedral subdivisions of the simplex.

# Generalization to $r$-colored permutations

We let $\Lambda$ denote the 2-fold edgewise subdivision of the barycentric subdivision of the simplex $2^V$.

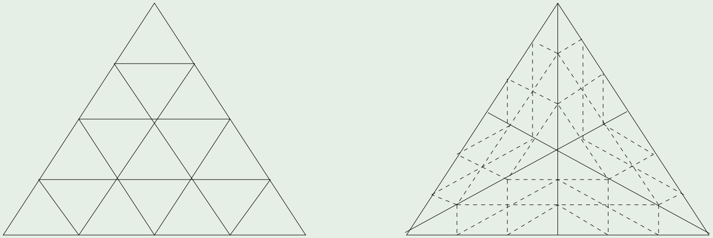
**Proposition (A–Savvidou, 2012)**

*For every $n$-element set $V$,*

$$\ell_V(K, x) \ = \ \ell_V(\Lambda, x).$$

This makes it natural to consider the $r$-fold edgewise subdivision $\Lambda_r$ of the barycentric subdivision of the simplex $2^V$.

## Example



The subdivision $\Lambda_3$ of the 2-simplex on the right.

Recall that a permutation $w = (w_1, w_2, \ldots, w_n) \in \mathfrak{S}_n$ is $r$-colored if each $w_i$ has been colored with one of the elements of $\{0, 1, \ldots, r-1\}$. We let

- $\mathfrak{S}_n^r$ be the group of $r$-colored permutations of $[n]$

and for $w \in \mathfrak{S}_n^r$ as above

- $\mathrm{exc}_A(w) \ := \ \# \{i \in [n-1] : \ w(i) > i \text{ has zero color}\}$,
- $\mathrm{csum}(w)$ be the sum of the colors of the entries of $w$.

The flag-excedance number of $w$ is defined by Bagno–Garber as

$$\mathrm{fex}(w) \ = \ r \cdot \mathrm{exc}_A(w) + \mathrm{csum}(w).$$

We call $w$ balanced if $\mathrm{fex}(w)$ is divisible by $r$.

**Theorem (A, 2014)**

*We have*

$$\begin{aligned} \ell_V(\Lambda_r, x) &= \sum_{w} x^{\mathrm{fex}(w)/r} \\ &= \sum_{i=0}^{\lfloor n/2 \rfloor} \xi^{+}_{n,r,i}\, x^i (1+x)^{n-2i}, \end{aligned}$$

*where the first sum runs over all balanced derangements $w \in \mathfrak{S}_n^r$ and $\xi^{+}_{n,r,i}$ is the number of elements of $\mathfrak{S}_n^r$ with $i$ descending runs, none of size one, and last coordinate of zero color.*

# Cluster subdivision

We let

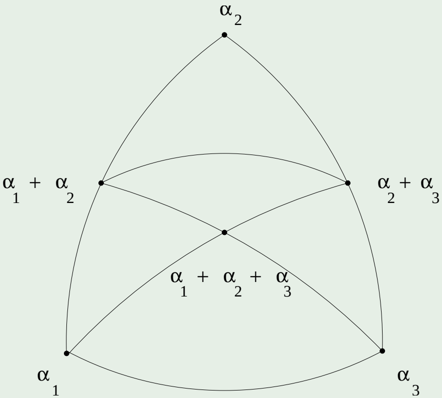
- $W$ be a finite Coxeter group of rank $n$,
- $S$ be a genarating set of simple reflections,
- $W_J$ be the standard parabolic subgroup corresponding to $J \subseteq S$.

The cluster complex $\Delta_W$ has a positive part $\Delta_W^+$ which naturally defines a triangulation of the simplex $2^S$, called the cluster subdivision and denoted by $\Gamma_W$.

Note: The cluster subdivision is flag.

## Example

The cluster subdivision for $\mathfrak{S}_3$:



Note: By definition, we have

$$\ell_S(\Gamma_W, x) \;=\; \sum_{J \subseteq S} (-1)^{|S \smallsetminus J|}\; h(\Delta^+_{W_J}, x)$$

where

$$h(\Delta^+_W, x) = \begin{cases} \displaystyle\sum_{i=0}^{n} \frac{1}{i+1}\binom{n}{i}\binom{n-1}{i} x^i, & \text{if } \; W = \mathfrak{S}_{n+1} \\[2ex] \displaystyle\sum_{i=0}^{n} \binom{n}{i}\binom{n-1}{i} x^i, & \text{if } \; W = B_n \\[2ex] \displaystyle\sum_{i=0}^{n} \left(\binom{n}{i}\binom{n-2}{i} + \binom{n-2}{i-2}\binom{n-1}{i}\right) x^i, & \text{if } \; W = D_n. \end{cases}$$

**Theorem (A–Savvidou, 2012)**

*The local h-polynomial of the cluster subdivision $\Gamma_W$ is $\gamma$-nonnegative for every finite Coxeter group $W$.*

Problem: Find a proof which does not depend on the classification of finite Coxeter groups.

Note: Writing

$$\gamma(W,x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i(W)x^i, \qquad \xi(W,x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \xi_i(W)x^i,$$

where

$$C_W(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i(W)\, x^i(1+x)^{n-2i},$$

$$\ell_S(\Gamma_W, x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \xi_i(W)\, x^i(1+x)^{n-2i},$$

we have

$$\gamma(W,x) = \sum_{J \subseteq S} \xi(W_J, x).$$

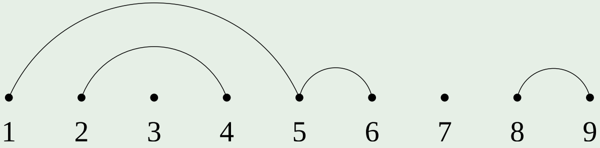
Let us also write

$$\ell_S(\Gamma_W, x) \;=\; \sum_{i=0}^{n} \ell_i(W) x^i.$$

We call a singleton block $\{b\}$ of a noncrossing partition $\pi$ of $[n]$ nested if some block of $\pi$ contains elements $a$ and $c$ such that $a < b < c$; otherwise the block $\{b\}$ is nonnested.

## Example

A noncrossing partition of [9] with nested singleton block $\{3\}$ and a non-nested singleton block $\{7\}$:




Note: An analogous definition exists for type $B$ noncrossing partitions.

**Theorem** (A–Savvidou, 2012)

*The coefficient $\ell_i(W)$ is equal to:*

- *the number of noncrossing partitions $\pi$ of $[n]$ with $i$ blocks, such that every singleton block of $\pi$ is nested, if $W = \mathfrak{S}_{n+1}$,*
- *the number of noncrossing partitions $\pi$ of type $B_n$ with no zero block and $i$ pairs $\{B, -B\}$ of nonzero blocks, such that every positive singleton block of $\pi$ is nested, if $W = B_n$,*
- *$n - 2$ times the number of noncrossing partitions of $[n-1]$ having $i$ blocks, if $W = D_n$.*

**Theorem (A–Savvidou, 2012)**

*The coefficient $\xi_i(W)$ is equal to:*

- *the number of noncrossing partitions of $[n]$ which have no singleton blocks and a total of $i$ blocks, if $W = \mathfrak{S}_{n+1}$,*
- *the number of noncrossing partitions of type $B_n$ which have no zero block, no singleton blocks and a total of $i$ pairs $\{B, -B\}$ of nonzero blocks, if $W = B_n$.*

**Corollary** (A–Savvidou, 2012)

*We have* $\xi_0(W) = 0$ *and*

$$\xi_i(W) = \begin{cases} \dfrac{1}{n-i+1}\dbinom{n}{i}\dbinom{n-i-1}{i-1}, & \text{if } W = \mathfrak{S}_{n+1} \\[2ex] \dbinom{n}{i}\dbinom{n-i-1}{i-1}, & \text{if } W = B_n \\[2ex] \dfrac{n-2}{i}\dbinom{2i-2}{i-1}\dbinom{n-2}{2i-2}, & \text{if } W = D_n \end{cases}$$

*for* $1 \le i \le \lfloor n/2 \rfloor$.

For a summery of these results see

- C.A. Athanasiadis, *A survey of subdivisions and local h-vectors*, in "The Mathematical Legacy of Richard P. Stanley", AMS, 2016.

# III. Methods

# Methods

Methods to prove $\gamma$-nonnegativity include:

- valley hopping (Foata–Schützenberger–Strehl)
- combinatorial expansions (Bränden, Shin–Zeng, Stembridge)
- symmetric functions (Shareshian–Wachs)
- poset decompositions (Simion–Ullman, Petersen, Mühle)
- poset homology, shellability (Linusson–Shareshian–Wachs)
- enriched $P$-partitions (Stembridge, Petersen)
- combinatorics of subdivisions (A–Savvidou).

# Valley hopping

We have seen several applications of valley hopping. For more see, for instance:

- P. Bränden, *Actions on permutations and unimodality of descent polynomials*, European J. Combin. **29** (2008), 514–531.
- A. Postnikov, V. Reiner and L. Williams, *Faces of generalized permutohedra*, Doc. Math. **13** (2008), 207–273.
- Z. Lin and J. Zeng, *The $\gamma$-positivity of basic Eulerian polynomials via group actions*, J. Combin. Theory Series A **135** (2015), 112–129.

# Symmetric functions

For $\Theta \subseteq \mathbb{Z}$ we let

- $\mathrm{Stab}(\Theta)$ be the set of subsets of $\Theta$ which do not contain two successive integers.

**Theorem** (Shareshian–Wachs, 2010)

*We have*

$$\sum_{w \in \mathfrak{S}_n} q^{\mathrm{maj}(w)-\mathrm{exc}(w)} t^{\mathrm{exc}(w)} = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \gamma_{n,i}(q)\, t^i (1+t)^{n-1-2i}$$

*where*

$$\gamma_{n,i}(q) = \sum q^{\mathrm{maj}(w^{-1})},$$

*the sum running over all permutations $w \in \mathfrak{S}_n$ with $i$ descents, such that $\mathrm{Des}(w) \in \mathrm{Stab}([n-2])$.*

The proof of Shareshian–Wachs uses symmetric functions. Recall the polynomials $T_\lambda(t)$ defined by

$$\sum_\lambda T_\lambda(t)\, s_\lambda(x) \;=\; \frac{\displaystyle\sum_{k\geq 1} (1+t+\cdots+t^{k-1})\, s_k(x)}{1-\displaystyle\sum_{k\geq 2} (t+t^2+\cdots+t^{k-1})\, s_k(x)}$$

and define the $R_\lambda(t)$ similarly by the equality

$$\sum_\lambda R_\lambda(t)\, s_\lambda(x) \;=\; \frac{1}{1-\displaystyle\sum_{k\geq 2} (t+t^2+\cdots+t^{k-1})\, s_k(x)}.$$

Note: We have

$$\sum_{\lambda \vdash n} f^{\lambda} T_{\lambda}(t) \;=\; A_n(t)$$

and

$$\sum_{\lambda \vdash n} f^{\lambda} R_{\lambda}(t) \;=\; d_n(t),$$

where $f^{\lambda}$ is the number of standard Young tableaux of shape $\lambda$.

Note: The symmetry and unimodality of the polynomials $R_{\lambda}(t)$ and $T_{\lambda}(t)$ was shown by Brenti (1990).

Note: We have

$$\sum_{\lambda} T_\lambda(t)\, s_\lambda(x) \;=\; \frac{(1-t)H(x,1)}{H(x,t)-tH(x;1)}$$

and

$$\sum_{\lambda} R_\lambda(t)\, s_\lambda(x) \;=\; \frac{1-t}{H(x,t)-tH(x;1)},$$

where

$$H(x,z) \;=\; \sum_{n\geq 0} s_n(x) z^n \;=\; \frac{1}{\prod_{i\geq 1}(1-x_i z)}.$$

Note: The generating functions on the right-hand sides have several important algebraic-geometric and combinatorial interpretations.

The proof uses the following result. For a word $w = a_1 a_2 \cdots a_n$ on the alphabet $\mathbb{Z}_{>0} = \{1, 2, \dots\}$ we set $x_w = x_{a_1} x_{a_2} \cdots x_{a_n}$.

**Theorem** (Gessel, unpublished)

*We have*

$$\frac{(1-t)H(x,1)}{H(x,t) - tH(x;1)} = \sum_{n \geq 1} \sum_{w \in U_n} x_w \, t^{\mathrm{des}(w)} (1+t)^{n-1-2\mathrm{des}(w)},$$

*where* $U_n$ *stands for the set of words* $w$ *of length* $n$ *on the alphabet* $\mathbb{Z}_{>0}$ *such that* $\mathrm{Des}(w) \in \mathrm{Stab}([n-2])$*, and*

$$\frac{1-t}{H(x,t) - tH(x;1)} = 1 + \sum_{n \geq 2} \sum_{w \in \widetilde{U}_n} x_w \, t^{\mathrm{des}(w)+1} (1+t)^{n-2-2\mathrm{des}(w)},$$

*where* $\widetilde{U}_n$ *stands for the set of words* $w$ *of length* $n$ *on the alphabet* $\mathbb{Z}_{>0}$ *such that* $\mathrm{Des}(w) \in \mathrm{Stab}(\{2, \dots, n-2\})$.

**Corollary**

*We have*

$$T_\lambda(t) \;=\; \sum t^{\mathrm{des}(Q)}(1+t)^{n-1-2\mathrm{des}(Q)},$$

*where the sum ranges over all standard Young tableaux* $Q \in \mathrm{SYT}(\lambda)$ *such that* $\mathrm{Des}(Q) \in \mathrm{Stab}([n-2])$, *and*

$$R_\lambda(t) \;=\; \sum t^{\mathrm{des}(Q)+1}(1+t)^{n-2\mathrm{des}(Q)-2},$$

*where the sum ranges over all standard Young tableaux* $Q \in \mathrm{SYT}(\lambda)$ *such that* $\mathrm{Des}(Q) \in \mathrm{Stab}(\{2, \ldots, n-2\})$.

Sketch of proof. Use Gessel's result, interpret the elements of $U_n$ and $\widetilde{U}_n$ as reading words of semistandard ribbon skew tableaux, express the resulting ribbon skew Schur functions in terms of ordinary skew Schur functions and extract the coefficient of $s_\lambda(x)$ to get the desired expressions for $T_\lambda(t)$ and $R_\lambda(t)$. $\square$

Sketch of proof of Shareshian–Wachs. Let us write

$$A_n(q,t) \;:=\; \sum_{w \in \mathfrak{S}_n} q^{\mathrm{maj}(w)-\mathrm{exc}(w)} t^{\mathrm{exc}(w)}.$$

The "Eulerian quasisymmetric function" expansion of

$$\frac{(1-t)H(x,1)}{H(x,t)-tH(x;1)} \;=\; \sum_{\lambda} T_\lambda(t)\, s_\lambda(x),$$

due to Shareshian–Wachs (2010), gives

$$\sum_{w \in \mathfrak{S}_n} F_{n,\mathrm{DEX}(w)}(x)\, t^{\mathrm{exc}(w)} \;=\; \sum_{\lambda \vdash n} T_\lambda(t)\, s_\lambda(x).$$

Taking the stable principal specialization of both hand sides, we get

$$\frac{A_n(q,t)}{(1-q)(1-q^2)\cdots(1-q^n)} \; = \; \sum_{\lambda \vdash n} T_\lambda(t) \, \frac{f^\lambda(q)}{(1-q)(1-q^2)\cdots(1-q^n)}$$

and conclude that

$$A_n(q,t) \; = \; \sum_{\lambda \vdash n} T_\lambda(t) \, f^\lambda(q),$$

where

$$f^\lambda(q) \; := \sum_{Q \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(Q)}.$$

The $\gamma$-expansion of $T_\lambda(t)$, given in the corollary, as well as standard manipulations and properties of the Robinson–Schensted correspondence, yield the desired expansion for $A_n(q,t)$. □

Note. Similarly, the quasisymmetric function expansion

$$\frac{1-t}{H(x,t)-tH(x;1)} \;=\; \sum_{n\geq 0}\sum_{w\in\mathcal{D}_n} F_{n,\mathrm{DEX}(w)}(x)\, t^{\mathrm{exc}(w)},$$

due to Shareshian–Wachs (2010), gives

$$\sum_{w\in\mathcal{D}_n} F_{n,\mathrm{DEX}(w)}(x)\, t^{\mathrm{exc}(w)} \;=\; \sum_{\lambda\vdash n} R_\lambda(t)\, s_\lambda(x).$$

Taking the stable principal specialization yields the following result:

### Theorem

*We have*

$$\sum_{w\in\mathcal{D}_n} q^{\mathrm{maj}(w)-\mathrm{exc}(w)}\, t^{\mathrm{exc}(w)} = \sum_{\lambda\vdash n} R_\lambda(t)\, f^\lambda(q),$$

$$= \sum_{i=0}^{\lfloor (n-2)/2\rfloor} \xi_{n,i}(q)\, t^{i+1}(1+t)^{n-2i-2},$$

*where*

$$\xi_{n,i}(q) = \sum q^{\mathrm{maj}(w^{-1})},$$

*the sum running over all permutations* $w\in\mathfrak{S}_n$ *with* $i$ *descents, such that* $\mathrm{Des}(w)\in\mathrm{Stab}(\{2,\ldots,n-2\})$.

Using nonstable principal specialization instead yields the following refinement:

## Theorem

*We have*

$$\sum_{w\in\mathcal{D}_n} p^{\mathrm{des}(w)} q^{\mathrm{maj}(w)-\mathrm{exc}(w)} t^{\mathrm{exc}(w)} = p\cdot\sum_{\lambda\vdash n} R_\lambda(t)\, f^\lambda(p,q),$$

$$= \sum_{i=0}^{\lfloor (n-2)/2\rfloor} \xi_{n,i}(p,q)\, t^{i+1}(1+t)^{n-2i-2}$$

*where*

$$f^\lambda(p,q) := \sum_{Q\in\mathrm{SYT}(\lambda)} p^{\mathrm{des}(Q)} q^{\mathrm{maj}(Q)}$$

## Theorem

*and*

$$\xi_{n,i}(p,q) \;=\; p \cdot \sum p^{\mathrm{des}(w^{-1})} q^{\mathrm{maj}(w^{-1})},$$

*the sum running over all permutations $w \in \mathfrak{S}_n$ with $i$ descents, such that* $\mathrm{Des}(w) \in \mathrm{Stab}(\{2,\ldots,n-2\})$.

Similarly:

## Theorem

*We have*

$$\sum_{w\in\mathfrak{S}_n} p^{\mathrm{des}^*(w)} q^{\mathrm{maj}(w)-\mathrm{exc}(w)} t^{\mathrm{exc}(w)} = \sum_{\lambda\vdash n} T_\lambda(t)\, f^\lambda(p,q),$$

$$= \sum_{i=0}^{\lfloor (n-1)/2\rfloor} \gamma_{n,i}(p,q)\, t^i (1+t)^{n-1-2i},$$

*where*

$$\mathrm{des}^*(w) = \begin{cases} \mathrm{des}(w), & \textit{if } \; w(1)=1 \\ \mathrm{des}(w)-1, & \textit{if } \; w(1)\neq 1 \end{cases}$$

*for* $w \in \mathfrak{S}_n$ *and*

$$\gamma_{n,i}(p,q) = \sum p^{\mathrm{des}(w^{-1})} q^{\mathrm{maj}(w^{-1})},$$

*the sum running over all permutations* $w \in \mathfrak{S}_n$ *with* $i$ *descents, such that* $\mathrm{Des}(w) \in \mathrm{Stab}([n-2])$.

# Combinatorics of subdivisions

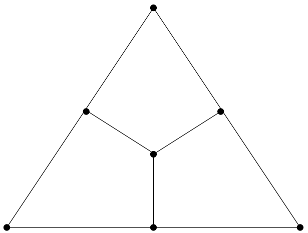
Consider again the polynomial

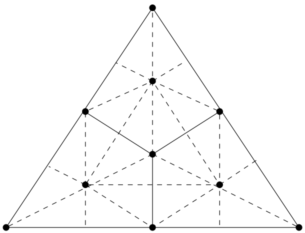
$$f_n^+(x) \;=\; \sum_w x^{\mathrm{fex}(w)/2},$$

where the first sum runs over all derangements $w \in B_n$ with an even number of negative signs. Let us use the fact that

$$f_n^+(x) \;=\; \ell_V(K, x)$$

to find a formula for $f_n^+(x)$ which implies $\gamma$-nonnegativity.





Let us recall the definition of $\ell_V(\Gamma, x)$. We let

- $V$ be an $n$-element set,
- $\Gamma$ be a triangulation of the simplex $2^V$ on the vertex set $V$.

**Definition** (Stanley, 1992)

*The local h-polynomial of* $\Gamma$ *(with respect to* $V$*) is defined as*

$$\ell_V(\Gamma, x) \;=\; \sum_{F \subseteq V} (-1)^{n-|F|} \, h(\Gamma_F, x),$$

*where* $\Gamma_F$ *is the restriction of* $\Gamma$ *to the face* $F$ *of the simplex* $2^V$.

We also recall that the link of a simplicial complex $\Delta$ at a face $F \in \Delta$ is defined as $\mathrm{link}_\Delta(F) := \{G \smallsetminus F : F \subseteq G \in \Delta\}$.

**Proposition** (Stanley, 1992)

*For every triangulation $\Delta'$ of a pure simplicial complex $\Delta$,*

$$h(\Delta', x) \;=\; \sum_{F \in \Delta} \ell_F(\Delta'_F, x)\, h(\mathrm{link}_\Delta(F), x).$$

### Corollary

*For every triangulation $\Delta$ of the boundary complex $\Sigma_n$ of the n-dimensional cross-polytope we have*

$$h(\Delta, x) \;=\; \sum_{F \in \Sigma_n} \ell_F(\Delta_F, x)\,(1+x)^{n-|F|}.$$

*In particular, if $\ell_F(\Delta_F, x)$ is $\gamma$-nonnegative for every $F \in \Sigma_n$, then so is $h(\Delta, x)$.*

### Example

The polynomial $h(\mathrm{esd}_r(\Sigma_n), x)$ is $\gamma$-nonnegative for all $n, r$.

We let

- $V$ be an $n$-element set,
- $\Gamma$ be a triangulation of the simplex $2^V$ on the vertex set $V$,
- $E$ be a face of $\Gamma$.

**Definition (A, 2012)**

*The relative local h-polynomial of $\Gamma$ (with respect to $V$) at $E \in \Gamma$ is defined as*

$$\ell_V(\Gamma, E, x) \ = \sum_{\sigma(E) \subseteq F \subseteq V} (-1)^{d-|F|} \, h(\mathrm{link}_{\Gamma_F}(E), x),$$

*where $\sigma(E)$ is the smallest face of $2^V$ containing $E$.*

Note: $\ell_V(\Gamma, \varnothing, x) = \ell_V(\Gamma, x)$.

## Example

We let

- $\Gamma$ be the barycentric subdivision of $2^V$,
- $E$ be a face of $\Gamma$ given by the chain $S_1 \subset S_2 \subset \cdots \subset S_k$ of nonempty subsets of $V$.

Then

$$\ell_V(\Gamma, E, x) \;=\; d_{n_0}(x)\, A_{n_1}(x) A_{n_2}(x) \cdots A_{n_k}(x),$$

where $d_0(x) := 1$, $n_0 = |V \smallsetminus S_k|$ and $n_i = |S_i \smallsetminus S_{i-1}|$ for $1 \le i \le k$.

**Theorem (A, 2012)**

*The polynomial $\ell_V(\Gamma, E, x)$*

- *is symmetric, and*
- *has nonnegative coefficients.*

**Theorem (Katz–Stapledon, 2016)**

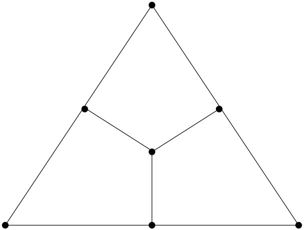
*The polynomial $\ell_V(\Gamma, E, x)$ is unimodal for every regular triangulation $\Gamma$ of $2^V$ and every $E \in \Gamma$.*

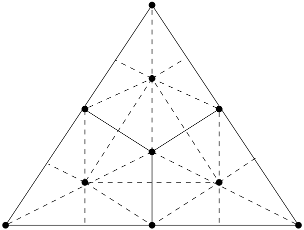
**Proposition (A, 2012)**

*For every triangulation $\Gamma$ of the simplex $2^V$ and every triangulation $\Gamma'$ of $\Gamma$,*

$$\ell_V(\Gamma', x) \ = \ \sum_{E \in \Gamma} \ell_E(\Gamma'_E, x) \, \ell_V(\Gamma, E, x).$$

We now note that $K$ is a subdivision of the simplicial barycentric subdivision of $2^V$





and apply the previous formula when

- $\Gamma$ is the simplicial barycentric subdivision of $2^V$,
- $\Gamma' = K$.

Note: Each face $E \in \Gamma$ is subdivided by $\Gamma'$ into $2^{\dim(E)}$ simplices of the same dimension. This implies that

$$\ell_E(\Gamma'_E, x) \;=\; \begin{cases} x^{|E|/2}, & \text{if } \;|E| \text{ is even} \\ 0, & \text{otherwise.} \end{cases}$$

We deduce the following formula for $\ell_V(K, x) = f_n^+(x)$, which implies its $\gamma$-nonnegativity.

## Proposition

$$f_n^+(x) \;=\; \sum \binom{n}{r_0, r_1, \ldots, r_{2k}} \, x^k \, d_{r_0}(x) \, A_{r_1}(x) \cdots A_{r_{2k}}(x),$$

*where the sum ranges over all $k \geq 0$ and over all sequences $(r_0, r_1, \ldots, r_{2k})$ of integers which satisfy $r_0 \geq 0$, $r_1, \ldots, r_{2k} \geq 1$ and sum to $n$.*

## Example

Applying the same formula to the second barycentric subdivision $\Gamma^2$ of $2^V$ we get

$$\ell_V(\Gamma^2, x) \; = \; \sum \binom{n}{r_0, r_1, \ldots, r_k} d_k(x)\, d_{r_0}(x)\, A_{r_1}(x) \cdots A_{r_k}(x),$$

where the sum ranges over all $k \geq 0$ and over all sequences $(r_0, r_1, \ldots, r_k)$ of integers which satisfy $r_0 \geq 0$, $r_1, \ldots, r_k \geq 1$ and sum to $n$.

Note: This implies the $\gamma$-nonnegativity of $\ell_V(\Gamma^2, x)$.

# Poset homology

We let

- $P$ be a finite graded poset with rank function $\rho_P$,
- $Q$ be a finite graded poset with rank function $\rho_Q$.

**Definition** (Björner–Welker, 2005)

*The Rees product of $P$ and $Q$ is defined as*

$$P * Q \;=\; \{(p, q) \in P \times Q : \; \rho_P(p) \geq \rho_Q(q)\},$$

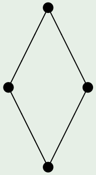
*with partial order defined by setting $(p_1, q_1) \leq (p_2, q_2)$ if and only if:*

- $p_1 \leq p_2$ *holds in* $P$,
- $q_1 \leq q_2$ *holds in* $Q$, *and*
- $\rho_P(p_2) - \rho_P(p_1) \geq \rho_Q(q_2) - \rho_Q(q_1)$.

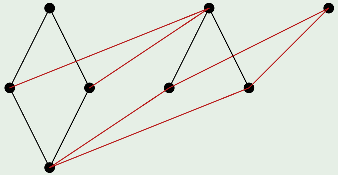
Note: Equivalently, $(p_1, q_1)$ is covered by $(p_2, q_2)$ if and only if

- $p_1$ is covered by $p_2$ in $P$, and
- either $q_1 = q_2$, or $q_1$ is covered by $q_2$ in $Q$.

## Example

 $*$  $=$

## Example



For a graded poset $P$ of rank $n+1$ with minimum element $\hat{0}$, maximum element $\hat{1}$ and rank function $\rho : P \to \{0, 1, \dots, n+1\}$, we let

- $\bar{P} \;=\; P \smallsetminus \{\hat{0}, \hat{1}\}$,
- $\mu(\bar{P}) \;=\; \mu_P(\hat{0}, \hat{1})$,

where $\mu_P$ is the Möbius function of $P$. For $S \subseteq [n]$ we set

- $\beta_P(S) \;=\; (-1)^{|S|-1} \mu(\bar{P}_S)$,

where

$$\bar{P}_S \;=\; \{x \in P : \rho(x) \in S\}$$

is a rank-selected subposet.

For positive integers $n, x$ we let

- $T_{x,n}$ be the poset whose Hasse diagram is a complete $x$-ary tree of height $n-1$, with root at the bottom.

**Theorem** (Linusson–Shareshian–Wachs, 2012)

*For every EL-shellable poset $P$ of rank $n+1$ and every positive integer $x$ we have*

$$|\mu(\bar{P} * T_{x,n})| = \sum_{S \in \mathrm{Stab}(\{2,\ldots,n-1\})} \beta_P([n] \setminus S)\, x^{|S|}\, (1+x)^{n-1-2|S|} + \sum_{S \in \mathrm{Stab}(\{2,\ldots,n-2\})} \beta_P([n-1] \setminus S)\, x^{|S|+1}\, (1+x)^{n-2-2|S|},$$

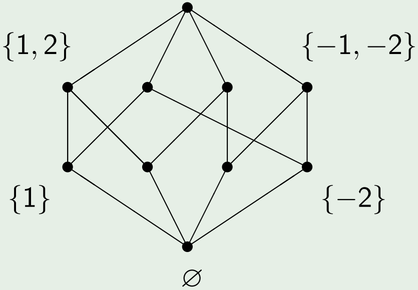
*where $\mathrm{Stab}(\Theta)$ denotes the set of all subsets of $\Theta$ which do not contain two consecutive integers.*

We will apply this to the set

- $B_n^r$ of subsets of $[n]$, with each element $r$-colored, partially ordered by inclusion, with a maximum element $\hat{1}$ attached

## Example

$B_2^2 \quad =$



to prove the following result, mentioned yesterday:

**Theorem (A, 2014)**

*We have*

$$\begin{aligned} f^{+}_{n,r}(x) \quad &:= \quad \sum_{w} x^{\mathrm{fex}(w)/r} \\ &= \quad \sum_{i=0}^{\lfloor n/2 \rfloor} \xi^{+}_{n,r,i}\, x^{i} (1+x)^{n-2i}, \end{aligned}$$

*where the first sum runs over all balanced derangements $w \in \mathfrak{S}^r_n$ and $\xi^{+}_{n,r,i}$ is the number of elements of $\mathfrak{S}^r_n$ with $i$ descending runs, none of size one, and last coordinate of zero color.*

Using the definition of the Möbius function and a result of Shareshian–Wachs (2009), one can show that

$$|\mu(\bar{B}_n^r * T_{x,n})| \ = \ x^n d_n^r(1/x),$$

where

$$d_n^r(x) \ = \ f_{n,r}^{+}(x) \ + \ \sum x^{\lceil \frac{\mathrm{fex}(w)}{r} \rceil},$$

the sum ranging over all nonbalanced derangements $w \in \mathfrak{S}_n^r$. Comparing with the expression provided by the result of Linusson–Shareshian–Wachs, one can conclude that

$$x^n \, f_{n,r}^{+}(1/x) \ = \sum_{S \in \mathrm{Stab}(\{2,\dots,n-1\})} \beta_P([n] \smallsetminus S) \, x^{|S|} \, (1+x)^{n-1-2|S|},$$

where $P = B_n^r$. An easy EL-labeling for $P$ gives a combinatorial interpretation to the numbers $\beta_P(S)$ and yields the desired $\gamma$-expansion for $f_{n,r}^{+}(x)$.

Thank you for your attention!