Double regularization of polyzetas at negative multi-indices and polylogarithmic trans-series

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Introduction : from analytic continuation to double regularization of polyzetas at negative multi-indices

Polyzetas with complex multi-indices

$$\zeta(s) = \sum_{n\geq 1} n^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}dt}{e^t - 1}, \qquad s \in \mathbb{N}, s > 1.$$

Riemann extended $\zeta(s)$, on

$$\mathcal{H}_1 = \{s \in \mathbb{C} | \Re(s) > 1\},$$

as a meromorphic function. This series converges absolutely in $\mathcal{H}_1$.

In the same vein, for $r \in \mathbb{N}_+$, the polyzeta

$$\zeta(s_1, \ldots, s_r) = \sum_{n_1 > \ldots > n_r > 0} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}}, \qquad s_1, \ldots, s_r \in \mathbb{C}^r.$$

converges absolutely in

$$\mathcal{H}_r = \{(s_1, \ldots, s_r) \in \mathbb{C}^r | \forall m = 1, \ldots, r, \Re(s_1) + \ldots + \Re(s_m) > m\}.$$

After a theorem by Abel, it can be obtained as

$$\zeta(s_1, \ldots, s_r) = \lim_{z \to 1} \mathrm{Li}_{s_1, \ldots, s_r}(z) = \lim_{N \to +\infty} \mathrm{H}_{s_1, \ldots, s_r}(N),$$

where, for any $z \in \mathbb{C}, |z| < 1$ and $N \in \mathbb{N}$,

$$\mathrm{Li}_{s_1, \ldots, s_r}(z) = \sum_{n_1 > \ldots > n_r > 0} \frac{z^{n_1}}{n_1^{s_1} \ldots n_r^{s_r}}, \quad \mathrm{H}_{s_1, \ldots, s_r}(N) = \sum_{N \geq n_1 > \ldots > n_r > 0} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}}$$

which are well defined for $(s_1, \ldots, s_r) \in \mathbb{C}^r$.

To find a structure for $\{\zeta(s_1, \ldots, s_r)\}_{(s_1, \ldots, s_r) \in \mathcal{H}_r}$?

Encoding integer multi-indices by words

Let X^*, Y^* and Y_0^* be the free **monoids** (admitting $1_{X^*}, 1_{Y^*}$ and $1_{Y_0^*}$ as units) generated respectively by $X = \{x_0, x_1\}$, $Y = \{y_k\}_{k\geq 1}$ and $Y_0 = Y \cup \{y_0\}$.
Let $\mathcal{L}ynY_0, \mathcal{L}ynY$ and $\mathcal{L}ynX$ denote the sets of Lyndon words respectively over Y_0, Y and X, totally ordered by

$$y_0 > y_1 > y_2, \cdots \quad \text{and} \quad x_0 < x_1.$$

The **length** and the **weight** of $w = y_{s_1} \ldots y_{s_r} \in Y^*$ or Y_0^* (resp. $x_{s_1} \ldots x_{s_r} \in X^*$) are respectively $|w| = r$, for Y^* or Y_0^*, (resp. X^*) and $(w) = s_1 + \ldots + s_r$, for Y^* and Y_0^*.

- $(s_1, \ldots, s_r) \in \mathbb{N}^r \leftrightarrow y_{s_1} \ldots y_{s_r} \in Y_0^*$
- $(s_1, \ldots, s_r) \in \mathbb{N}_+^r \leftrightarrow y_{s_1} \ldots y_{s_r} \in Y^* \underset{\pi_Y}{\overset{\pi_X}{\rightleftharpoons}} x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1 \in X^*$

  For $s_1 > 1$, the associated **words** in $x_0X^*x_1$ or $(Y - \{y_1\})Y^*$ are said to be convergent.
  For $r \geq k \geq 1$, a divergent word is of the following form
  $$(\{1\}^k, s_{k+1}, \ldots, s_r) \leftrightarrow y_1^k y_{s_{k+1}} \ldots y_{s_r} \underset{\pi_Y}{\overset{\pi_X}{\rightleftharpoons}} x_1^k x_0^{s_{k+1}-1}x_1 \ldots x_0^{s_r-1}x_1.$$

Indexing polylogarithms and harmonic sums by words

For $(s_1,\ldots,s_r)\in\mathbb{N}_+^r$,

$$\begin{aligned}
\mathrm{Li}_{s_1,\ldots,s_r}(z) &= \mathrm{Li}_{y_{s_1}\ldots y_{s_r}}(z) &= \mathrm{Li}_{x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1}(z),\\
\mathrm{H}_{s_1,\ldots,s_r}(N) &= \mathrm{H}_{y_{s_1}\ldots y_{s_r}}(N) &= \mathrm{H}_{x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1}(N),\\
\zeta(s_1,\ldots,s_r) &= \zeta(y_{s_1}\ldots y_{s_r}) &= \zeta(x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1).
\end{aligned}$$

$\mathcal{Z}=\mathbb{Q}$-algebra generated by convergent polyzetas.

How can we extend the following isomorphisms of algebras

$$\begin{aligned}
\mathrm{Li}_\bullet:(\mathbb{C}\langle X\rangle,ш,1_{X^*}) &\longrightarrow (\mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*},.,1), & u\longmapsto \mathrm{Li}_u,\\
\mathrm{H}_\bullet:(\mathbb{C}\langle Y\rangle,\uplus,1_{Y^*}) &\longrightarrow (\mathbb{C}\{\mathrm{H}_w\}_{w\in Y^*},.,1), & u\longmapsto \mathrm{H}_u.
\end{aligned}$$

respectively over[1] $(\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle,ш,1_{X^*})$ and $(\mathbb{C}^{\mathrm{rat}}\langle\langle Y\rangle\rangle,\uplus,1_{Y^*})$?

For $(s_1,\ldots,s_r)\in\mathbb{N}^r$,

$$\begin{aligned}
\mathrm{Li}^-_{y_{s_1}\ldots y_{s_r}}(z) &:= \mathrm{Li}_{-s_1,\ldots,-s_r}(z) &= \sum_{n_1>\ldots>n_r>0} n_1^{s_1}\ldots n_r^{s_r}\, z^{n_1},\\
\mathrm{H}^-_{y_{s_1}\ldots y_{s_r}}(N) &:= \mathrm{H}_{-s_1,\ldots,-s_r}(N) &= \sum_{N\ge n_1>\ldots>n_r>0} n_1^{s_1}\ldots n_r^{s_r},\\
\zeta^-(y_{s_1}\ldots y_{s_r}) &:= \zeta(-s_1,\ldots,-s_r) &\leftrightarrow \sum_{n_1>\ldots>n_r>0} n_1^{s_1}\ldots n_r^{s_r}.
\end{aligned}$$

[1] $\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$ (resp. $\mathbb{C}^{\mathrm{rat}}\langle\langle Y\rangle\rangle$) the closure of $\mathbb{C}.X$ (resp. $\mathbb{C}.Y$) by scaling and $\{+,\mathrm{conc},{}^*\}$. They are also shuffle (resp. stuffle) closed.

Continuity over Chen series

Continuity and growth condition (1/2)

For $i = 1, 2$, let $(\mathbb{K}_i, \|.\|_i)$ be a semi-normed space and $g_i \in \mathbb{Z}$.

Definition (continuity and indiscernability, HNM, 1990)

Let $\mathcal{C}l$ be a class of $\mathbb{K}_1\langle\langle X\rangle\rangle$ ($\mathcal{C}l \subset \mathbb{K}_1\langle\langle X\rangle\rangle$) and $S \in \mathbb{K}_2\langle\langle X\rangle\rangle$.

1. S is said to be *continuous* over $\mathcal{C}l$ if, for any $\Phi \in \mathcal{C}l$, the sum $\sum_{w\in X^*} \|\langle S|w\rangle\|_2 \|\langle \Phi|w\rangle\|_1$ is convergent.
   We will denote $\langle S \parallel \Phi\rangle$ the sum $\sum_{w\in X^*}\langle S|w\rangle\langle\Phi|w\rangle$.
   $\mathbb{K}_2\langle\langle X\rangle\rangle^{\mathrm{cont}} \equiv$ set of continuous power series over $\mathcal{C}l$.
2. S is said to be *indiscernable* over $\mathcal{C}l$ iff, for any $\Phi \in \mathcal{C}l$, $\langle S \parallel \Phi\rangle = 0$.

Definition (growth condition, HNM, 1990)

Let χ_1 and χ_2 be real positive morphisms over X^*. Let $S \in \mathbb{K}_1\langle\langle X\rangle\rangle$.

1. S satisfies the χ_1*−growth condition* of order g_1 if it satisfies
   $$\exists K \in \mathbb{R}_+, \exists n \in \mathbb{N}, \forall w \in X^{\geq n}, \quad \|\langle S|w\rangle\|_1 \leq K\chi_1(w)\ |w|!^{g_1}.$$
   We denote by $\mathbb{K}_1^{(\chi_1,g_1)}\langle\langle X\rangle\rangle$ the set of formal power series in $\mathbb{K}_1\langle\langle X\rangle\rangle$ satisfying the χ_1−growth condition of order g_1.
2. If S is continuous over $\mathbb{K}_2^{(\chi_2,g_2)}\langle\langle X\rangle\rangle$ then it will be said to be (χ_2, g_2)-*continuous*. The set of formal power series which are (χ_2, g_2)-continuous is denoted by $\mathbb{K}_2^{(\chi_2,g_2)}\langle\langle X\rangle\rangle^{\mathrm{cont}}$.

Continuity and growth condition (2/2)

Lemma

Let χ_1 and χ_2 be real positive functions over X^*.
Let g_1 and $g_2 \in \mathbb{Z}$ such that $g_1 + g_2 \leq 0$.

1. *Let $S \in \mathbb{K}_1^{(\chi_1,g_1)}\langle\langle X \rangle\rangle$ and let $P \in \mathbb{K}_1\langle X \rangle$.*
   The right residual of S by P belongs to $\mathbb{K}_1^{(\chi_1,g_1)}\langle\langle X \rangle\rangle$.
2. *Let $R \in \mathbb{K}_2^{(\chi_2,g_2)}\langle\langle X \rangle\rangle$ and let $Q \in \mathbb{K}_2\langle X \rangle$.*
   The concatenation QR belongs to $\mathbb{K}_2^{(\chi_2,g_2)}\langle\langle X \rangle\rangle$.
3. *χ_1, χ_2 are morphisms over X^* satisfying $\sum_{x \in X} \chi_1(x)\chi_2(x) < 1$.*
   If $F_1 \in \mathbb{K}_1^{(\chi_1,g_1)}\langle\langle X \rangle\rangle$ (resp. $F_2 \in \mathbb{K}_2^{(\chi_2,g_2)}\langle\langle X \rangle\rangle$) then F_1 (resp. F_2) is continuous over $\mathbb{K}_2^{(\chi_2,g_2)}\langle\langle X \rangle\rangle$ (resp. $\mathbb{K}_1^{(\chi_1,g_1)}\langle\langle X \rangle\rangle$).

Lemma

Let $\mathcal{C}l \subset \mathbb{K}_1\langle\langle X \rangle\rangle$ be a monoid containing $\{e^{tx}\}_{x \in X}^{t \in \mathbb{K}_1}$.
Let $S \in \mathbb{K}_2\langle\langle X \rangle\rangle^{cont}$.

1. *If S is indiscernable over $\mathcal{C}l$ then for any $x \in X$, $x \triangleleft S$ and $S \triangleright x$ belong to $\mathbb{K}_2\langle\langle X \rangle\rangle^{cont}$ and they are indiscernable over $\mathcal{C}l$.*
2. *S is indiscernable over $\mathcal{C}l$ if and only if $S = 0$.*

Iterated integrals and Chen series

Ω : simply connected domain in $\mathbb{C}$. $\quad X = \{x_i\}_{i=0,..,m} \leftrightarrow \{\omega_i\}_{i=0,..,m}$.
An iterated integral associated to $w = x_{i_1} \cdots x_{i_k} \in X^*$, with respect to $\{\omega_i\}_{i=0,..,m}$ along the path $z_0 \rightsquigarrow z$ in Ω, is defined by $\alpha_{z_0}^z(1_{X^*}) = 1_\Omega$ and

$$\alpha_{z_0}^z(w) = \int_{z_0 \rightsquigarrow z} \omega_{i_1}(s)\alpha_0^z(x_{i_2} \ldots x_{i_k}) = \int_{z_0}^{z} \omega_{i_1}(z_1) \ldots \int_{z_0}^{z_{k-1}} \omega_{i_k}(z_k),$$

where $(z_0, z_1 \ldots, z_k, z)$ is a subdivision of $z_0 \rightsquigarrow z$.
For any $u, v \in X^*$, $\alpha_{z_0}^z(u ш v) = \alpha_{z_0}^z(u)\alpha_{z_0}^z(v)$ (**Chen**'s lemma, 1954).
For any $R, S \in \mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$, one has (subjected to be convergent)

$$\alpha_{z_0}^z(R) = \sum_{w \in X^*} \langle R|w\rangle \alpha_{z_0}^z(w), \qquad \alpha_{z_0}^z(R ш S) = \alpha_{z_0}^z(R)\alpha_{z_0}^z(S).$$

Chen series[2] : $\quad C_{z_0 \rightsquigarrow z} = \sum_{w \in X^*} \alpha_{z_0}^z(w) w = \prod_{l \in \mathcal{L}ynX} \exp(\alpha_{z_0}^z(S_l)P_l).$

$$(DE) \quad dC_{z_0 \rightsquigarrow z} = M(z)C_{z_0 \rightsquigarrow z}, \text{ where}^3\ M(z) = (\omega_0(z)x_0 + \ldots + \omega_m(z)x_m).$$

Lemma

Let $R \in \mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$ of linear representation (ρ, μ, ν) of dimension n. Then, for any $w \in X^+$, one has $|\langle R|w\rangle| \leq n^2 \|\rho\|_\infty^{n,1} \|\mu(w)\|_\infty^{n,n} \|\nu\|_\infty^{1,n}$.

[2] $\Delta_ш C_{z_0 \rightsquigarrow z} = C_{z_0 \rightsquigarrow z} \otimes C_{z_0 \rightsquigarrow z}$.
[3] $\Delta_ш M(z) = M(z) \otimes 1_{X^*} + 1_{X^*} \otimes M(z)$.

Set of exchangeable power series, $\mathbb{C}_{\mathrm{exc}}\langle\langle X\rangle\rangle$

Here, $X=\{x_0,\cdots,x_m\}$. The power series S belongs to $\mathbb{C}_{\mathrm{exc}}\langle\langle X\rangle\rangle$, iff $(\forall u,v\in X^*,x\in X,|u|_x=|v|_x)\Rightarrow(\langle S|u\rangle=\langle S|v\rangle)$,

$$S=\sum_{i_0,\ldots,i_m\geq 0}s_{i_0,\ldots,i_m}x_0^{i_0}ш\ldots шx_m^{i_m}\Rightarrow\alpha_{z_0}^z(S)=\sum_{i_0,\ldots,i_m\geq 0}s_{i_0,\ldots,i_m}\frac{\alpha_{z_0}^z(x_0)^{i_0}\ldots\alpha_{z_0}^z(x_m)^{i_m}}{i_0!\ldots i_m!}$$

(**Fliess**, 1972).

Let $\mathbb{C}_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle X\rangle\rangle:=\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle\bigcap\mathbb{C}_{\mathrm{exc}}\langle\langle X\rangle\rangle$. It is shuffle closed.

Lemma

1. $\mathbb{C}_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle X\rangle\rangle=\mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle ш\ldots ш\mathbb{C}^{\mathrm{rat}}\langle\langle x_m\rangle\rangle$.
2. *For any $x\in X$, one has*[4] $\mathbb{C}^{\mathrm{rat}}\langle\langle x\rangle\rangle=\mathrm{span}_{\mathbb{C}}\{(ax)^*ш\mathbb{C}\langle x\rangle|a\in\mathbb{C}\}$.
3. *The family $\{x_0^*,\ldots,x_m^*\}$ is algebraically independent over $(\mathbb{C}\langle X\rangle,ш,1_{X^*})$ within $(\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle,ш,1_{X^*})$.*

In particular, for $X=\{x_0,x_1\}$, the module $(\mathbb{C}\langle X\rangle,ш,1_{X^*})[x_0^*,x_1^*,(-x_0)^*]$ is $\mathbb{C}\langle X\rangle$-free and $\{(x_0^*)^{шk}ш(x_1^*)^{шl}\}^{(k,l)\in\mathbb{Z}\times\mathbb{N}}$ forms a $\mathbb{C}\langle X\rangle$-basis of it. Hence, $\{wш(x_0^*)^{шk}ш(x_1^*)^{шl}\}_{w\in X^*}^{(k,l)\in\mathbb{Z}\times\mathbb{N}}$ is a $\mathbb{C}$-basis of it.

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[4] $\Delta_ш(ax)^*=(ax)^*\otimes(ax)^*$.

Iterated integrals and power series over $X = \{x_0, x_1\}$

$\Omega := \mathbb{C} - (]-\infty, 0] \cup [1, +\infty[)$ and $\omega_0(z) := dz/z, \omega_1(z) := dz/(1-z)$.
Let $\gamma_0(\varepsilon)$ and $\gamma_1(\varepsilon)$ be the circular paths of radius ε encircling 0 and 1 clockwise, respectively. In particular, $\gamma_0(\varepsilon, \beta) = \varepsilon e^{\mathrm{i}\beta_0} \rightsquigarrow \varepsilon e^{\mathrm{i}\beta_1} \subset \gamma_0(\varepsilon)$ and $\gamma_1(\varepsilon, \beta) = 1 - \varepsilon e^{\mathrm{i}\beta_0} \rightsquigarrow 1 - \varepsilon e^{\mathrm{i}\beta_1} \subset \gamma_1(\varepsilon)$, where $\beta = \beta_1 - \beta_0$.
Setting $z = \varepsilon e^{\mathrm{i}\beta}$ and using (DE), one has

$$\frac{dC_{\gamma_0(\varepsilon,\beta)}}{d\beta} = \mathrm{i}\left(x_0 + \frac{\varepsilon e^{\mathrm{i}\beta} x_1}{1 - e^{\mathrm{i}\beta}}\right) C_{\gamma_0(\varepsilon,\beta)}, \frac{dC_{\gamma_1(\varepsilon,\beta)}}{d\beta} = -\mathrm{i}\left(\frac{\varepsilon e^{\mathrm{i}\beta} x_0}{1 - \varepsilon e^{\mathrm{i}\beta}} + x_1\right) C_{\gamma_1(\varepsilon,\beta)}.$$

Lemma

For any $i = 0, 1$ and $w \in X^+$, one has $|\langle C_{\gamma_i(\varepsilon,\beta)} | w\rangle| \leq \varepsilon^{|w|_{x_i}} \beta^{|w|} |w|!^{-1}$.
Therefore, $C_{\gamma_i(\varepsilon,\beta)} = e^{\mathrm{i}\beta x_i} + o(\varepsilon)$ and $C_{\gamma_i(\varepsilon)} = e^{2\mathrm{i}\pi x_i} + o(\varepsilon)$.

Theorem (HNM, 1995)

Let $R \in \mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$ of linear representation (ρ, μ, ν). Then

$$\alpha_{z_0}^z(R) = \langle R \parallel C_{z_0 \rightsquigarrow z}\rangle = \rho\left(\prod_{l \in \mathcal{L}ynX} \exp(\alpha_{z_0}^z(S_l)\mu(P_l))\right)\nu.$$

Example

$$\begin{aligned}
\alpha_{z_0}^z(\mathbb{C}[x_0^*, x_1^*, (-x_0)^*]) &\subset \mathrm{span}_{\mathbb{C}}\{z^k(1-z)^{-l}\}^{(k,l)\in\mathbb{Z}\times\mathbb{N}} =: \mathcal{C},\\
\alpha_{z_0}^z(\mathbb{C}[x_0^*, x_1^*, (-x_0)^*]\text{Ш}\mathbb{C}\langle x_1\rangle) &\subset \mathrm{span}_{\mathbb{C}}\{z^k(1-z)^{-l}\log^m((1-z)^{-1})\}_{m\in\mathbb{N}}^{(k,l)\in\mathbb{Z}\times\mathbb{N}},\\
\alpha_{z_0}^z(\mathbb{C}[x_0^*, x_1^*, (-x_0)^*]\text{Ш}\mathbb{C}\langle X\rangle) &\subset \mathrm{span}_{\mathbb{C}}\{z^k(1-z)^{-l}\mathrm{Li}_w(z)\}_{w\in X^*}^{(k,l)\in\mathbb{Z}\times\mathbb{N}}.
\end{aligned}$$

Double regularization of polyzetas at negative multi-indices

First global renormalization of divergent polyzetas

$$\mathrm{L} := \sum_{w\in X^*} \mathrm{Li}_w\, w = \prod_{l\in\mathcal{L}ynX} e^{\mathrm{Li}_{S_l}\, P_l}, \qquad Z_{ш} := \prod_{l\in\mathcal{L}ynX-X} e^{\zeta(S_l)P_l}.$$

$$\mathrm{H} := \sum_{w\in Y^*} \mathrm{H}_w w = \prod_{l\in\mathcal{L}ynY} e^{\mathrm{H}_{\Sigma_l}\Pi_l}, \qquad Z_{⧢} := \prod_{l\in\mathcal{L}ynY-\{y_1\}} e^{\zeta(\Sigma_l)\Pi_l}.$$

L, $Z_{ш}$ are group-like, for $\Delta_{ш}$, and H, $Z_{⧢}$ are group-like, for $\Delta_{⧢}$.

Theorem (Abel like theorem, HNM, 2005)

$$\lim_{z\to 1} \exp\left[-y_1 \log\frac{1}{1-z}\right]\pi_Y \mathrm{L}(z) = \lim_{N\to\infty} \exp\left[\sum_{k\geq 1} \mathrm{H}_{y_k}(N)\frac{(-y_1)^k}{k}\right]\mathrm{H}(N) = \pi_Y Z_{ш}.$$

For any $w \in Y^*$, there exists $a_i, b_{i,j} \in \mathcal{Z}$ and $\gamma_w, \alpha_i, \beta_{i,j} \in \mathcal{Z}[\gamma]$ such that

(**Costermans, Enjalbert, HNM**, 2004)

$$\mathrm{Li}_{\pi_X w}(z) \underset{z\to 1}{\asymp} \sum_{i=1}^{(w)} a_i \log^i(1-z) + \langle Z_{ш}|\pi_X w\rangle + \sum_{i\in\mathbb{N}_+, j\in\mathbb{N}_-} b_{i,j}\frac{\log^i(1-z)}{(1-z)^j},$$

$$\text{and } \mathrm{H}_w(N) \underset{N\to+\infty}{\asymp} \sum_{i=1}^{|w|} \alpha_i \log^i(N) + \gamma_w + \sum_{i,j\in\mathbb{N}_+} \beta_{i,j}\frac{\log^i(N)}{N^j}.$$

Example

$$\begin{aligned}
\mathrm{Li}_{2,1}(z) &= \zeta(3) + (1-z)\log(1-z) - 1 - \tfrac{1}{2}(1-z)\log^2(1-z) + (1-z)^2[-\tfrac{1}{4}\log^2(1-z) + \tfrac{1}{4}\log(1-z)] + \ldots,\\
\mathrm{H}_{2,1}(N) &= \zeta(3) - (\log(N) + 1 + \gamma)/N + \log(N)/2N + \ldots,\\
\mathrm{Li}_{1,2}(z) &= 2 - 2\zeta(3) - \zeta(2)\log(1-z) - 2(1-z)\log(1-z) + (1-z)\log^2\tfrac{1}{1-z} + (1-z)^2[\tfrac{\log^2(1-z)}{2} - \tfrac{\log(1-z)}{2}] + \ldots,\\
\mathrm{H}_{1,2}(N) &= \zeta(2)\gamma - 2\zeta(3) + \zeta(2)\log(N) + (\zeta(2)+2)/2N + \ldots,\\
\zeta(2)\gamma &= .949481711114981524545564102231704933640005949473666\ldots
\end{aligned}$$

Euler-Mac Laurin constants associated to $\{\zeta(w)\}_{w\in y_1Y^*}$

The map $\gamma_\bullet : (\mathbb{C}\langle Y\rangle, ⧢, 1_{Y^*}) \to (\mathbb{C}, ., 1)$ is a character and its graph

$$Z_\gamma := \sum_{w\in Y^*} \gamma_w w,$$

as generating series, via ⧢-extended Friedrichs criterion, is group-like. Abel like theorem and ⧢-extended Schützenberger factorization yield

$$Z_\gamma = B(y_1)\pi_Y Z_{ш}, \quad \text{or equivalently,} \quad Z_{⧢} = B'(y_1)\pi_Y Z_{ш} \quad (\mathbf{HNM}, 2009),$$

where

$$B(y_1) = \exp\left[\gamma y_1 - \sum_{k\geq 2}\zeta(k)\frac{(-y_1)^k}{k}\right] \text{ and } B'(y_1) = \exp\left[-\sum_{k\geq 2}\zeta(k)\frac{(-y_1)^k}{k}\right].$$

Hence, for $k \in \mathbb{N}_+, w \in Y^+$, one has (**Costermans, HNM**, 2005)

$$\langle Z_\gamma | y_1^k\rangle = \sum_{s_1,\ldots,s_k>0, s_1+\ldots+ks_k=k} \frac{(-1)^k}{s_1!\ldots s_k!}(-\gamma)^{s_1}\left(-\frac{\zeta(2)}{2}\right)^{s_2}\ldots\left(-\frac{\zeta(k)}{k}\right)^{s_k},$$

$$\langle Z_\gamma | y_1^k w\rangle = \sum_{i=0}^{k} \frac{\zeta(x_0[(-x_1)^{k-i} ш \pi_X w])}{i!}\left(\sum_{j=1}^{i} B_{i,j}(\gamma, -\zeta(2), 2\zeta(3), \ldots)\right),$$

where the $B_{i,j}$'s are Bell polynomials.

$\{\mathrm{Li}^-_w\}_{w\in Y_0^*}$ and $\{\mathrm{H}^-_w\}_{w\in Y_0^*}$ as polynomials

Theorem (Bùi, Duchamp, HNM, Ngô, 2014)

1. $\{\mathrm{Li}^-_{y_k}\}_{k\ge 0}$ *(resp.* $\{\mathrm{H}^-_{y_k}\}_{k\ge 0}$*) are linearly independent.*

2. *For* $w \in Y_0^*$, $\mathrm{Li}^-_w(z) \in \mathbb{Z}[(1-z)^{-1}] \subsetneq \mathcal{C}$ *and* $\mathrm{H}^-_w(N) \in \mathbb{Q}[N]$ *of degree* $|w| + (w)$*. Hence, there exists* $B^-_w \in \mathbb{N}$, $C^-_w \in \mathbb{Q}$ *s.t.*

$$\mathrm{H}^-_w(N) \underset{N\to+\infty}{\widetilde{\quad}} C^-_w N^{|w|+(w)} \quad \text{and} \quad \mathrm{Li}^-_w(z) \underset{z\to 1}{\widetilde{\quad}} B^-_w (1-z)^{-|w|-(w)},$$

$$C^-_w = \prod_{w=uv,\, v\neq 1_{Y_0^*}} ((v)+|v|)^{-1} \quad \text{and} \quad B^-_w = (|w| + (w))!\, C^-_w .$$

Example

$$\begin{aligned}
\mathrm{H}^-_{y_2y_1}(N) &= N(N^2-1)(12N^2+15N+2)/120,\\
\mathrm{H}^-_{y_2^2}(N) &= N(N-1)(2N+1)(2N-1)(5N+6)(N+1)/360,\\
\mathrm{H}^-_{y_2y_3}(N) &= N(N-1)(N+1)(30N^4+35N^3-33N^2-35N+2)/840,\\
\mathrm{H}^-_{y_2y_4}(N) &= N(N-1)(N+1)(63N^5+72N^4-133N^3-138N^2+49N+30)/2520.
\end{aligned}$$

$$\begin{aligned}
\mathrm{Li}^-_{y_1^2}(z) &= -(1-z)^{-1}+5(1-z)^{-2}-7(1-z)^{-3}+3(1-z)^{-4},\\
\mathrm{Li}^-_{y_2y_1}(z) &= (1-z)^{-1}-11(1-z)^{-2}+31(1-z)^{-3}-33(1-z)^{-4}+12(1-z)^{-5},\\
\mathrm{Li}^-_{y_1y_2}(z) &= (1-z)^{-1}-9(1-z)^{-2}+23(1-z)^{-3}-23(1-z)^{-4}+8(1-z)^{-5}.
\end{aligned}$$

w	C^-_w	B^-_w	w	C^-_w	B^-_w
y_n	$\frac{1}{n+1}$	$n!$	y_my_n	$\frac{1}{(n+1)(m+n+2)}$	$n!m!\binom{m+n+1}{n+1}$
y_0^2	$\frac{1}{2}$	1	$y_2y_2y_3$	$\frac{1}{280}$	12960
y_0^n	$\frac{1}{n!}$	1	$y_2y_{10}y_1^2$	$\frac{1}{2160}$	9686476800
y_1^2	$\frac{1}{8}$	3	$y_2^2y_4y_3y_{11}$	$\frac{1}{2612736}$	416761182546508800000

Second global renormalization of divergent polyzetas

$$\mathrm{L}^- := \sum_{w\in Y_0^*} \mathrm{Li}_w^- \; w, \quad \mathrm{H}^- := \sum_{w\in Y_0^*} \mathrm{H}_w^- \; w, \quad C^- := \sum_{w\in Y_0^*} C_w^- \; w.$$

Theorem (Second Abel like theorem)

$$\lim_{z\to 1} \Lambda^{\odot -1}((1-z)^{-1}) \odot \mathrm{Li}^-(z) = \lim_{N\to+\infty} \Upsilon^{\odot -1}(N) \odot \mathrm{H}^-(N) = C^-,$$

where

$$\Lambda(t) := \sum_{w\in Y_0^*} ((w)+|w|)!\, t^{(w)+|w|} w \quad \text{and} \quad \Upsilon(t) := \sum_{w\in Y_0^*} t^{(w)+|w|} w.$$

Moreover, H^- *and* C^- *are group-like, respectively, for* $\Delta_{\uplus}$ *and* $\Delta_{\sqcup\!\sqcup}$.

Theorem (Section orbit theorem)

1. *The following maps are surjective morphisms of algebras*

$$\begin{array}{lclcll} \mathrm{H}_\bullet^- & : & (\mathbb{Q}\langle Y_0\rangle, \uplus) & \longrightarrow & (\mathbb{Q}\{\mathrm{H}_w^-\}_{w\in Y_0^*}, .), & w \longmapsto \mathrm{H}_w^-, \\ \mathrm{Li}_\bullet^- & : & (\mathbb{Q}\langle Y_0\rangle, \top) & \longrightarrow & (\mathbb{Q}\{\mathrm{Li}_w^-\}_{w\in Y_0^*}, .), & w \longmapsto \mathrm{Li}_w^-, \end{array}$$

   where $\top$ *is a law of algebra in* $\mathbb{Q}\langle Y_0\rangle$ *not dualizable.*

   Moreover, $\ker \mathrm{H}_\bullet^- = \ker \mathrm{Li}_\bullet^- = \mathbb{Q}\{w - w\top 1_{Y_0^*} | w \in Y_0^*\}$.

2. *Let* $\top' : \mathbb{Q}\langle Y_0\rangle \times \mathbb{Q}\langle Y_0\rangle \longrightarrow \mathbb{Q}\langle Y_0\rangle$ *be a law such that* $\mathrm{Li}_\bullet^-$ *is a morphism for* $\top'$ *and* $(1_{Y_0^*}\top'\mathbb{Q}\langle Y_0\rangle) \cap \ker(\mathrm{Li}_\bullet^-) = \{0\}$.

   Then $\top' = g \circ \top$, *where* $g \in GL(\mathbb{Q}\langle Y_0\rangle)$ *such that* $\mathrm{Li}_\bullet^- \circ g = \mathrm{Li}_\bullet^-$.

Bi-integro-differential algebra $\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$

Let $\lambda(z) := z(1-z)^{-1}$. Let us equip $\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$ with the basis

$$\begin{aligned}\mathcal{B} &:= (z^k \mathrm{Li}_{ux_1}(z)\,\mathrm{Li}_{x_0^n}(z))_{(k,n,u)\in\mathbb{Z}\times\mathbb{N}\times X^*}\\ &\sqcup ((1-z)^{-l}\,\mathrm{Li}_{ux_1}(z)\,\mathrm{Li}_{x_0^n}(z))_{(l,n,u)\in\mathbb{N}^+\times\mathbb{N}\times X^*}\\ &\sqcup (z^k \mathrm{Li}_{x_0^n}(z))_{(k,n)\in\mathbb{Z}\times\mathbb{N}} \sqcup ((1-z)^{-l}\,\mathrm{Li}_{x_0^n}(z))_{(l,n)\in\mathbb{N}^+\times\mathbb{N}}\end{aligned}$$

and let us define $\mathrm{ind} : \mathcal{B} \to \mathbb{Z}$ by $\mathrm{ind}(z^k(1-z)^{-l}\,\mathrm{Li}_{x_0^n}(z)) = k$ and $\mathrm{ind}(z^k(1-z)^{-l}\,\mathrm{Li}_{ux_1}(z)\log^n(z)) = k + |ux_1|$.

Lemma

Let us consider the following operators over $\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$

$$\theta_1 = (1-z)\frac{d}{dz}, \theta_0 = z\frac{d}{dz}, \quad \iota_1(f) = \int_0^z f(t)\omega_1(t), \iota_0(f) = \int_{z_0}^z f(s)\omega_0(s),$$

where, $z_0 = 0$ if $\mathrm{ind}(b) \geq 1$ and $z_0 = 1$ if $\mathrm{ind}(b) \leq 0$. Then

1. *$\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$ is closed under the action of $\{\theta_0, \theta_1, \iota_0, \iota_1\}$.*
2. *The operators $\{\theta_0, \theta_1, \iota_0, \iota_1\}$ satisfy in particular,*
$$\begin{aligned}\forall k = 0,1, \theta_k\iota_k = \mathrm{Id} \quad &\text{and} \quad \theta_1 + \theta_0 = [\theta_1, \theta_0] = \partial_z,\\ [\theta_0\iota_1, \theta_1\iota_0] = 0 \quad &\text{and} \quad (\theta_0\iota_1)(\theta_1\iota_0) = (\theta_1\iota_0)(\theta_0\iota_1) = \mathrm{Id}.\end{aligned}$$
3. *$\theta_0\iota_1$ and $\theta_1\iota_0$ are scalar operators within $\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$, with eigenvalues λ and $1/\lambda$, i.e. $(\theta_0\iota_1)f = \lambda f$ and $(\theta_1\iota_0)f = f/\lambda$.*

Let $\Im, \Theta$ be the algebra morphisms $\mathbb{C}\langle X\rangle \to \mathrm{End}_{\mathbb{C}}(\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*})$ defined by $\Im(1_{X^*}) = \Theta(1_{X^*}) = \mathrm{Id}$ and $\Im(vx_i) = \Im(v)\iota_i$ and $\Theta(vx_i) = \Theta(v)\theta_i$.

Towards indexing polylogarithms by rational series

Theorem (extension of $\mathrm{Li}_\bullet$)

$$\begin{array}{rcl}\mathrm{Li}_\bullet : (\mathbb{C}[x_0^*]ш\mathbb{C}[(-x_0)^*]ш\mathbb{C}[x_1^*]ш\mathbb{C}\langle X\rangle, ш, 1_{X^*}) & \longrightarrow & (\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}, \times, 1),\\ T & \longmapsto & \Im(T)1_\Omega.\end{array}$$

$\mathrm{Li}_\bullet$ *is surjective and* $\ker \mathrm{Li}_\bullet$ *is the ideal generated by* $x_0^* ш x_1^* - x_1^* + 1$*.*

By linearity and continuity, $\mathrm{Li}_\bullet$ can be extended over $\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$ as follows

$$\text{with } S = \sum_{n\ge 0}\langle S|x_0^n\rangle x_0^n + \sum_{k\ge 1}\sum_{w\in(x_0^*x_1)^k x_0^*}\langle S|w\rangle\, w,$$

$$\alpha_{z_0}^z(S) = \sum_{n\ge 0}\frac{\langle S|x_0^n\rangle}{n!}(\log(z)-\log(z_0))^n + \sum_{k\ge 1}\sum_{w\in(x_0^*x_1)^k x_0^*}\langle S|w\rangle \alpha_{z_0}^z(w),$$

one defines (if convergent)

$$\mathrm{Li}_S(z) = \sum_{n\ge 0}\frac{\langle S|x_0^n\rangle}{n!}\log^n(z) + \sum_{k\ge 1}\sum_{w\in(x_0^*x_1)^k x_0^*}\langle S|w\rangle\,\mathrm{Li}_w(z).$$

As an orbit operator, $\Im$ can be extended over $\mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle$ and the previous extension of $\mathrm{Li}_\bullet$ is valid for $\mathbb{C}\langle X\rangle ш\mathbb{C}[x_0^*]ш\mathbb{C}[(-x_0)^*]ш\mathbb{C}[x_1^*]$ and $\mathbb{C}\langle X\rangle ш \mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle$:

$$\begin{aligned}\Im(\mathbb{C}\langle X\rangle ш\mathbb{C}[x_0^*]ш\mathbb{C}[(-x_0)^*]ш\mathbb{C}[x_1^*])1_\Omega &= \mathrm{span}_{\mathbb{C}}\left\{z^a(1-z)^{-b}\,\mathrm{Li}_w(z)\right\}_{w\in X^*}^{a\in\mathbb{Z},b\in\mathbb{N}}\\ &= \mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*},\\ \Im(\mathbb{C}\langle X\rangle ш\mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle)1_\Omega &= \mathrm{span}_{\mathbb{C}}\left\{z^a(1-z)^b\mathrm{Li}_w(z)\right\}_{w\in X^*}^{a,b\in\mathbb{C}}\end{aligned}$$

Examples

For any $a, b \in \mathbb{C}$, one has $\alpha_1^z((ax_0)^*) = z^a$ and $\alpha_0^z((bx_1)^*) = (1-z)^{-b}$. Then[5]

1. $\mathrm{Li}_{x_0^*}(z) = \Im(x_0^*)1_\Omega(z) = z$,
   $\mathrm{Li}_{x_1^*}(z) = \Im(x_1^*)1_\Omega(z) = (1-z)^{-1}$,
   $\mathrm{Li}_{(x_0+x_1)^*}(z) = \Im((x_0+x_1)^*)1_\Omega(z) = z(1-z)^{-1}$.
2. For any $i \in \mathbb{N}_+$ and $x \in X$, since $(x^*)^{ш i} = (ix)^*$ then

$$\begin{array}{rcccl} \mathrm{Li}_{(x_0^*)^{ш i}}(z) & = & \Im((x_0^*)^{ш i})1_\Omega(z) & = & z^i, \\ \mathrm{Li}_{(x_1^*)^{ш i}}(z) & = & \Im((x_1^*)^{ш i})1_\Omega(z) & = & (1-z)^{-i}. \\ \mathrm{Li}_{(x_0^*)^{ш j} ш (x_1^*)^{ш i}}(z) & = & \Im((x_0^*)^{ш j} ш (x_1^*)^{ш i})1_\Omega(z) & = & z^j(1-z)^{-i}. \end{array}$$

3. For $a \in \mathbb{C}, x \in X, i \in \mathbb{N}_+$, since $(ax)^{*i} = (ax)^* ш (1+ax)^{i-1}$ then

$$\mathrm{Li}_{(ax_0)^{*i}}(z) = \Im((ax_0)^{*i})1_\Omega(z) = z^a \sum_{k=0}^{i-1} \binom{i-1}{k} \frac{(a \log z)^k}{k!},$$

$$\mathrm{Li}_{(ax_1)^{*i}}(z) = \Im((ax_1)^{*i})1_\Omega(z) = \frac{1}{(1-z)^a} \sum_{k=0}^{i-1} \binom{i-1}{k} \frac{1}{k!} \left(a \log \frac{1}{1-z}\right)^k.$$

---

[5] For any $c_0, c_1 \in \mathbb{C}$, one also has $(c_0x_0 + c_1x_1)^* = (c_0x_0)^* ш (c_1x_1)^*$.

Example

$$\mathrm{Li}^{-}_{y_{s_1}\ldots y_{s_r}} = \sum_{k_1=0}^{s_1}\sum_{k_2=0}^{s_1+s_2-k_1}\ldots\sum_{k_r=0}^{\substack{(s_1+\ldots+s_r)-\\(k_1+\ldots+k_{r-1})}}\binom{s_1}{k_1}\binom{s_1+s_2-k_1}{k_2}\ldots$$
$$\binom{s_1+\ldots+s_r-k_1-\ldots-k_{r-1}}{k_r}(\theta_0^{k_1}\lambda)\ldots(\theta_0^{k_r}\lambda),$$
$$\theta_0^{k_i}\lambda(z) = \begin{cases}\lambda(z), & \text{if} \quad k_i=0,\\ \dfrac{1}{1-z}\displaystyle\sum_{j=1}^{k_i}S_2(k_i,j)j!(\lambda(z))^j, & \text{if} \quad k_i>0,\end{cases}$$

where $S_2(k_i,j)$ denote the Stirling numbers of second kind.
Hence, $\mathrm{Li}^{-}_{y_{s_1}\ldots y_{s_r}} = \mathrm{Li}_F = \Im(F)1_\Omega$, where F is the following exchangeable rational series

$$F = \sum_{k_1=0}^{s_1}\sum_{k_2=0}^{s_1+s_2-k_1}\ldots\sum_{k_r=0}^{\substack{(s_1+\ldots+s_r)-\\(k_1+\ldots+k_{r-1})}}\binom{s_1}{k_1}\binom{s_1+s_2-k_1}{k_2}\ldots\binom{\sum_{i=1}^{r}s_i-\sum_{i=1}^{r-1}k_i}{k_r}$$
$$F_{k_1}\,ш\ldots ш\,F_{k_r},$$
$$F_{k_i} = \begin{cases}x_1^*-1_{X^*}, & \text{if} \quad k_i=0,\\ x_1^*\,ш\displaystyle\sum_{j=1}^{k_i}S_2(k_i,j)j!(x_1^*-1_{X^*})^{шj}, & \text{if} \quad k_i>0.\end{cases}$$

Double regularization of $\{\zeta(-s_1,\ldots,-s_r)\}_{(s_1,\ldots,s_r)\in\mathbb{N}^r}$

$$\mathrm{Li}_{-1}(z) = \mathrm{Li}^-_{y_1}(z) = \mathrm{Li}_{R_1}(z) = z(1-z)^{-2} \sim (1-z)^{-2}, \quad \text{for} \quad z \to 1.$$
$$\mathrm{H}_{-1}(N) = \mathrm{H}^-_{y_1}(N) = \mathrm{H}_{\pi_Y(R_1)}(N) = N(N+1)/2 \sim N^2/2, \quad \text{for} \quad N \to +\infty.$$

Then Li_{-1} can be encoded by $R_1 = (2x_1)^* - x_1^*$.

More generally, for any $k \geq 1$, there exists $R_k \in \mathbb{C}[x_1^*]$ such that

$$\mathrm{Li}_{-k}(z) = \mathrm{Li}^-_{y_k}(z) = \mathrm{Li}_{R_k}(z) \sim k!/(1-z)^{k+1}, \quad \text{for} \quad z \to 1.$$
$$\mathrm{H}_{-k}(N) = \mathrm{H}^-_{y_k}(N) = \mathrm{H}_{\pi_Y(R_k)}(N) \sim N^{k+1}/(k+1), \quad \text{for} \quad N \to +\infty.$$

Let $\zeta_{ш} : (\mathbb{C}\langle X\rangle, ш, 1_{X^*}) \to (\mathbb{C}, ., 1)$ and $\gamma_\bullet : (\mathbb{C}\langle Y\rangle, \,⧢\!\!\!\!-\,, 1_{Y^*}) \to (\mathbb{C}, ., 1)$ be the morphisms, satisfying $\zeta_{ш}(l) = \gamma_{\pi_Y l} = \zeta(l)$, $l \in \mathcal{L}ynX - X$, and

$$\begin{aligned} \zeta_{ш}(x_1) &= \mathrm{f.p.}_{z\to 1}\mathrm{Li}_{x_1}(z) &= 0, \quad &\{(1-z)^a \log^b(1/(1-z))\}_{a\in\mathbb{Z}, b\in\mathbb{N}} \\ \gamma_{y_1} &= \mathrm{f.p.}_{N\to\infty}\mathrm{H}_{y_1}(N) &= \gamma, \quad &\{N^a \log^b N\}_{a\in\mathbb{Z}, b\in\mathbb{N}}. \end{aligned}$$

How to extend $\zeta_{ш}$ and $\gamma_\bullet$ respectively over $(\mathbb{C}\langle X\rangle ш \mathbb{C}[x_1^*], ш, 1_{X^*})$ and $(\mathbb{C}\langle Y\rangle \,⧢\!\!\!\!-\, \mathbb{C}[y_1^*], \,⧢\!\!\!\!-\,, 1_{Y^*})$?

$$\pi_Y(tx_1)^* = \sum_{k\geq 0}(ty_1)^k \quad \text{and} \quad \mathrm{H}_{\pi_Y(tx_1)^*} = \sum_{n\geq 0}\mathrm{H}_{y_1^n}t^n = \exp\left(-\sum_{n\geq 1}\mathrm{H}_{y_n}\frac{(-t)^n}{n}\right).$$

Theorem (Double regularization at negative multi-indices)

$$\zeta_{ш}((tx_1)^*) = 1 \quad \textit{and} \quad \gamma_{\pi_Y(tx_1)^*} = \exp\left(\gamma t - \sum_{n\geq 2}\zeta(n)\frac{(-t)^n}{n}\right) = \Gamma^{-1}(1+t).$$

Euler-Mac Laurin constants $\{\gamma_{-s_1,\ldots,-s_r}\}_{(s_1,\ldots,s_r)\in\mathbb{N}^r}$

Since

$$
\begin{aligned}
\mathrm{Li}_{-1,-1} &= -\,\mathrm{Li}_{x_1^*} + 5\,\mathrm{Li}_{(2x_1)^*} - 7\,\mathrm{Li}_{(3x_1)^*} + 3\,\mathrm{Li}_{(4x_1)^*}\,,\\
\mathrm{Li}_{-2,-1} &= \mathrm{Li}_{x_1^*} - 11\,\mathrm{Li}_{(2x_1)^*} + 31\,\mathrm{Li}_{(3x_1)^*} - 33\,\mathrm{Li}_{(4x_1)^*} + 12\,\mathrm{Li}_{(5x_1)^*}\,,\\
\mathrm{Li}_{-1,-2} &= \mathrm{Li}_{x_1^*} - 9\,\mathrm{Li}_{(2x_1)^*} + 23\,\mathrm{Li}_{(3x_1)^*} - 23\,\mathrm{Li}_{(4x_1)^*} + 8\,\mathrm{Li}_{(5x_1)^*}\,,
\end{aligned}
$$

then

$$
\begin{aligned}
\mathrm{H}_{-1,-1} &= -\mathrm{H}_{\pi_Y(x_1^*)} + 5\mathrm{H}_{\pi_Y((2x_1)^*)} - 7\mathrm{H}_{\pi_Y((3x_1)^*)} + 3\mathrm{H}_{\pi_Y((4x_1)^*)},\\
\mathrm{H}_{-2,-1} &= \mathrm{H}_{\pi_Y(x_1^*)} - 11\mathrm{H}_{\pi_Y((2x_1)^*)} + 31\mathrm{H}_{\pi_Y((3x_1)^*)} - 33\mathrm{H}_{\pi_Y((4x_1)^*)} + 12\mathrm{H}_{\pi_Y((5x_1)^*}\\
\mathrm{H}_{-1,-2} &= \mathrm{H}_{\pi_Y(x_1^*)} - 9\mathrm{H}_{\pi_Y((2x_1)^*)} + 23\mathrm{H}_{\pi_Y((3x_1)^*)} - 23\mathrm{H}_{\pi_Y((4x_1)^*)} + 8\mathrm{H}_{\pi_Y((5x_1)^*)}.
\end{aligned}
$$

As a consequence,

$$
\begin{aligned}
\gamma_{-1,-1} &= -\Gamma^{-1}(2) + 5\Gamma^{-1}(3) - 7\Gamma^{-1}(4) + 3\Gamma^{-1}(5)\\
&= 11/24,\\
\gamma_{-2,-1} &= \Gamma^{-1}(2) - 11\Gamma^{-1}(3) + 31\Gamma^{-1}(4) - 33\Gamma^{-1}(5) + 12\Gamma^{-1}(6)\\
&= -73/120,\\
\gamma_{-1,-2} &= \Gamma^{-1}(2) - 9\Gamma^{-1}(3) + 23\Gamma^{-1}(4) - 23\Gamma^{-1}(5) + 8\Gamma^{-1}(6)\\
&= -67/120.
\end{aligned}
$$

One more extension of $\mathrm{Li}_\bullet$

Theorem

1. $\Im(\mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle)1_\Omega = \mathrm{span}_\mathbb{C}\{z^a(1-z)^b\}_{a,b\in\mathbb{C}}$.
2. *The family* $\{z^a(1-z)^b\}_{\{\Re(b)\geq 0\} \text{ and } \{0\leq\Re(b)<1\}\cup \{0\leq\Re(a)<1\}}$ *is a linear basis.*
3. *The following map is surjective and its kernel contains the ideal generated by* $x_0^* ш x_1^* - x_1^* + 1$,
$$\begin{array}{rcl}\mathrm{Li}_\bullet : (\mathbb{C}\langle X\rangle ш \mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle, ш, 1_{X^*}) & \twoheadrightarrow & (\mathbb{C}\left\{z^a(1-z)^b\,\mathrm{Li}_w\right\}_{w\in X^*}^{a,b\in\mathbb{C}}, \times, 1),\\ T & \mapsto & \Im(T)1_\Omega.\end{array}$$
4. *The family* $\{\mathrm{Li}_w\}_{w\in X^*}$ *is free over* $\mathbb{C}\{z^a(1-z)^b\}_{a,b\in\mathbb{C}}$.
5. $\mathbb{C}\{z^a(1-z)^b\,\mathrm{Li}_w\}_{w\in X^*}^{a,b\in\mathbb{C}}$ *is closed under the action of* $\{\theta_0,\theta_1,\iota_0,\iota_1\}$.

For any $s_1,\ldots,s_r\in\mathbb{C}^r$, one has

$$\begin{array}{llll}\iota_0(\mathrm{Li}_{s_1,\ldots,s_r}) = \mathrm{Li}_{s_1+1,\ldots,s_r} & \text{and} & \iota_1(\mathrm{Li}_{s_1,\ldots,s_r}) = \mathrm{Li}_{1,s_1,\ldots,s_r}, & \\ \theta_0(\mathrm{Li}_{s_1,\ldots,s_r}) = \lambda\,\mathrm{Li}_{s_2,\ldots,s_r} & \text{and} & \theta_1(\mathrm{Li}_{s_1,\ldots,s_r}) = \mathrm{Li}_{s_2,\ldots,s_r}, & \text{if } s_1 = 1,\\ \theta_0(\mathrm{Li}_{s_1,\ldots,s_r}) = \mathrm{Li}_{s_1-1,\ldots,s_r} & \text{and} & \theta_1(\mathrm{Li}_{s_1,\ldots,s_r}) = \mathrm{Li}_{s_1-1,\ldots,s_r}/\lambda, & \text{if } s_1 \neq 1.\end{array}$$

THANK YOU FOR YOUR ATTENTION