A refinement of the skew length statistic

Robin Sulzgruber

Universität Wien

77th Séminaire Lotharingien de Combinatoire
September 11th–14th 2016 • Strobl • Austria

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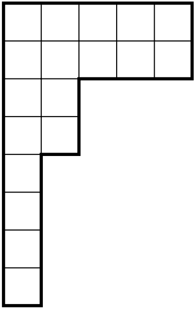

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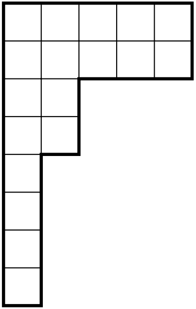

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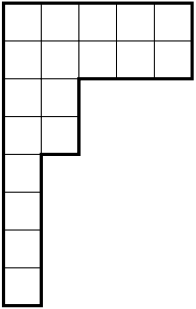

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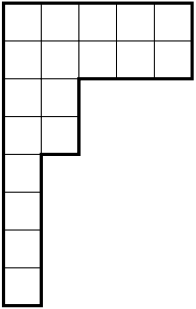

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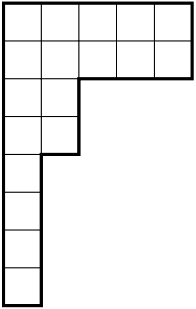

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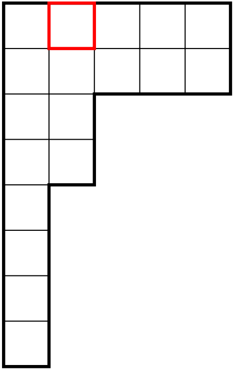

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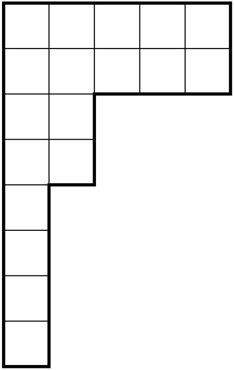

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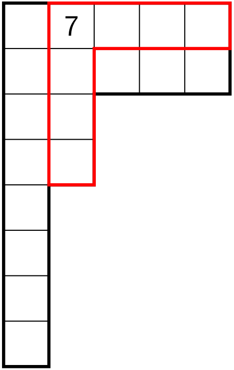

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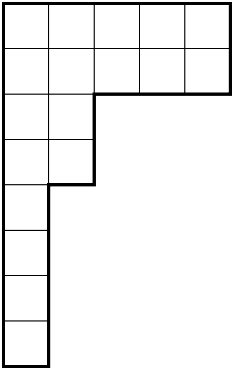

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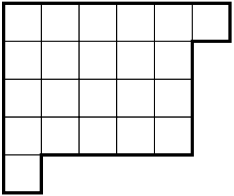

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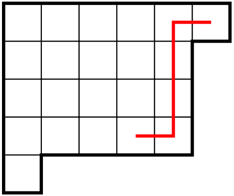

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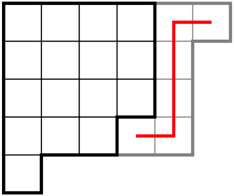

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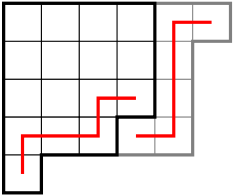

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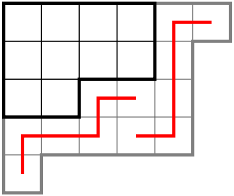

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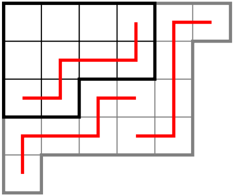

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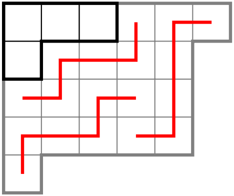

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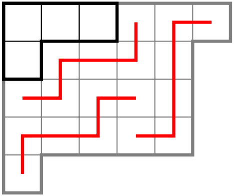

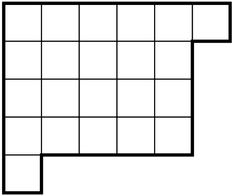

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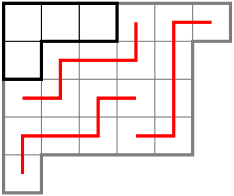

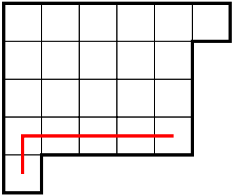

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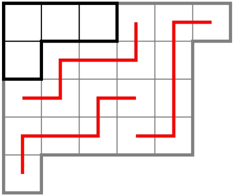

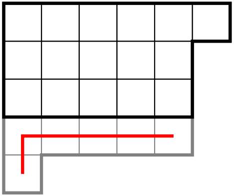

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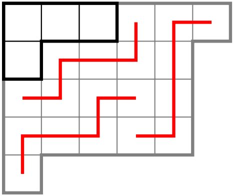

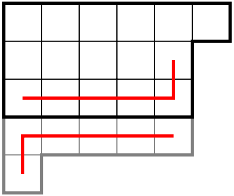

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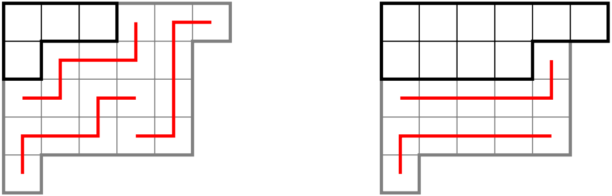

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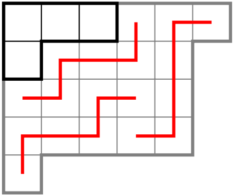

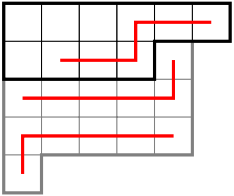

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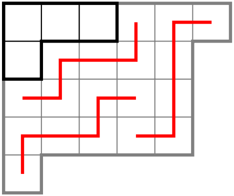

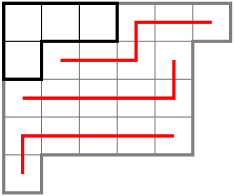

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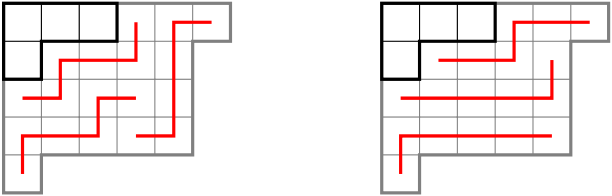

- Cores also appear in the modular representation theory of $\mathfrak{S}_n$.

Cores and reflection groups

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- Lapointe and Morse use $k+1$-cores in the study of k-Schur functions.
- Thiel and Williams generalise n, p-cores to other affine Weyl groups and have results on the maximal size, the expected size and the variance of the size in simply-laced types.

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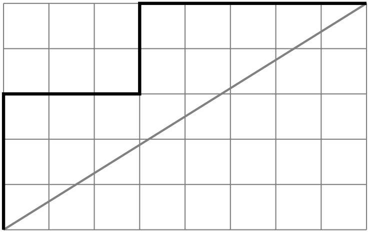

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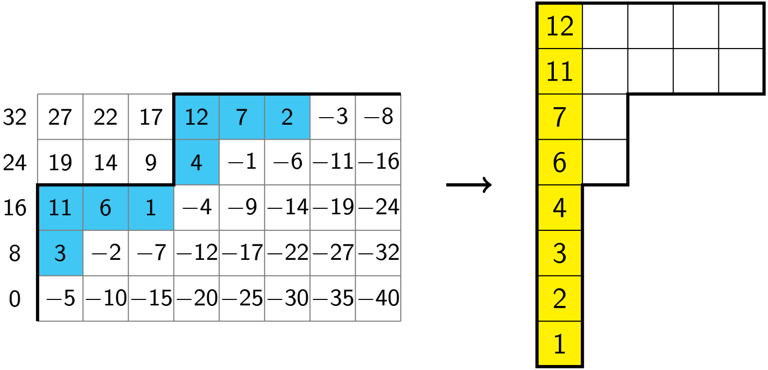

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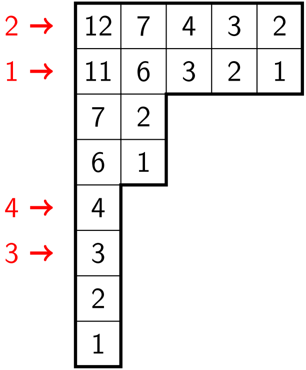

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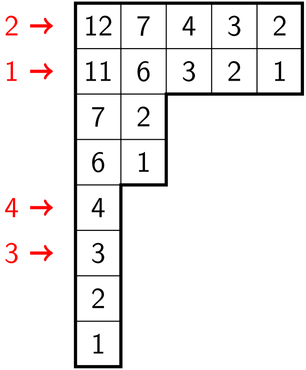
The *skew length* $\mathrm{skl}(\lambda)$ is the number of cells of λ that lie in an n-row and have hook length less than p.



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For relatively prime n and p define the rational q, t-Catalan numbers as

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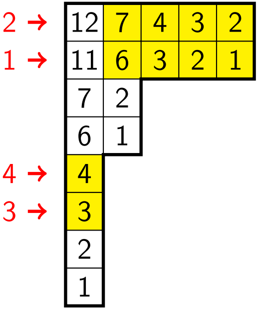

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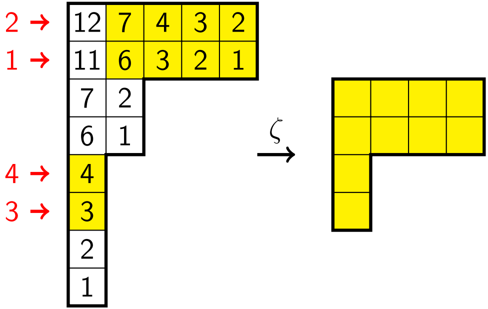

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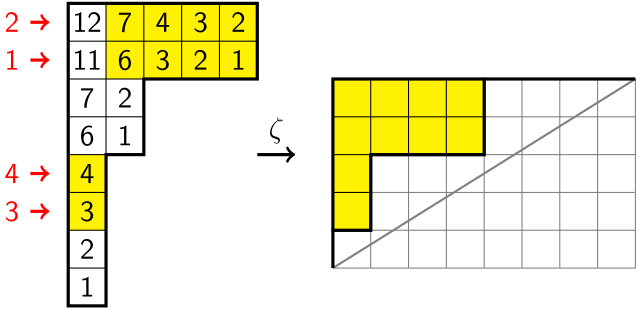

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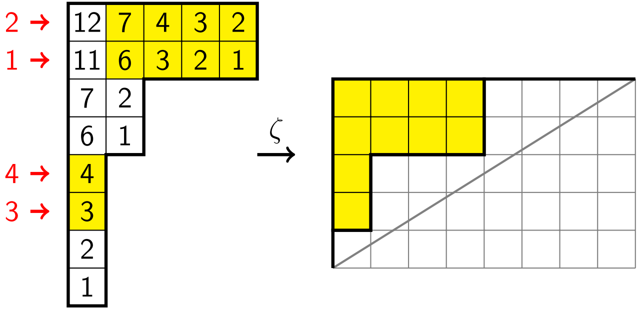

Theorem (Thomas, Williams) The map $\zeta : \mathfrak{C}_{n,p} \to \mathfrak{D}_{n,p}$ is a bijection.

Skew length and conjugation

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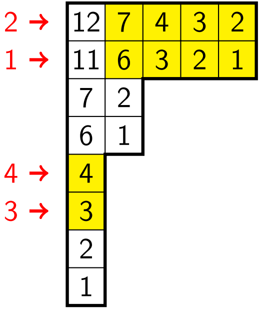

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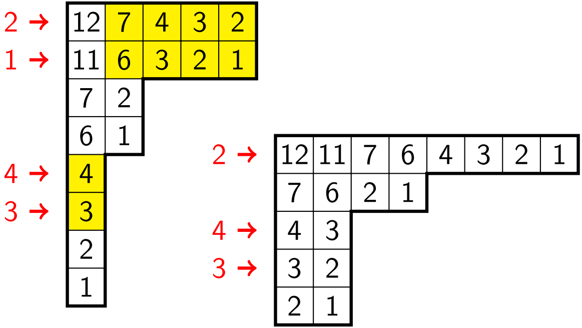

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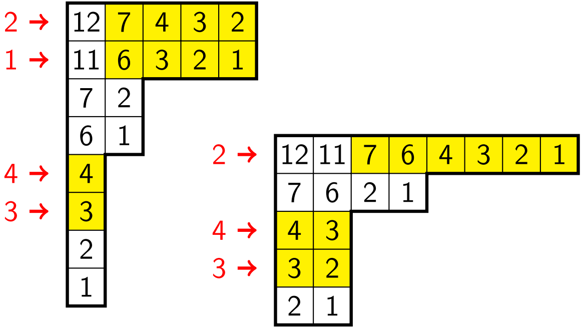

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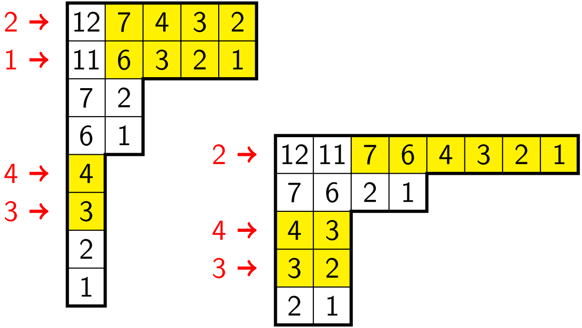

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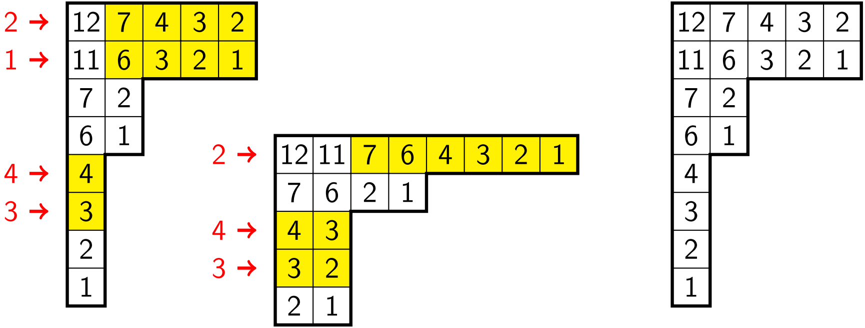

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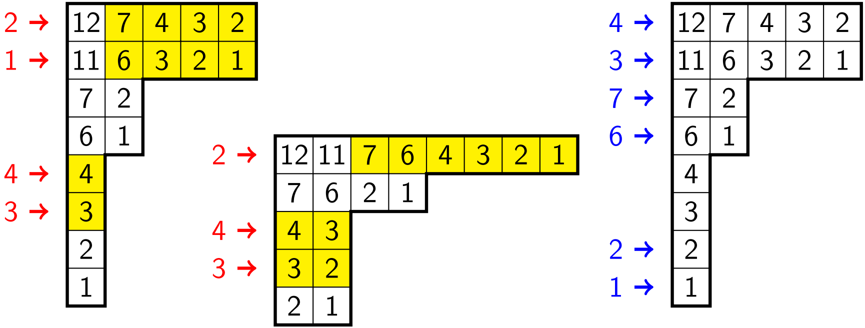

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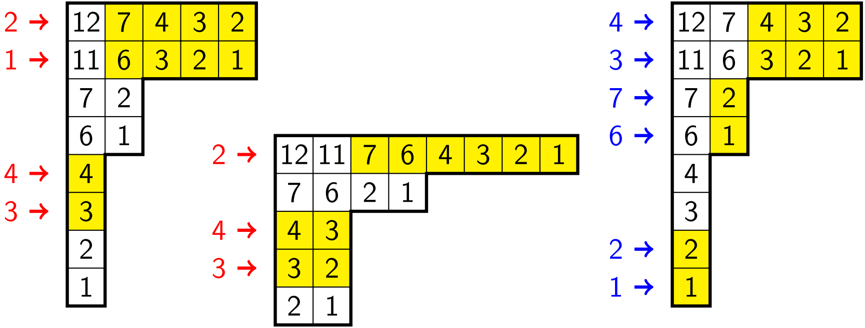

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 & & \downarrow & \downarrow & & \downarrow & \downarrow \\
\cline{3-7}
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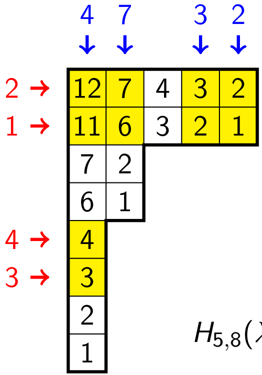

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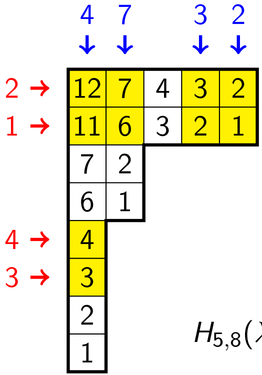

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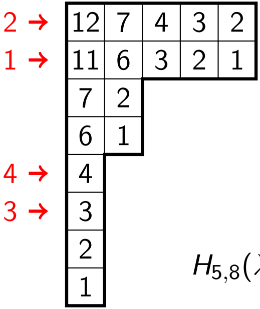

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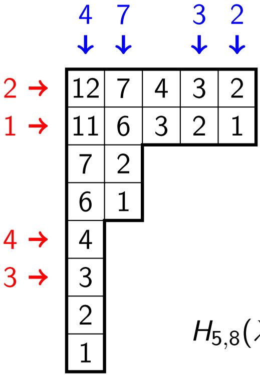

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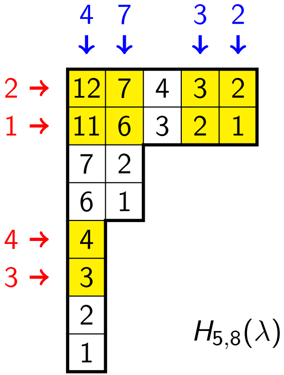

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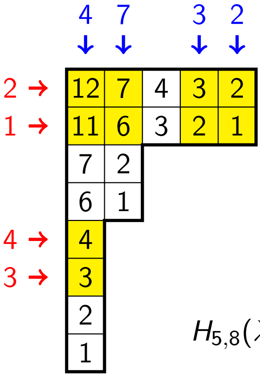

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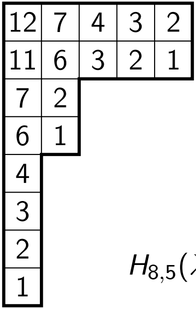

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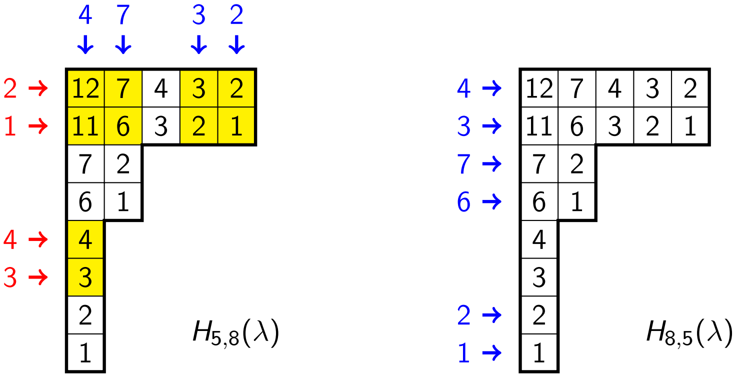

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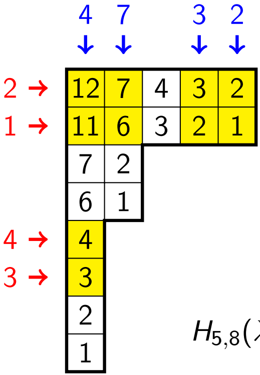

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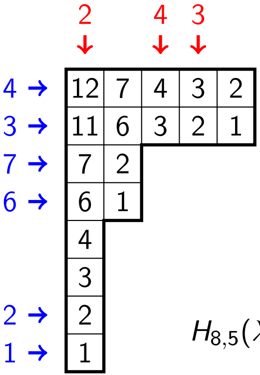

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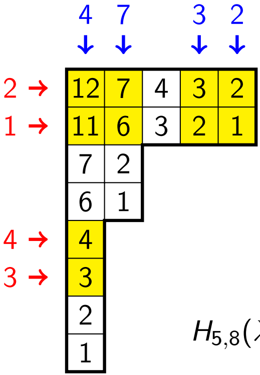

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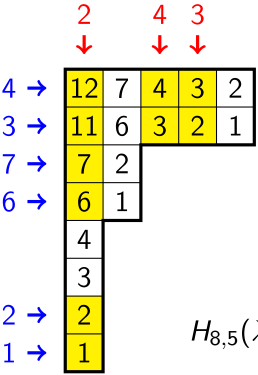

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The proof uses induction on the size of λ.

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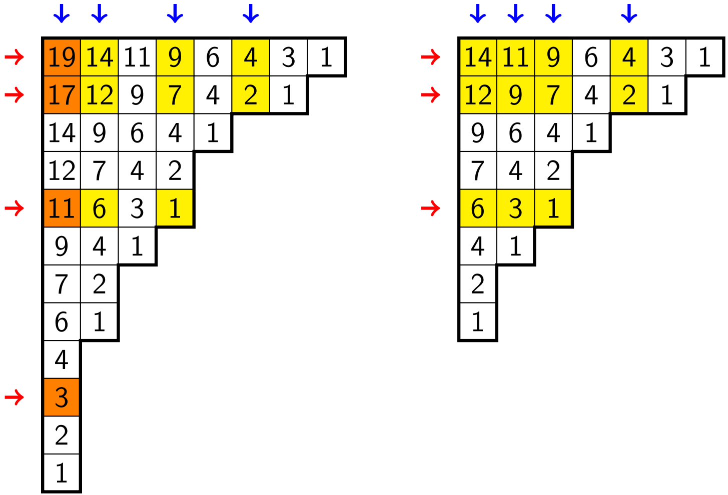

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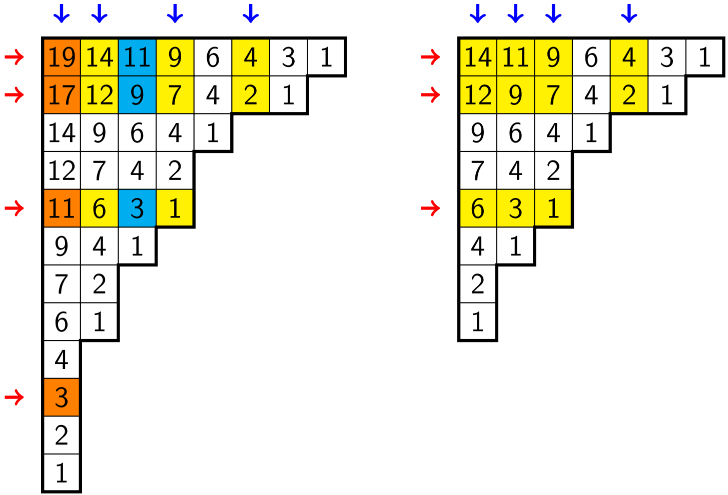

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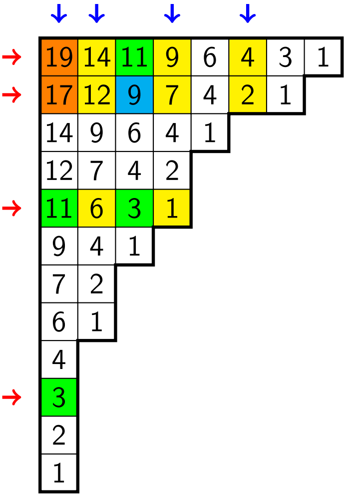

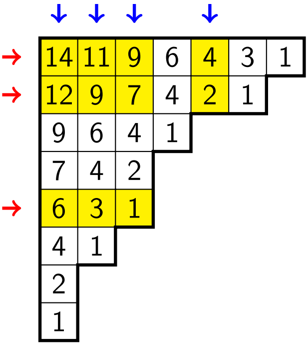

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11	6	3	1				
9	4	1					
7	2						
6	1						
4							
3							
2							
1							

19	14	11	9	6	4	3	1
17	12	9	7	4	2	1	
14	9	6	4	1			
12	7	4	2				
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The proof uses induction on the size of λ.

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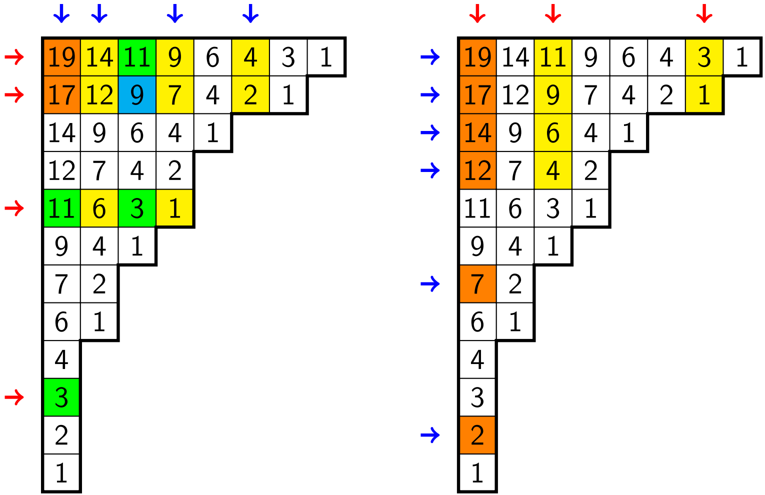

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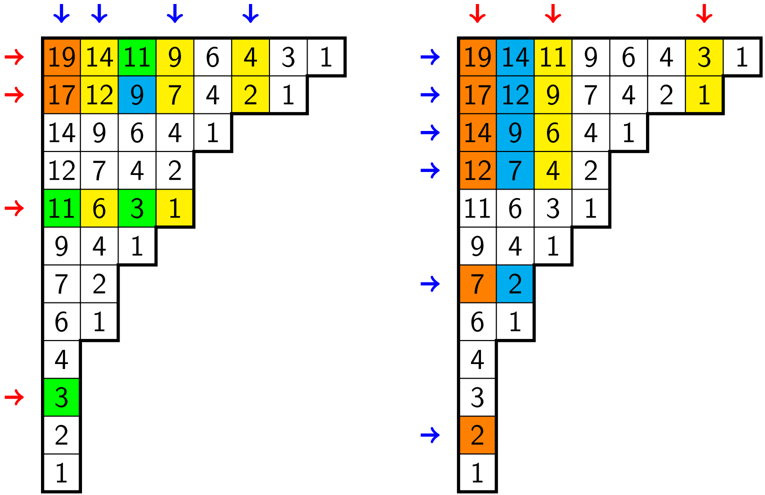

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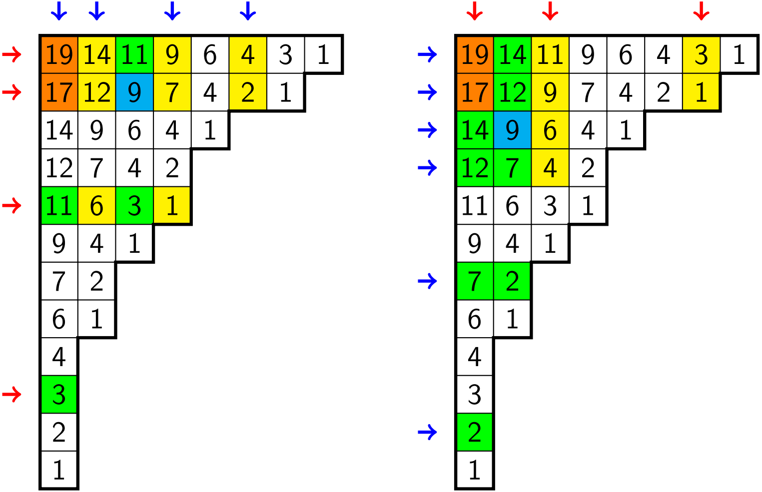

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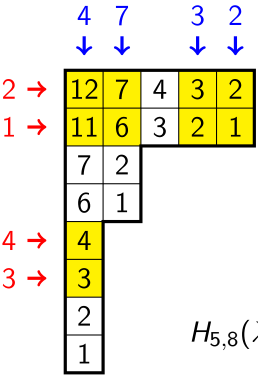

$H_{5,8}(\lambda)$

This is the end.

Thank you!