Outside nested decompositions and Schur function determinants

Emma Yu Jin

Technische Universität Wien

77th SLC, Strobl

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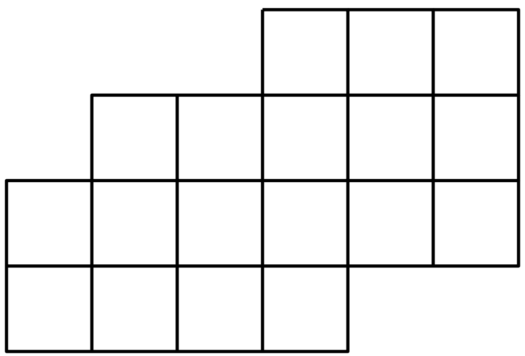
(Semi)standard Young tableaux

(Semi)standard Young tableaux (SSYT/SYT)



λ/μ
$= (6, 6, 6, 4)/(3, 1)$

			1	2	3
	2	2	3	3	4
1	3	4	5	5	6
3	4	5	6		

SSYT

$\longrightarrow$ weakly $\leq$

$\downarrow$ strictly $<$

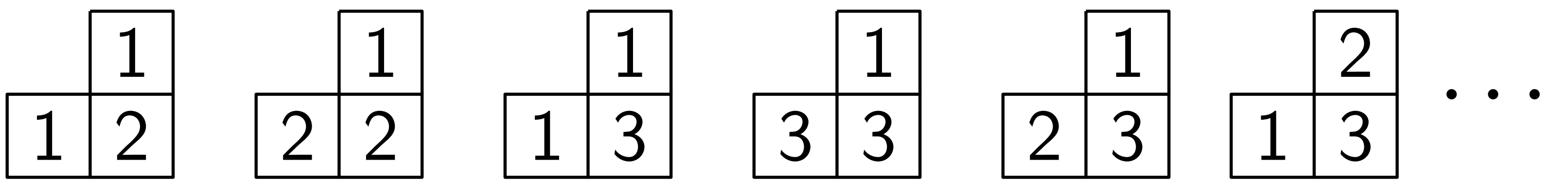
			1	2	3
	4	9	10	11	14
5	7	12	15	17	18
6	8	13	16		

SYT

$\longrightarrow$ strictly $<$

$\downarrow$ strictly $<$

Schur function $s_{\lambda/\mu}(X)$ and $f^{\lambda/\mu}$



$s_{\lambda/\mu}(X)$ is the generating function of SSYT.

$$s_{(2,2)/(1)}(X) = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + \boxed{2x_1x_2x_3} + \cdots$$

$\Longrightarrow$ the number $f^{\lambda/\mu}$ of SYT with entries from 1 to $|\lambda/\mu|$
is $f^{(2,2)/(1)} = [x_1x_2x_3]s_{(2,2)/(1)}(X) = 2$.

Determinantal formulas for $s_{\lambda/\mu}(X)$

$$s_{\ydiagram{2,2}}(X) = \det \begin{bmatrix} s_{\ydiagram{2}}(X) & s_{\ydiagram{3}}(X) \\ s_{\ydiagram{1}}(X) & s_{\ydiagram{2}}(X) \end{bmatrix}$$

Jacobi-Trudi determinant

$$s_{\ydiagram{2,2}}(X) = \det \begin{bmatrix} s_{\ydiagram{1,1}}(X) & s_{\ydiagram{1,1,1}}(X) \\ s_{\ydiagram{1}}(X) & s_{\ydiagram{1,1}}(X) \end{bmatrix}$$

Dual Jacobi-Trudi determinant

$$s_{\ydiagram{2,2}}(X) = \det \begin{bmatrix} s_{\ydiagram{2,1}}(X) & s_{\ydiagram{1,1}}(X) \\ s_{\ydiagram{2}}(X) & s_{\ydiagram{1}}(X) \end{bmatrix}$$

Giambelli determinant

$$s_{\ydiagram{2,2}}(X) = \det \begin{bmatrix} {\color{red} s_{\ydiagram{2}}(X)} & s_{\ydiagram{3}}(X) \\ s_{\ydiagram{1}}(X) & {\color{red} s_{\ydiagram{2}}(X)} \end{bmatrix}$$

Jacobi-Trudi determinant

$$s_{\ydiagram{2,2}}(X) = \det \begin{bmatrix} {\color{red} s_{\ydiagram{1,1}}(X)} & s_{\ydiagram{1,1,1}}(X) \\ s_{\ydiagram{1}}(X) & {\color{red} s_{\ydiagram{1,1}}(X)} \end{bmatrix}$$

Dual Jacobi-Trudi determinant

$$s_{\ydiagram{2,2}}(X) = \det \begin{bmatrix} {\color{red} s_{\ydiagram{2,1}}(X)} & s_{\ydiagram{1,1}}(X) \\ s_{\ydiagram{2}}(X) & {\color{red} s_{\ydiagram{1}}(X)} \end{bmatrix}$$

Giambelli determinant

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Jacobi-Trudi determinant

$$s_{\ydiagram{2,2}}(X) = \det \begin{bmatrix} s_{\ydiagram{1,1}}(X) & s_{\ydiagram{1,1,1}}(X) \\ s_{\ydiagram{1}}(X) & s_{\ydiagram{1,1}}(X) \end{bmatrix}$$

Dual Jacobi-Trudi determinant

$$s_{\ydiagram{2,2}}(X) = \det \begin{bmatrix} s_{\ydiagram{2,1}}(X) & s_{\ydiagram{1,1}}(X) \\ s_{\ydiagram{2}}(X) & s_{\ydiagram{1}}(X) \end{bmatrix}$$

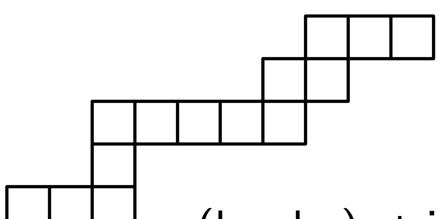
Giambelli determinant

Jacobi-Trudi determinant and its dual $s_{\lambda/\mu}(X)$,
Giambelli determinant $s_{\lambda}(X)$,
Lascoux and Pragacz determinant $s_{\lambda/\mu}(X)$,

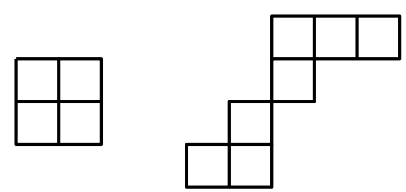
unified by

Hamel and Goulden determinant $s_{\lambda/\mu}(X)$,
(outside decompositions)

Reference: A.M. Hamel and I.P. Goulden, Planar decompositions of tableaux and Schur function determinants, Europ. J. Combinatorics, 16, 461-477, 1995.



(border) strip or ribbon



are not allowed

Hamel and Goulden's Determinant

Motivation: unify different determinantal expressions of $s_{\lambda/\mu}(X)$

Hamel and Goulden determinant,
(outside decompositions)

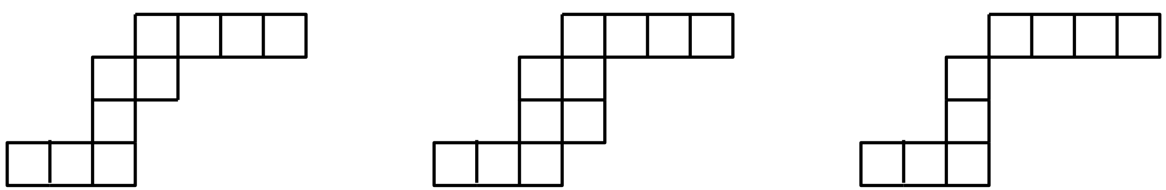
If the skew diagram of λ/μ is edgewise connected. Then, for any outside decomposition $\Phi = (\theta_1, \theta_2, \ldots, \theta_g)$ of the skew shape λ/μ, we have

$$s_{\lambda/\mu}(X) = \det[s_{\theta_i \# \theta_j}(X)]_{i,j=1}^{g}$$

where $s_{\emptyset}(X) = 1$ and $s_{\theta_i \# \theta_j}(X) = 0$ if $\theta_i \# \theta_j$ is undefined.

(Border) strips (or ribbons):

A skew diagram θ is a (border) strip if θ is edgewise connected and contains no 2×2 blocks of boxes.



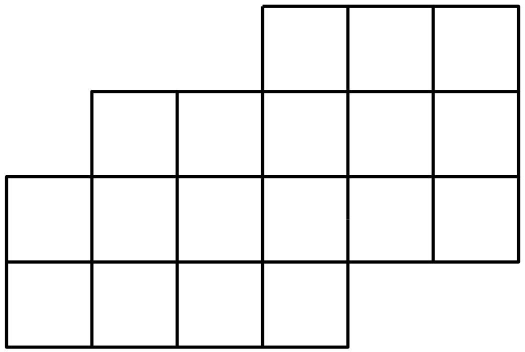
Yes No No

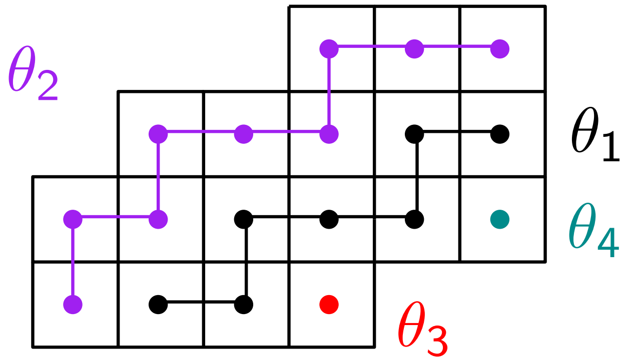
$\phi = (\theta_1, \theta_2, \theta_3, \theta_4)$ is an outside decomposition

(1) θ_i is a (border) strip for all i;

(2) the disjoint union of all (border) strips is the skew shape λ/μ;

(3) every starting box (resp. ending box) of θ_i is on the bottom or left (resp. the top or right) perimeter of the skew shape λ/μ.





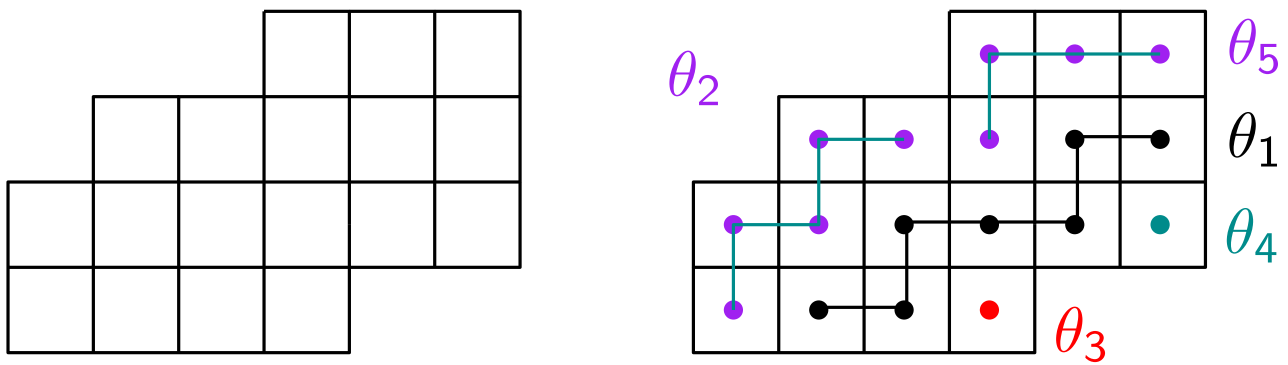
Yes

$\phi = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$ is not an outside decomposition

(1) θ_i is a (border) strip for all i;

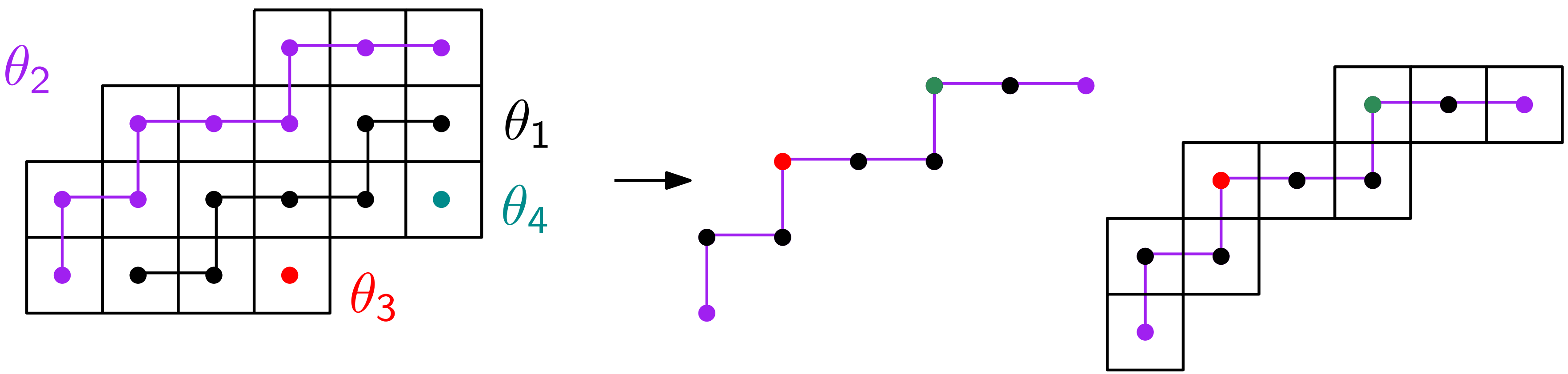
(2) the disjoint union of all (border) strips is the skew shape λ/μ;

(3) every starting box (resp. ending box) of θ_i is on the bottom or left (resp. the top or right) perimeter of the skew shape λ/μ.



No

Outside decomposition (is nested)$\rightarrow$ cutting strip$\rightarrow$ operator $\#$



E.g., $\theta_4 \# \theta_3 = (3,3)/(2)$ and $\theta_3 \# \theta_4$ is undefined.

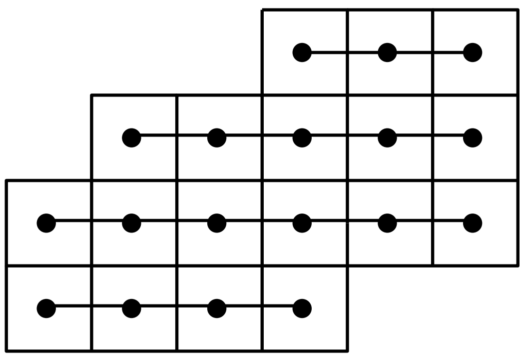
Simplify the definition of $\theta_i \# \theta_j$ from Hamel and Goulden's paper: W.Y.C. Chen, G.G Yan and A.L.B Yang, Transformations of border strips and Schur function determinants, J. Algebr. Comb. 21, 379-394, 2005.

Hamel and Goulden determinant,
(outside decompositions)

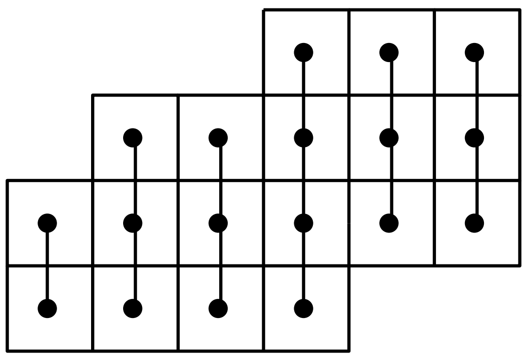
If the skew diagram of λ/μ is edgewise connected. Then, for any outside decomposition $\Phi = (\theta_1, \theta_2, \ldots, \theta_g)$ of the skew shape λ/μ, we have

$$s_{\lambda/\mu}(X) = \det[s_{\theta_i \# \theta_j}(X)]_{i,j=1}^{g},$$

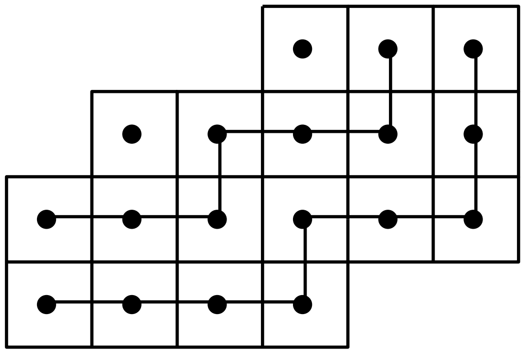
where $s_{\emptyset}(X) = 1$ and $s_{\theta_i \# \theta_j}(X) = 0$ if $\theta_i \# \theta_j$ is undefined.



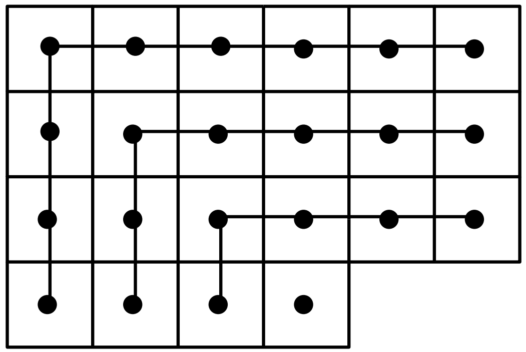
Jacobi-Trudi determinant



Dual Jacobi-Trudi determinant

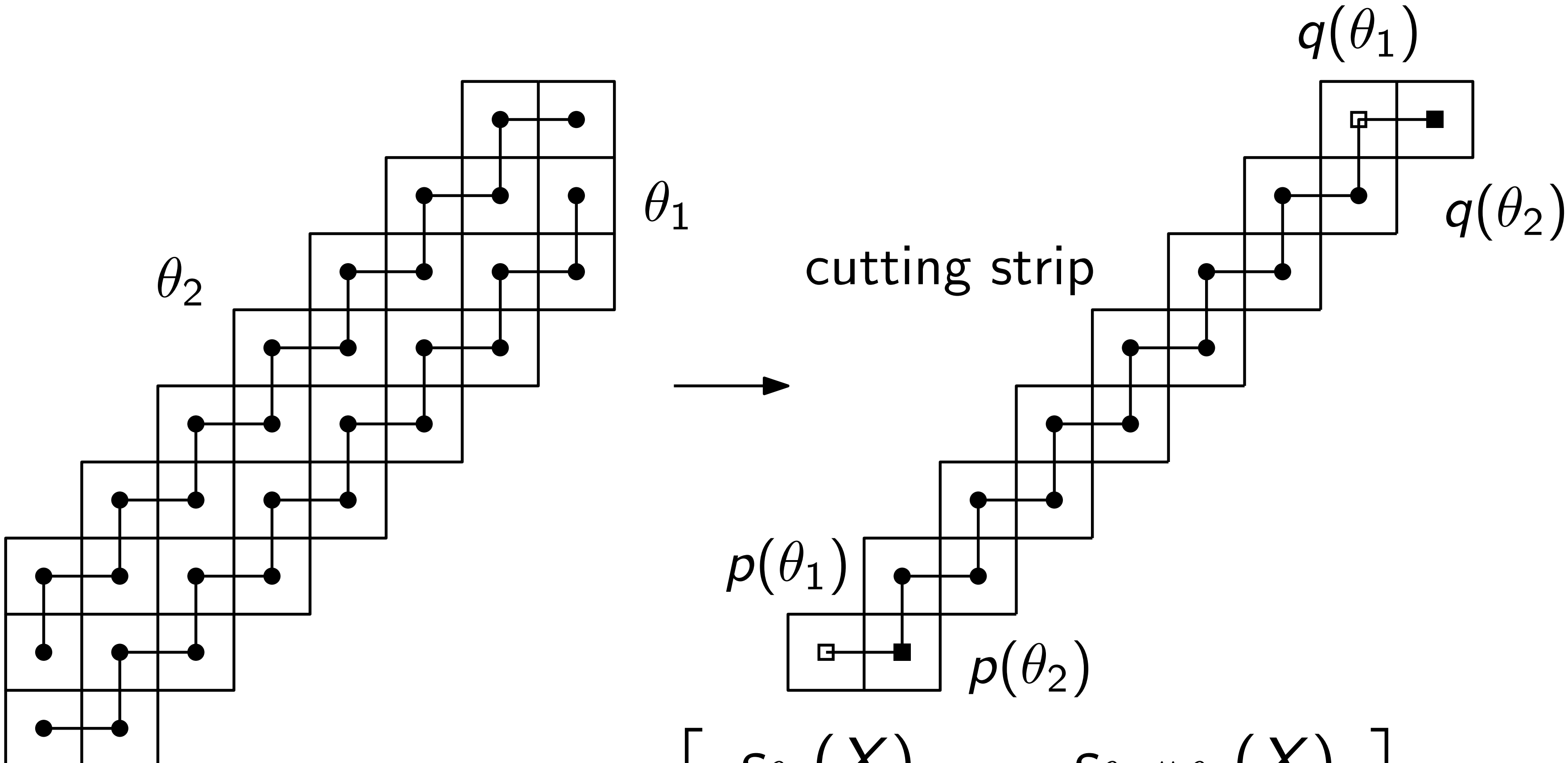


Lascoux-Pragacz determinant



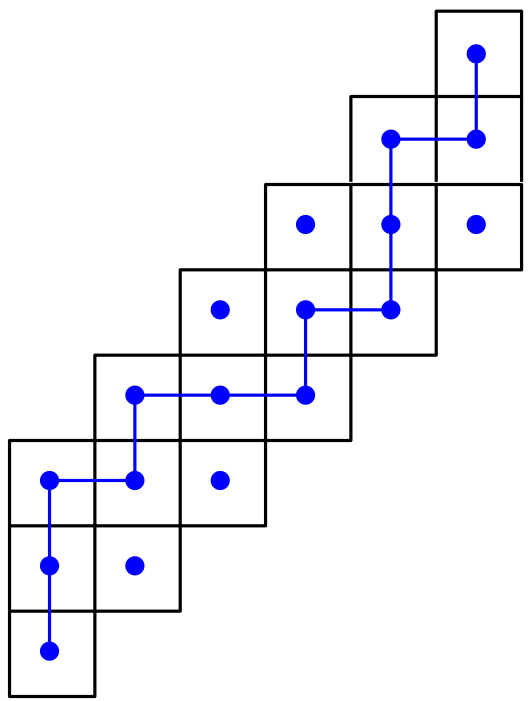
Giambelli determinant

Positive side: simplify some determinants

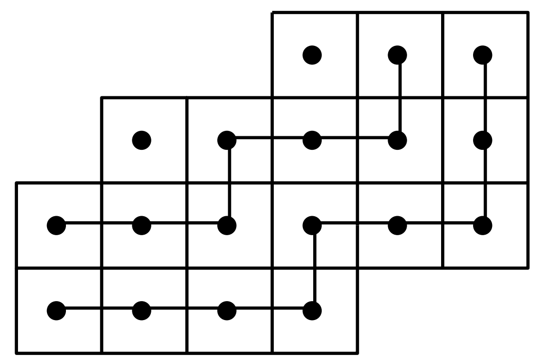


$$s_{\lambda/\mu}(X) = \det \begin{bmatrix} s_{\theta_1}(X) & s_{\theta_1\#\theta_2}(X) \\ s_{\theta_2\#\theta_1}(X) & s_{\theta_2}(X) \end{bmatrix}$$

Negative side: can not simplify some determinants

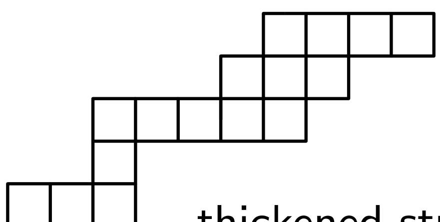


minimal strips= # columns

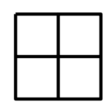


minimal strips= # rows

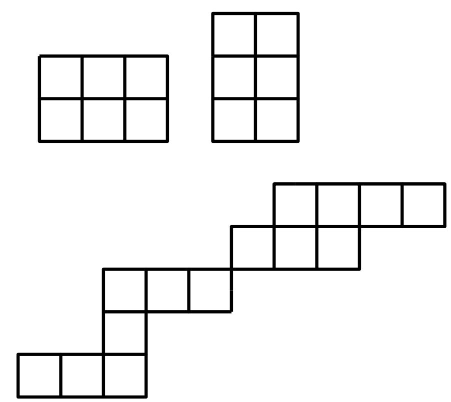
Main results
(outside nested decompositions)



thickened strip



is allowed



are not allowed

Main results:

a determinantal expression of $s_{\lambda/\mu}(X)p_{1^r}(X)$
(outside nested decompositions)

If the skew diagram of λ/μ is edgewise connected. Then, for any outside nested decomposition $\Phi = (\Theta_1, \Theta_2, \ldots, \Theta_g)$ of the skew shape λ/μ, we have

$$p_{1^r}(X)s_{\lambda/\mu}(X) = \det[s_{\Theta_i\#\Theta_j}(X)]_{i,j=1}^{g} \text{ where } p_{1^r}(X) = (\sum_{i=1}^{\infty} x_i)^r,$$

$s_{\emptyset}(X) = 1$, $s_{\Theta_i\#\Theta_j}(X) = 0$ if $\Theta_i\#\Theta_j$ is undefined
and r is the number of common special corners of Φ.

Main results:

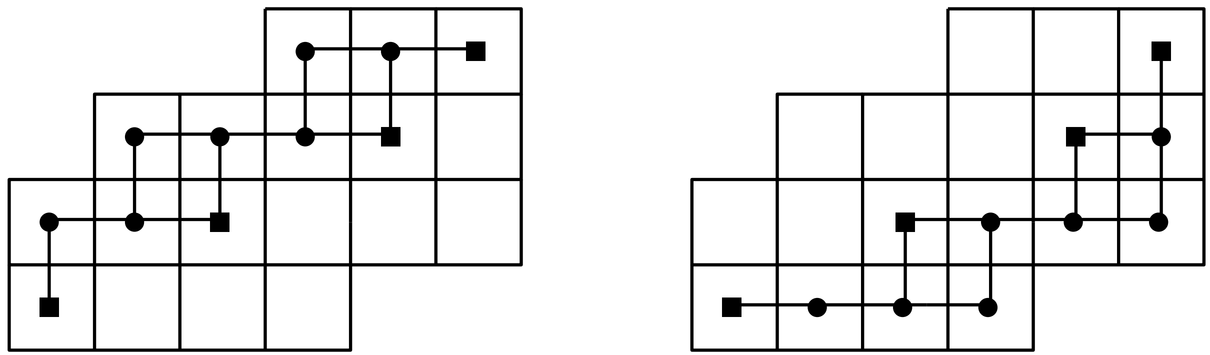
a determinantal expression of $s_{\lambda/\mu}(X)p_{1^r}(X)$ (outside nested decompositions) + exponential specialization

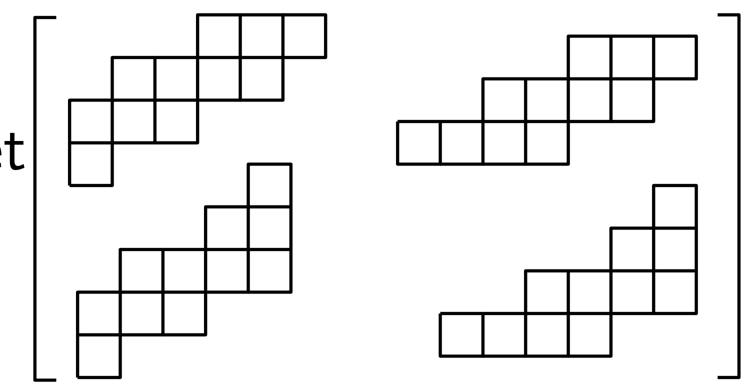
If the skew diagram of λ/μ is edgewise connected. Then, for any outside nested decomposition $\Phi = (\Theta_1, \Theta_2, \ldots, \Theta_g)$ of the skew shape λ/μ, we have

$$f^{\lambda/\mu} = |\lambda/\mu|!\,\det[(|\Theta_i\#\Theta_j|!)^{-1} f^{\Theta_i\#\Theta_j}]_{i,j=1}^{g}$$

where $f^{\emptyset} = 1$ and $f^{\Theta_i\#\Theta_j} = 0$ if $\Theta_i\#\Theta_j$ is undefined.

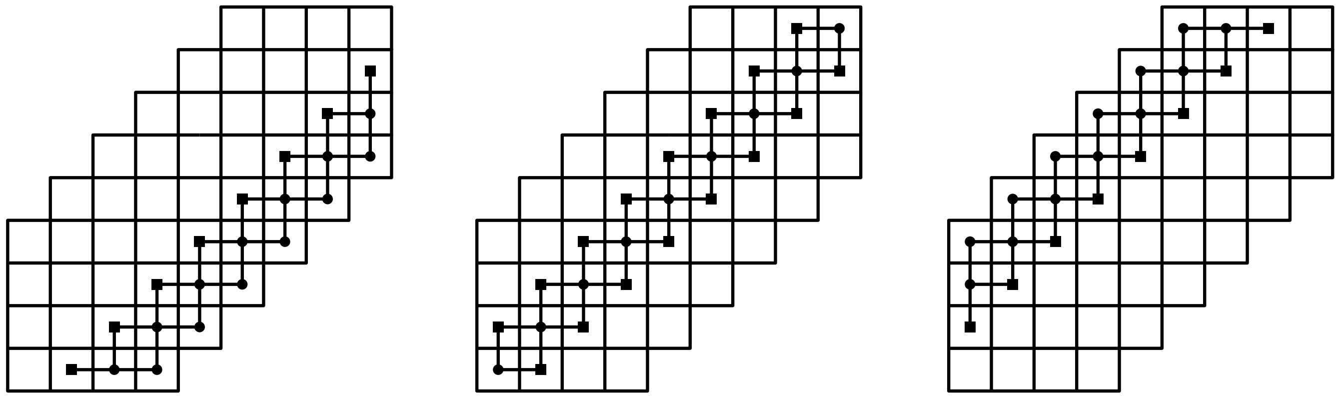
Outside nested decomposition:



$p_{1^4}(X)s_{(6,6,6,4)/(3,1)}(X) = \mathrm{det}$ 

Application to the m-strip tableaux:

Reference: Y. Baryshnikov and D. Romik, Enumeration formulas for Young tableaux in a diagonal strip, Israel Journal of Mathematics 178, 157-186, 2010.

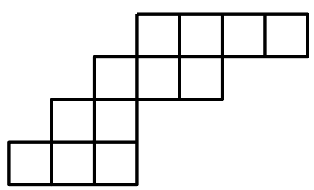


an outside nested decomposition of an m-strip diagram

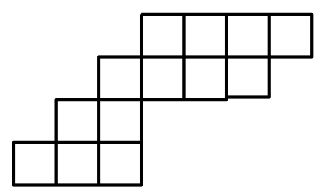
Outside nested decompositions

Thickened strips:

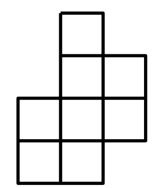
A skew diagram Θ is a thickened strip if Θ is edgewise connected and neither contains a 3×2 block of boxes nor a 2×3 block of boxes.



Yes

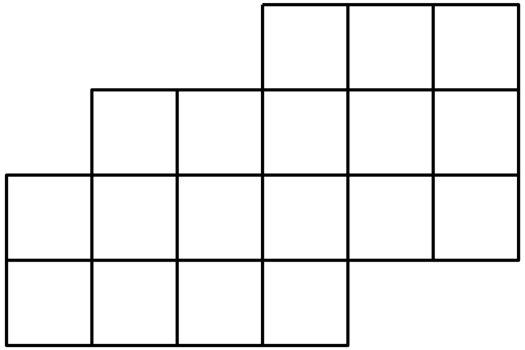


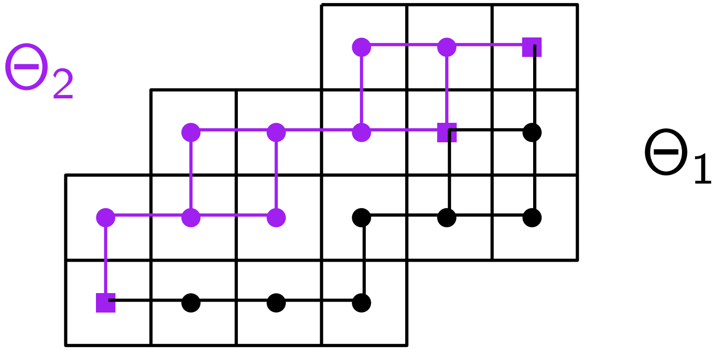
No



No

$\Phi = (\Theta_1, \Theta_2)$ is an outside thickened strip decomposition.





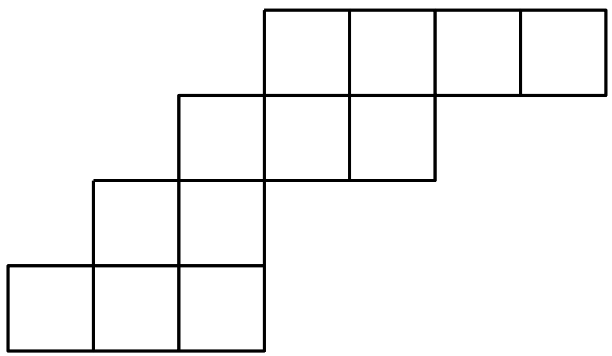
(1) Θ_i is a thickened strip for all i.

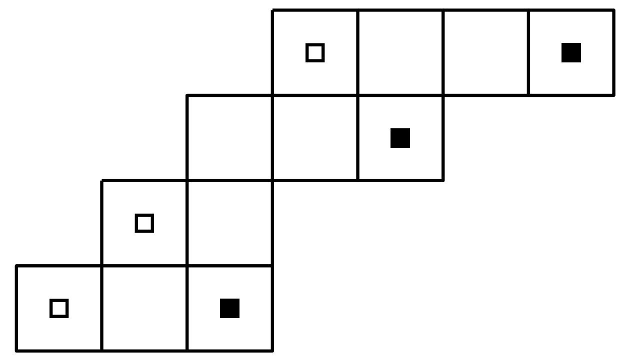
(2) the union of all thickened strips is the skew shape λ/μ.

(3) every starting box (resp. ending box) of Θ_i is on the bottom or left (resp. the top or right) perimeter of the skew shape λ/μ.

(4) allowed common special corners (next page)

Special corners: ■ and □

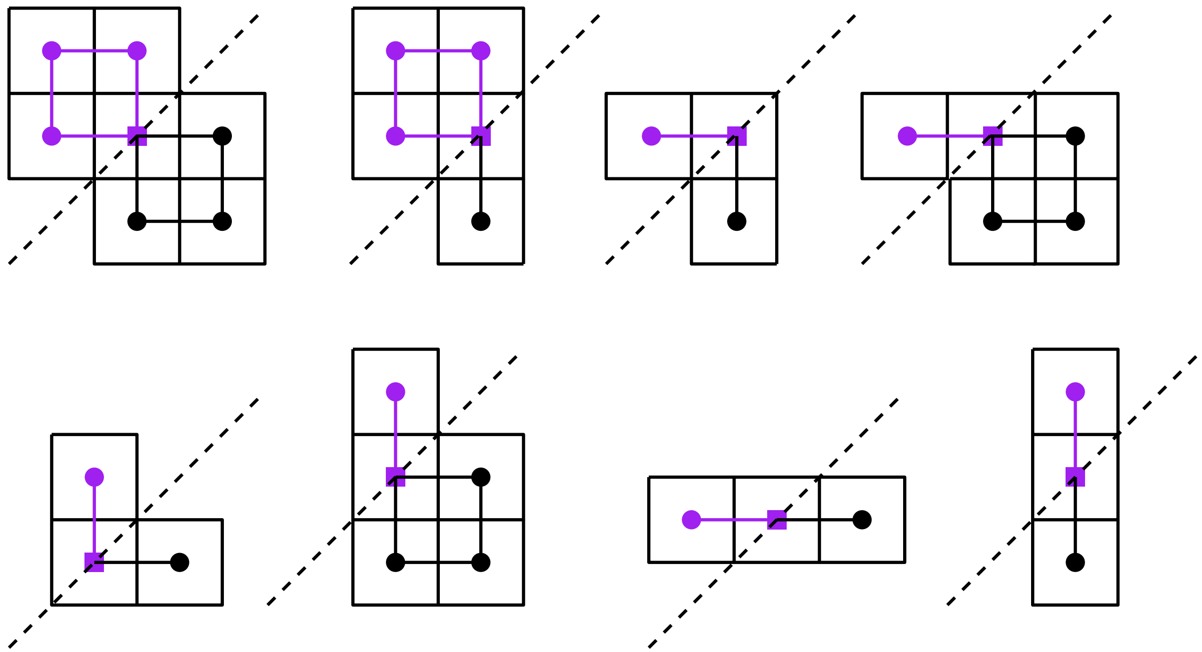




Special upper corners: □

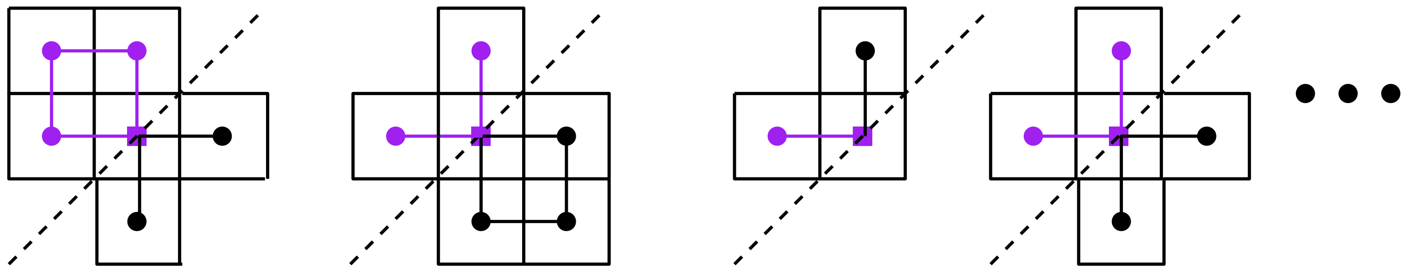
Special lower corners: ■

Outside thickened strip decompositions:

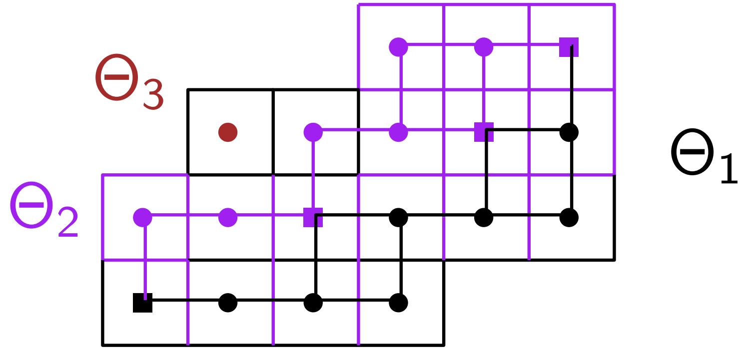


allowed common special corners

Non-outside thickened strip decomposition:

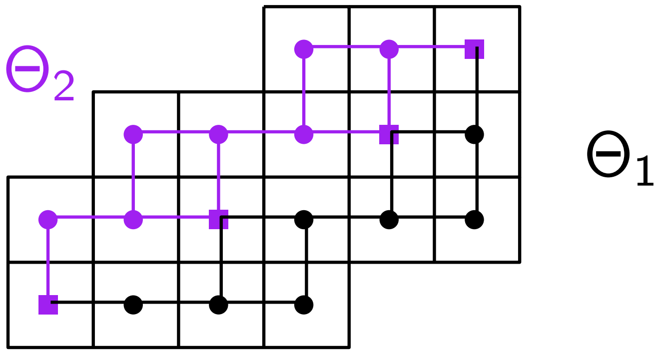


NOT allowed common special corners

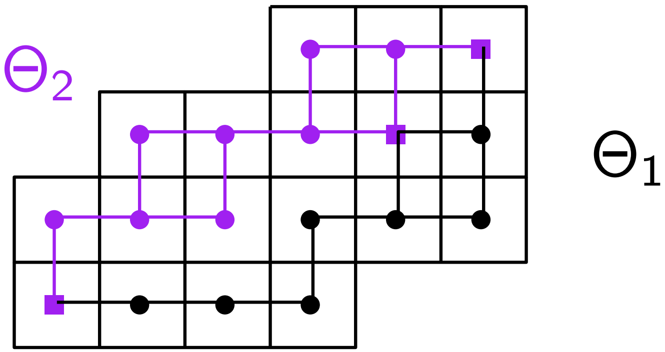


is not an outside thickened strip decomposition.

Outside nested decompositions:



Yes



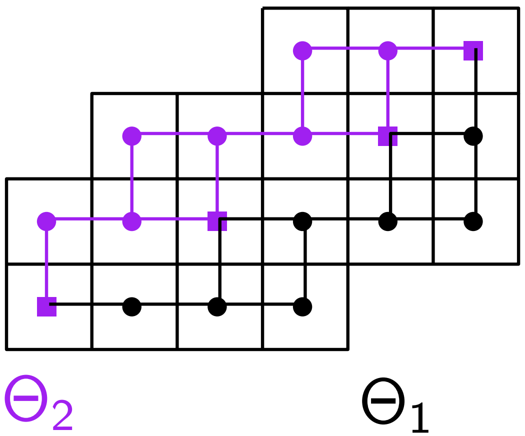
No

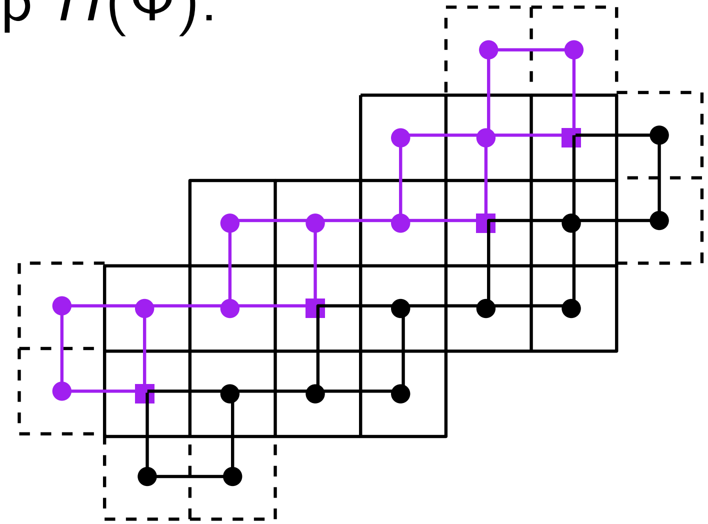
for all c, all boxes of content c all go up or all go right;

or all boxes of content c are all special corners;

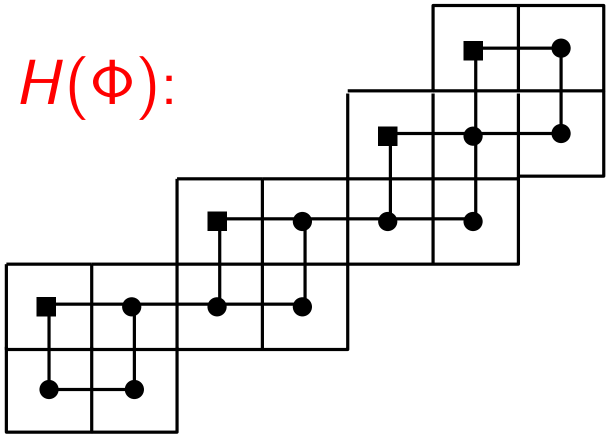
or all boxes of content $(c+1)$ are all special corners.

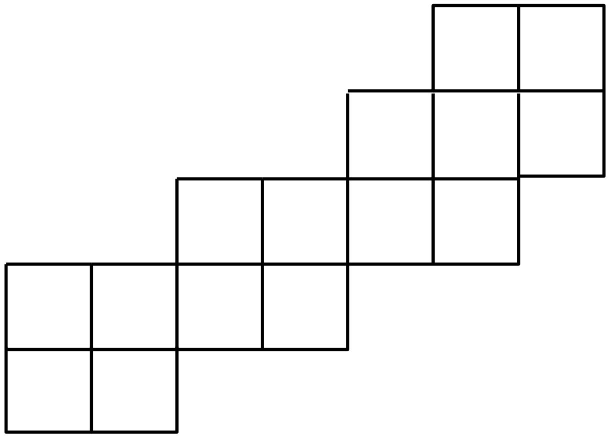
Thickened cutting strip $H(\Phi)$:



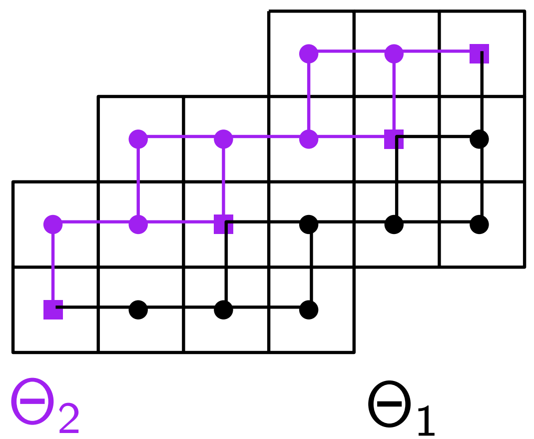


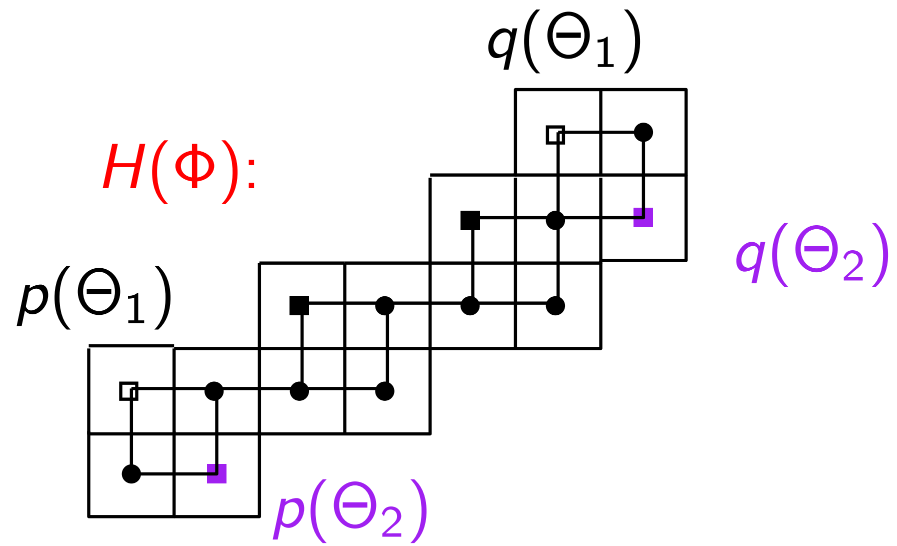
$\longrightarrow$ $H(\Phi)$:





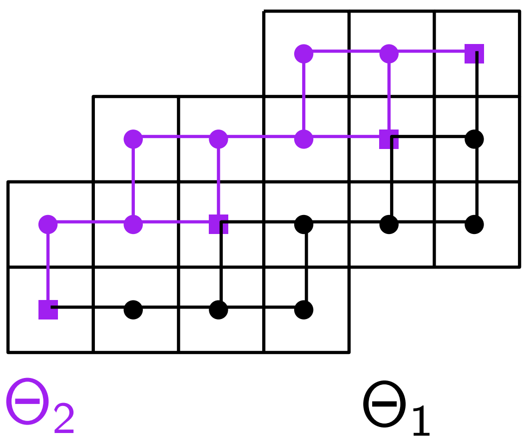
Define $\Theta_i \# \Theta_j = [p(\Theta_j), q(\Theta_i)]$

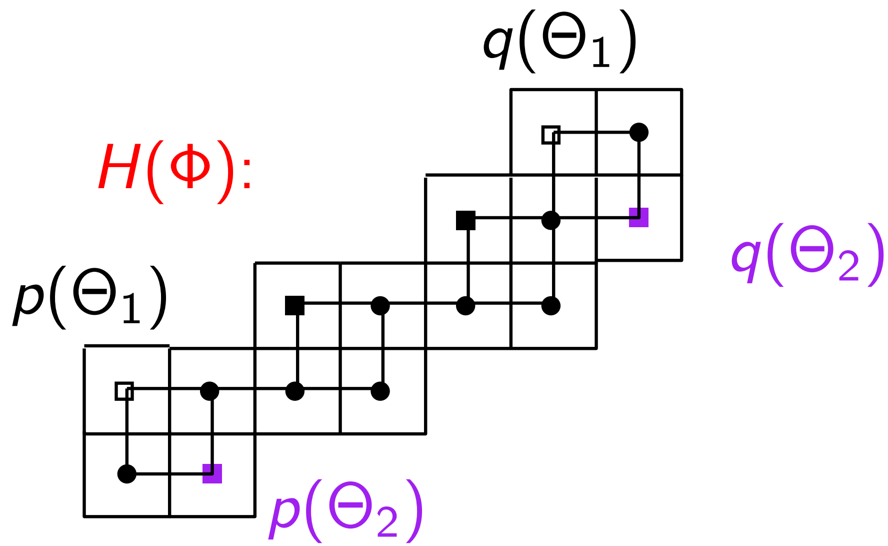




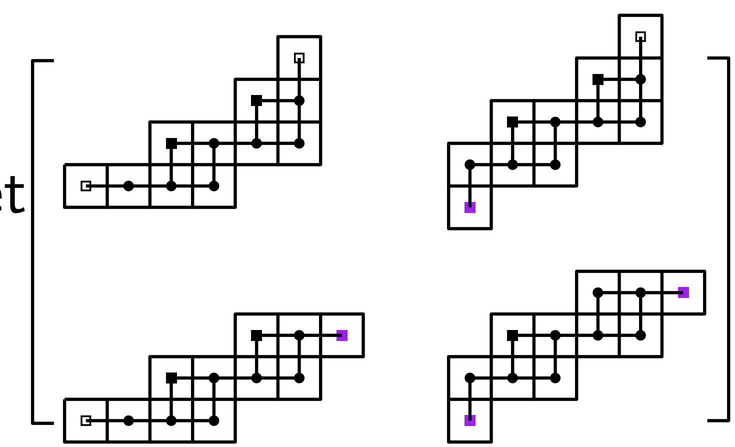
$$p_{1^4}(X) s_{(6,6,6,4)/(3,1)}(X) = \det \begin{bmatrix} s_{\Theta_1}(X) & s_{\Theta_1 \# \Theta_2}(X) \\ s_{\Theta_2 \# \Theta_1}(X) & s_{\Theta_2}(X) \end{bmatrix}$$

Define $\Theta_i \# \Theta_j = [p(\Theta_j), q(\Theta_i)]$

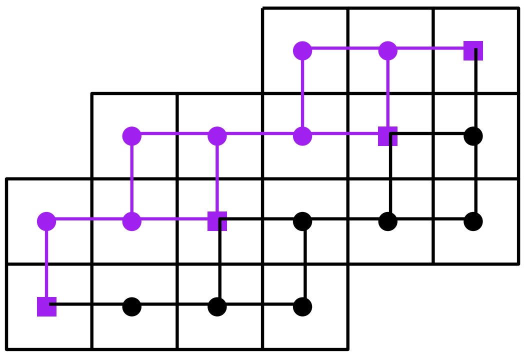


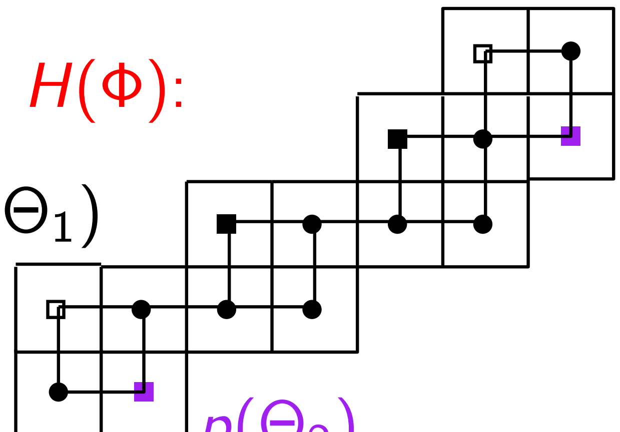


$p_{1^4}(X)s_{(6,6,6,4)/(3,1)}(X) = \det$

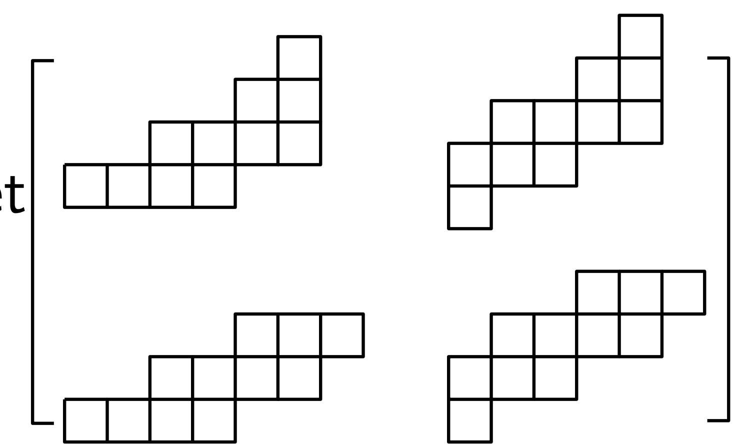


Define $\Theta_i \# \Theta_j = [p(\Theta_j), q(\Theta_i)]$

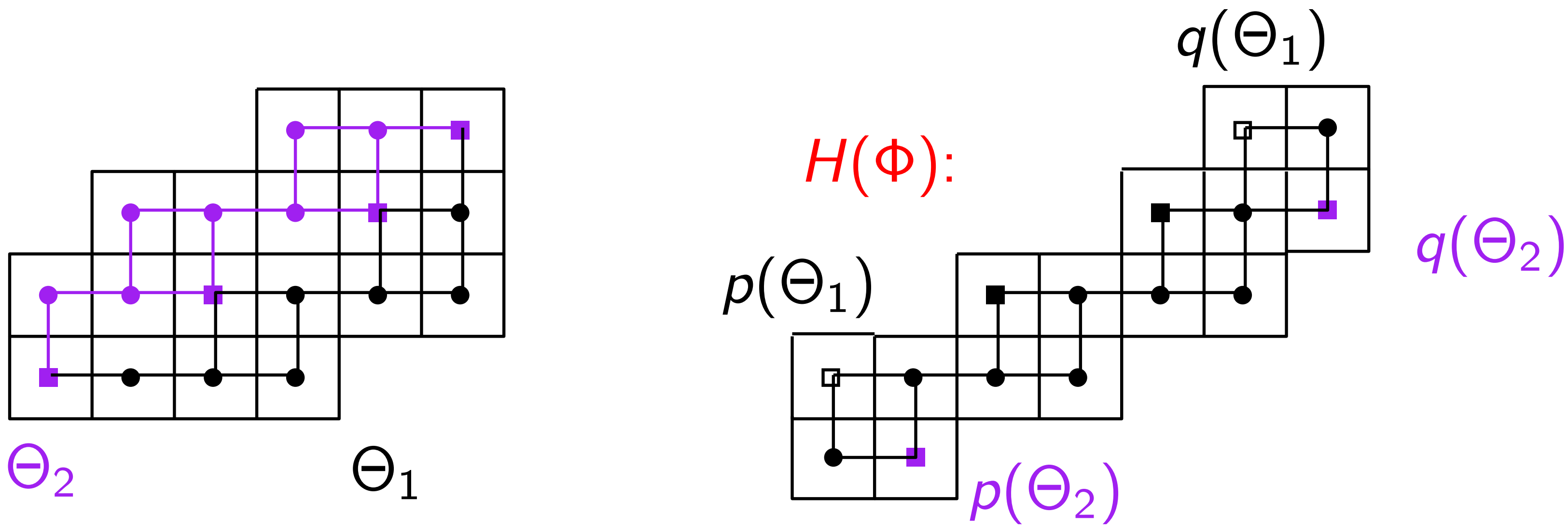




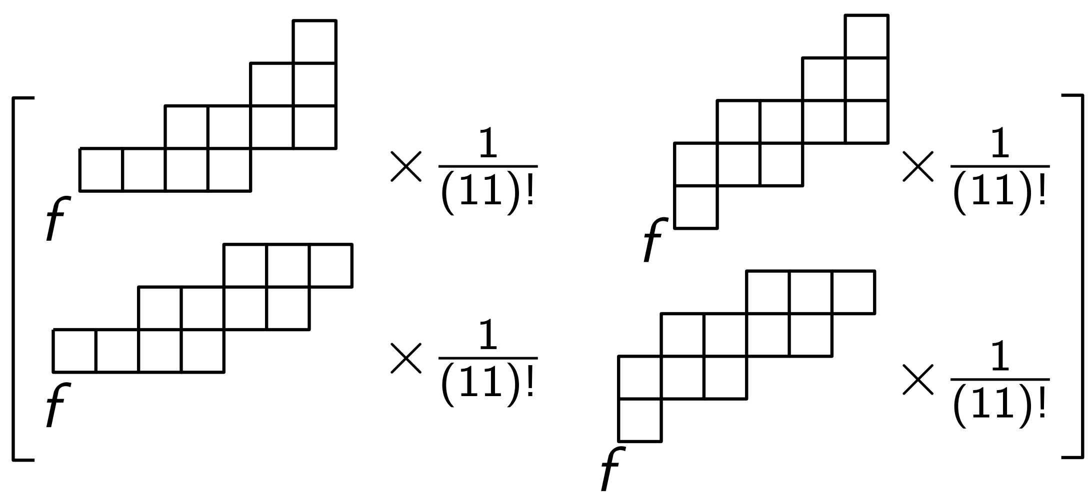
$p_{1^4}(X)s_{(6,6,6,4)/(3,1)}(X) = \det$



Define $\Theta_i \# \Theta_j = [p(\Theta_j), q(\Theta_i)]$



$f^{(6,6,6,4)/(3,1)} = (18)!\,\det$



Proof of the main results:

The proof consists of three main steps:

(1) SSYT $\rightarrow$ a sequence of non-crossing double lattice paths based on the bijection between SSYT and a sequence of non-intersecting lattice paths in Hamel and Goulden's paper.

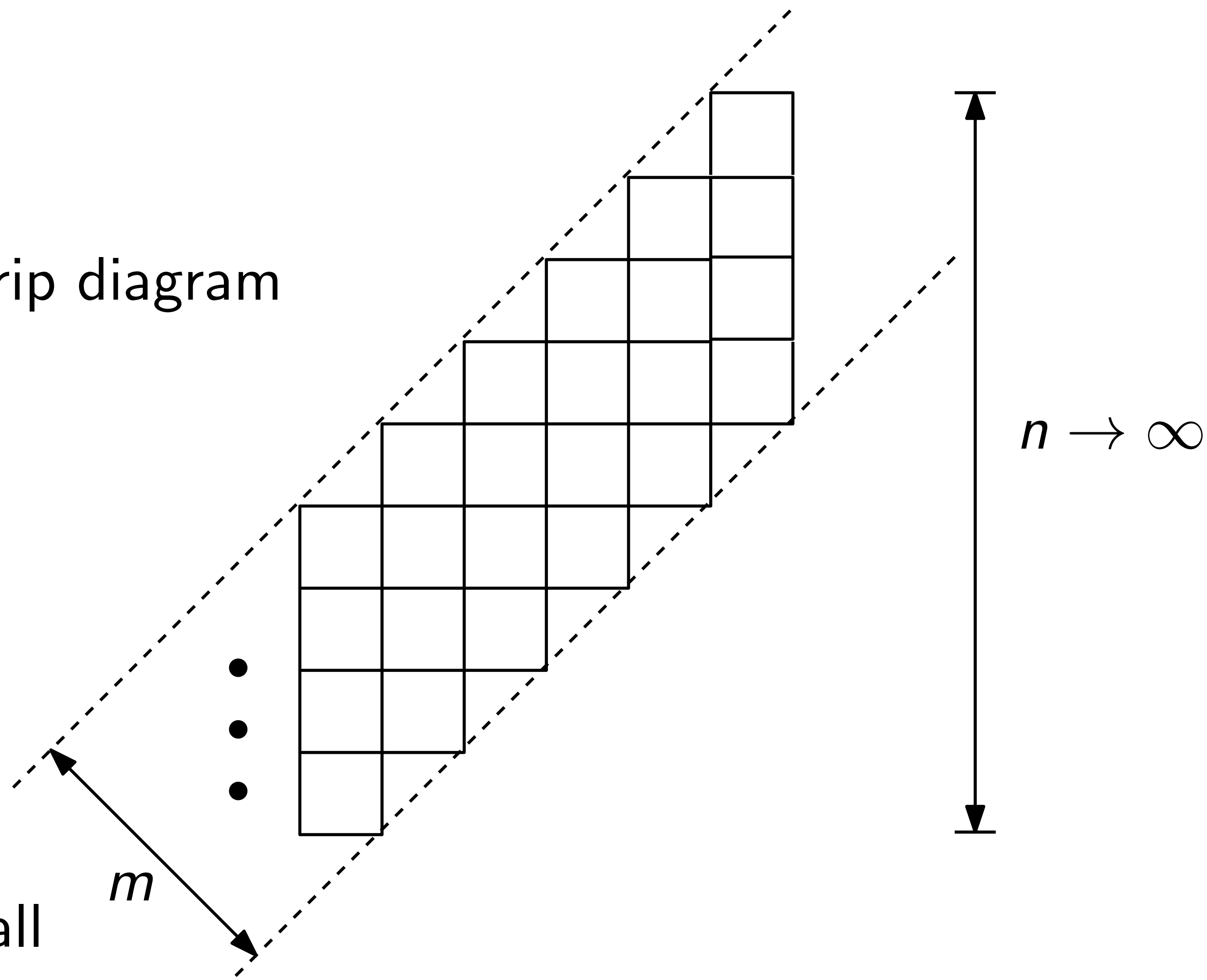
(2) Define a sequence of separable double lattice paths, whose generating function is $p_{1^r}(X)s_{\lambda/\mu}(X)$.

(3) Construct an involution on all non-separable sequences of double lattice paths, so that only the separable ones constribute the determinant $\det[s_{\Theta_i\#\Theta_j}(X)]$.

[Reference: J.R. Stembridge, Nonintersecting paths, pfaffians and plane partitions, Adv. Math., 83, 96-131, 1990.]

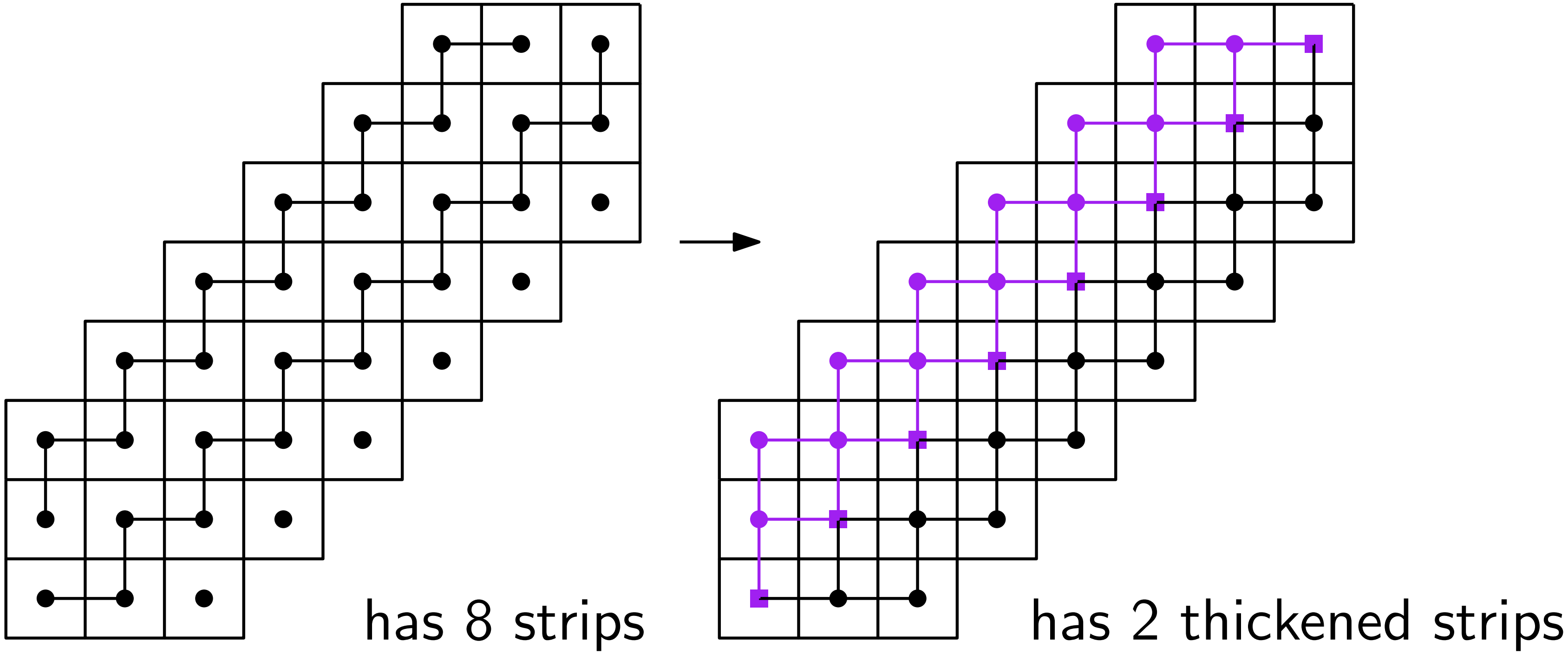
Applications

an m-strip diagram



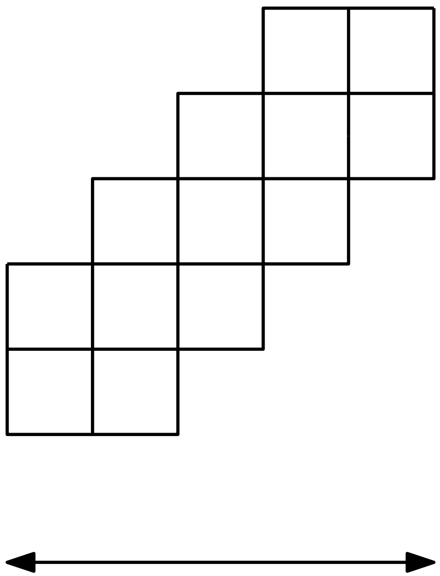
m is fixed
and small

Examples: a 5-strip diagram



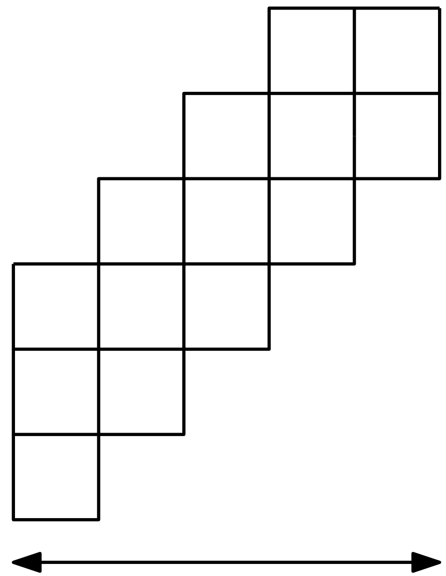
Counting 3-strip tableaux

$\mathcal{D}_{3n-2}$



n columns
$(3n-2)$ boxes

$\mathcal{D}^*_{3n-1}$

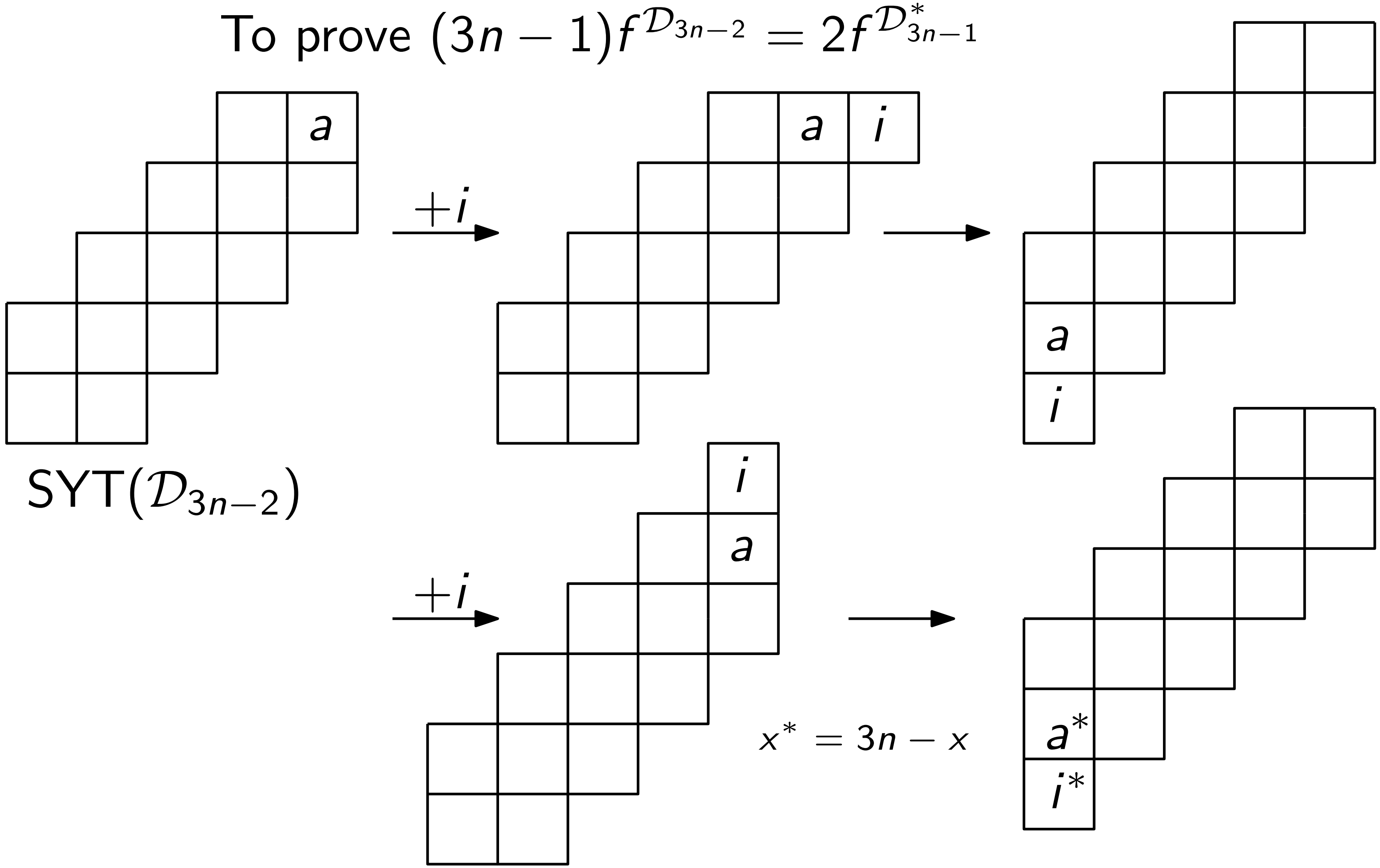


n columns
$(3n-1)$ boxes

To prove $(3n-1)f^{\mathcal{D}_{3n-2}} = 2f^{\mathcal{D}^*_{3n-1}}$

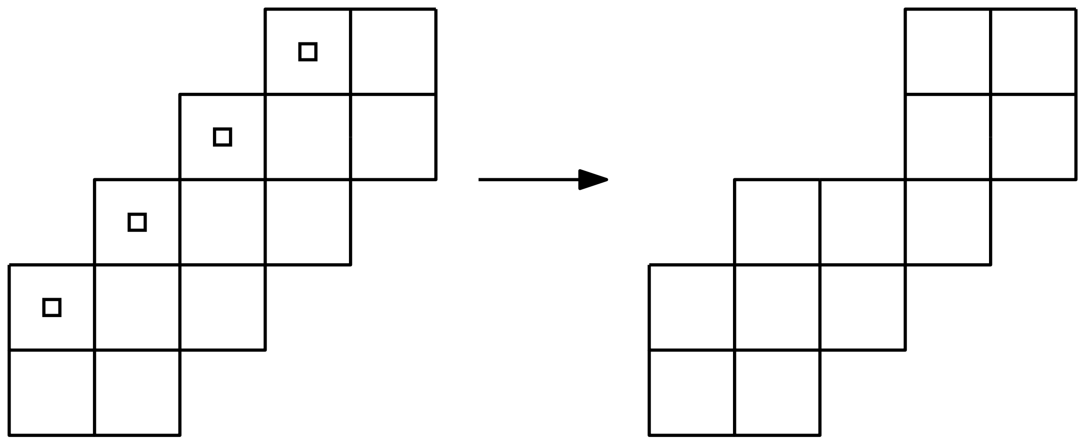
Counting 3-strip tableaux

To prove $(3n-1)f^{\mathcal{D}_{3n-2}} = 2f^{\mathcal{D}^*_{3n-1}}$



Counting 3-strip tableaux

$\mathcal{D}_{3n-2}$ $\qquad\qquad$ $\mathcal{D}_{3n-2,i}$



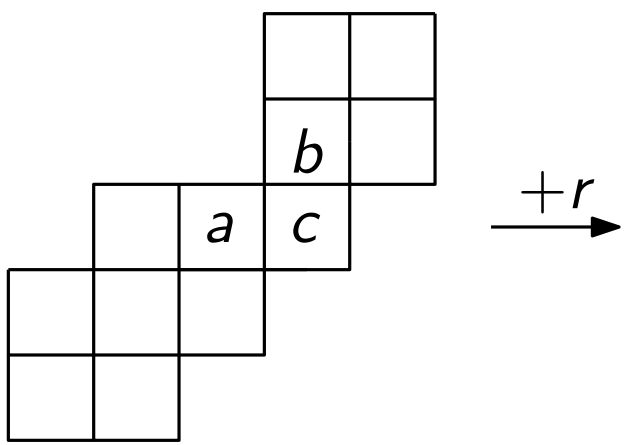
remove box $(i, n-i)$

$$f^{\mathcal{D}_{3n-2}} = \sum_{i=1}^{n-1} f^{\mathcal{D}_{3n-2,i}}.$$

To prove

$$(3n-2) f^{\mathcal{D}_{3n-2,i}} = f^{\mathcal{D}_{3n-2}} + \binom{3n-2}{3i-1} f^{\mathcal{D}^*_{3i-1}} f^{\mathcal{D}^*_{3n-3i-1}}.$$

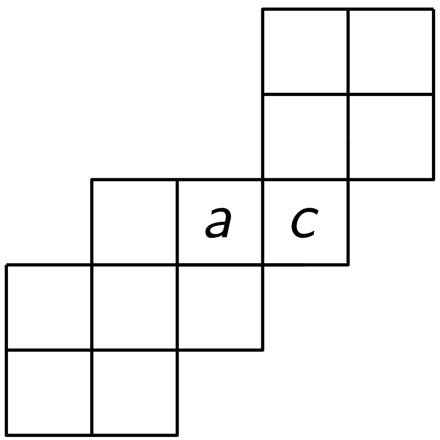
$$(3n-2)f^{\mathcal{D}_{3n-2,i}} = f^{\mathcal{D}_{3n-2}} + \binom{3n-2}{3i-1} f^{\mathcal{D}^*_{3i-1}} f^{\mathcal{D}^*_{3n-3i-1}}.$$



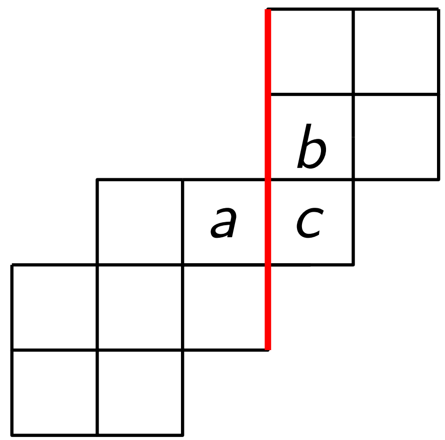
if $r < \min\{a, b\}$

$\mathrm{SYT}(\mathcal{D}_{3n-2,i})$

$$(3n-2)f^{\mathcal{D}_{3n-2,i}} = f^{\mathcal{D}_{3n-2}} + \binom{3n-2}{3i-1} f^{\mathcal{D}^*_{3i-1}} f^{\mathcal{D}^*_{3n-3i-1}}.$$

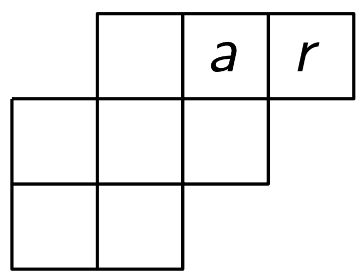


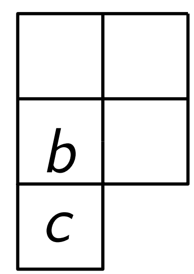
$\xrightarrow{+r}$



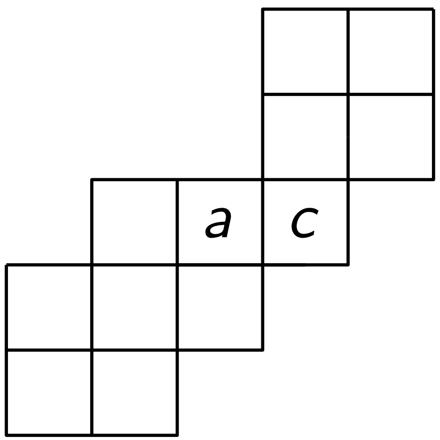
if $r > \min\{a, b\} = a$

$\mathrm{SYT}(\mathcal{D}_{3n-2,i})$ $=$

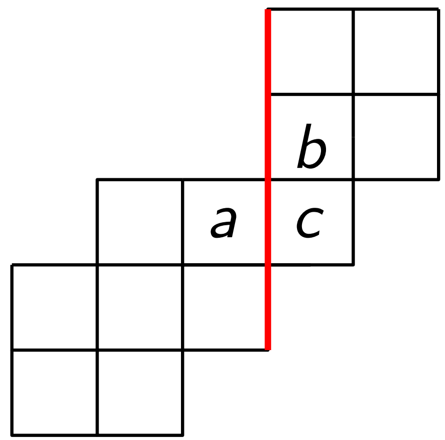




$$(3n-2)f^{\mathcal{D}_{3n-2,i}} = f^{\mathcal{D}_{3n-2}} + \binom{3n-2}{3i-1} f^{\mathcal{D}^*_{3i-1}} f^{\mathcal{D}^*_{3n-3i-1}}.$$

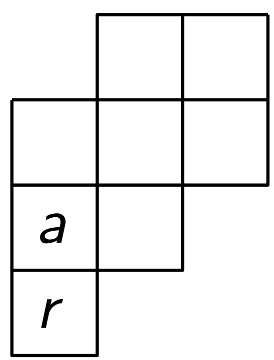


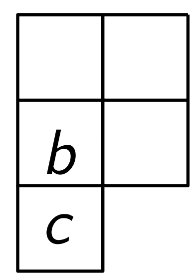
$\xrightarrow{+r}$



if $r > \min\{a, b\} = a$

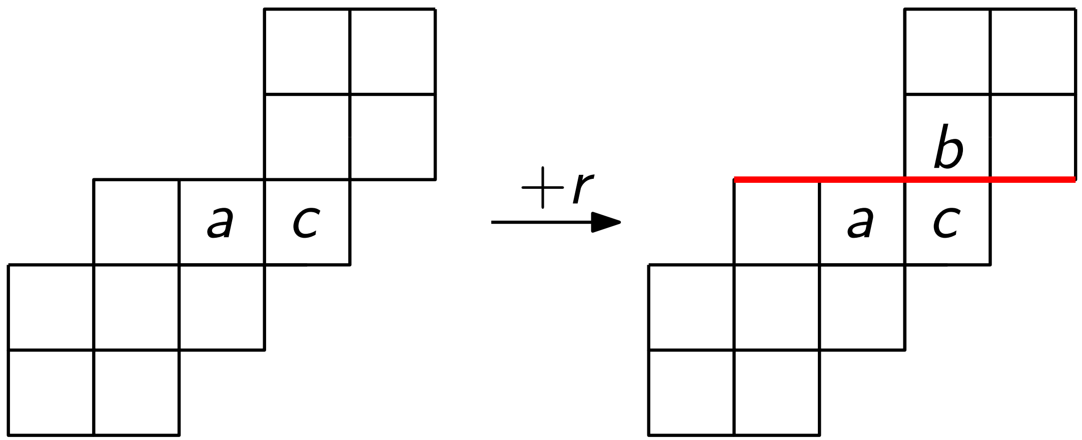
$\mathrm{SYT}(\mathcal{D}_{3n-2,i})$ $=$





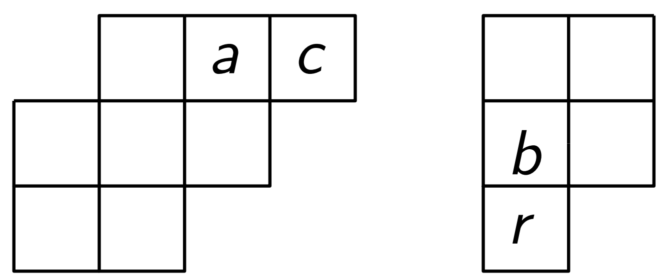
$\mathrm{SYT}(\mathcal{D}^*_{3n-3i-1})$ $\mathrm{SYT}(\mathcal{D}^*_{3i-1})$

$$(3n-2)f^{\mathcal{D}_{3n-2,i}} = f^{\mathcal{D}_{3n-2}} + \binom{3n-2}{3i-1} f^{\mathcal{D}^*_{3i-1}} f^{\mathcal{D}^*_{3n-3i-1}}.$$

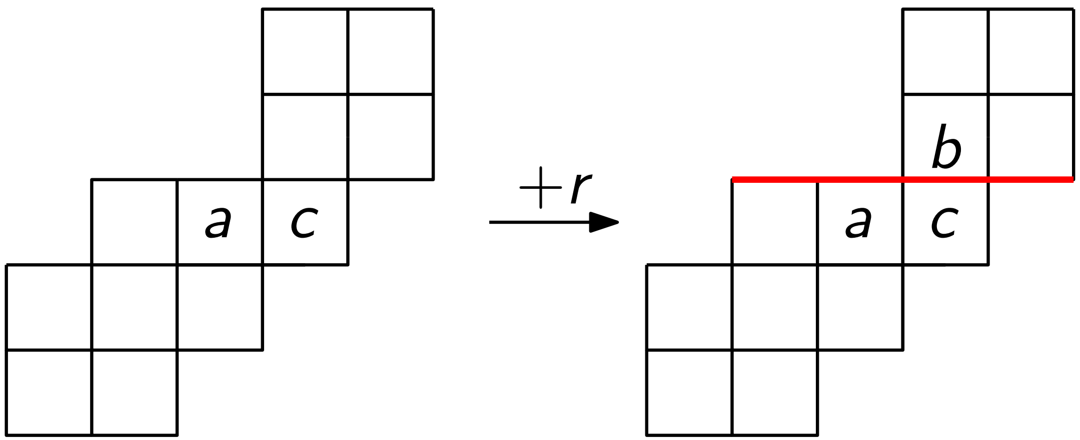


if $r > \min\{a, b\} = b$

$\mathrm{SYT}(\mathcal{D}_{3n-2,i})$ $=$

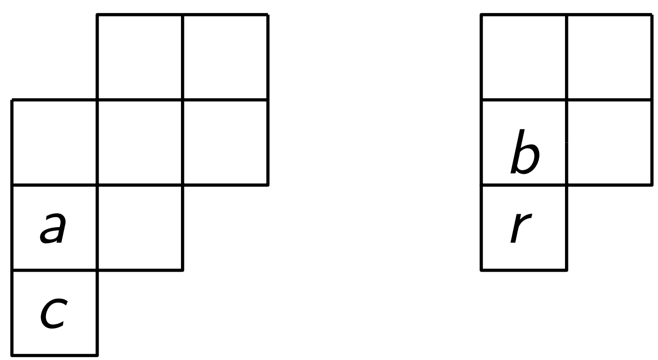


$$(3n-2)f^{\mathcal{D}_{3n-2,i}} = f^{\mathcal{D}_{3n-2}} + \binom{3n-2}{3i-1} f^{\mathcal{D}^*_{3i-1}} f^{\mathcal{D}^*_{3n-3i-1}}.$$



if $r > \min\{a, b\} = b$

$\mathrm{SYT}(\mathcal{D}_{3n-2,i})$ $=$



$\mathrm{SYT}(\mathcal{D}^*_{3n-3i-1})$ $\mathrm{SYT}(\mathcal{D}^*_{3i-1})$

$$(3n-1)f^{\mathcal{D}_{3n-2}} = 2f^{\mathcal{D}^*_{3n-1}}.$$

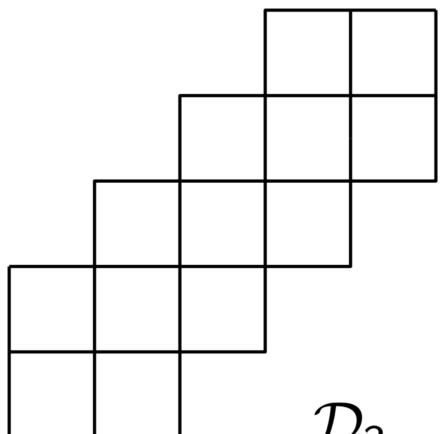
$$(3n-2)f^{\mathcal{D}_{3n-2,i}} = f^{\mathcal{D}_{3n-2}} + \binom{3n-2}{3i-1} f^{\mathcal{D}^*_{3i-1}} f^{\mathcal{D}^*_{3n-3i-1}}.$$

$$\overset{g.f.}{\Longrightarrow} \quad \begin{aligned} f(x) &= 2g(x), \\ f'(x) &= 1 + g(x)^2. \end{aligned} \quad \Longrightarrow \quad \begin{aligned} f(x) &= 2\tan(x/2), \\ g(x) &= \tan(x/2). \end{aligned}$$

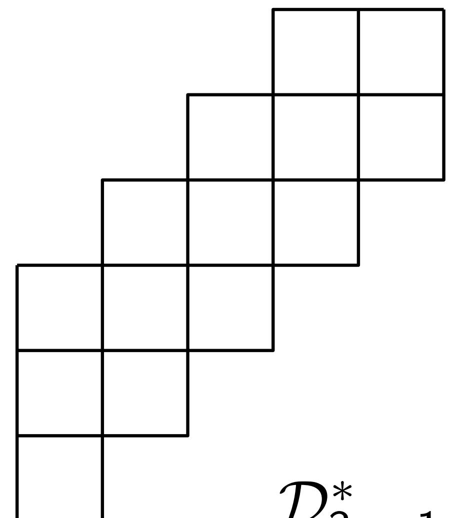
where

$$f(x) = \sum_{n\geq 1} \frac{f^{\mathcal{D}_{3n-2}}}{(3n-2)!} x^{2n-1}, \ g(x) = \sum_{n\geq 1} \frac{f^{\mathcal{D}^*_{3n-1}}}{(3n-1)!} x^{2n-1}.$$

Counting 3-strip tableaux



$\mathcal{D}_{3n-2}$



$\mathcal{D}^*_{3n-1}$

$$f^{\mathcal{D}_{3n-2}} = \frac{(3n-2)!E_{2n-1}}{(2n-1)!2^{2n-2}}, \quad f^{\mathcal{D}^*_{3n-1}} = \frac{(3n-1)!E_{2n-1}}{(2n-1)!2^{2n-1}}.$$

Summary:

Jacobi-Trudi determinant and its dual $s_{\lambda/\mu}(X)$,
Giambelli determinant $s_{\lambda}(X)$,
Lascoux and Pragacz determinant $s_{\lambda/\mu}(X)$,

unified by

Hamel and Goulden determinant $s_{\lambda/\mu}(X)$,
(outside decompositions)

generalized by

Jin, 2016, a determinant $p_{1^r}(X)s_{\lambda/\mu}(X)$,
(outside nested decompositions)

to count m-strip tableaux.

Vielen Dank!