Lattice paths with catastrophes

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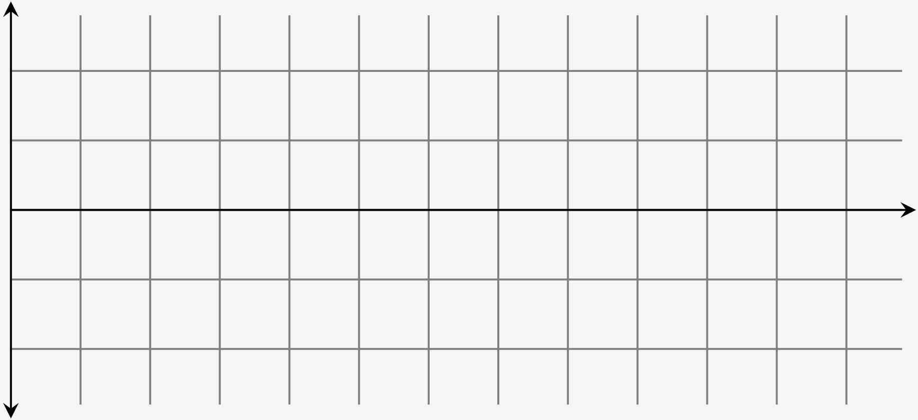
Lattice paths



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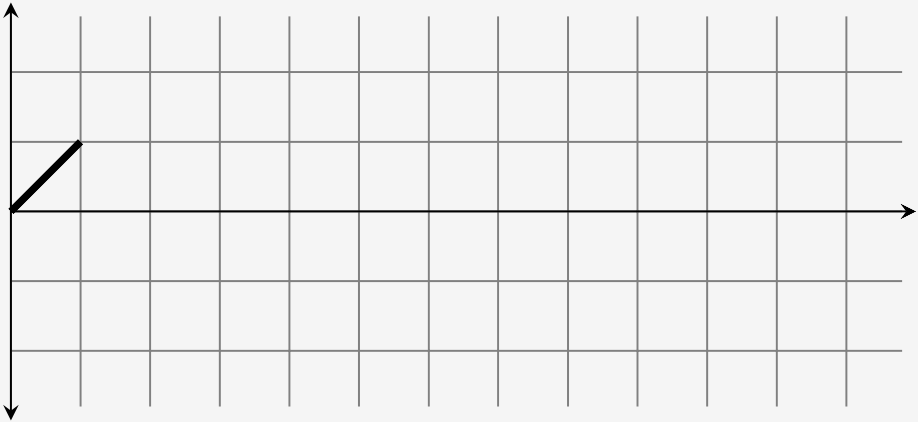

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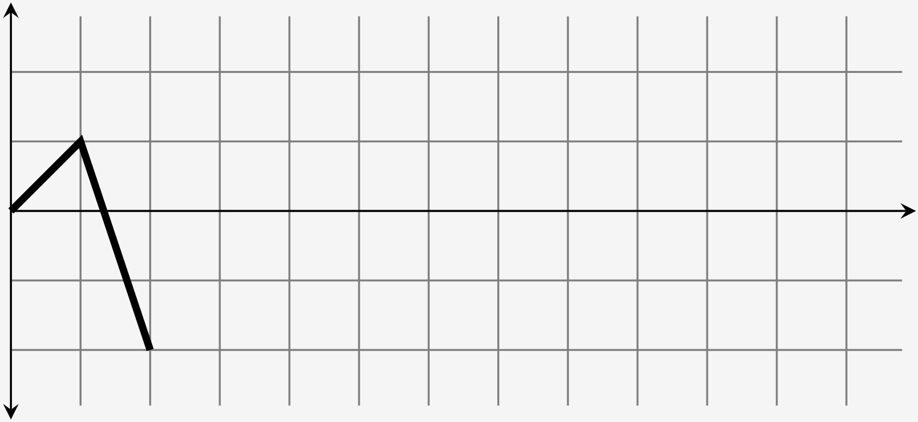

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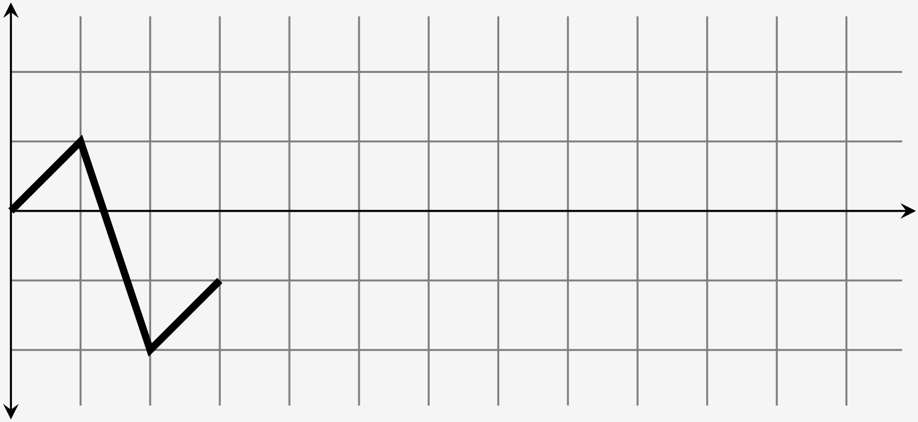

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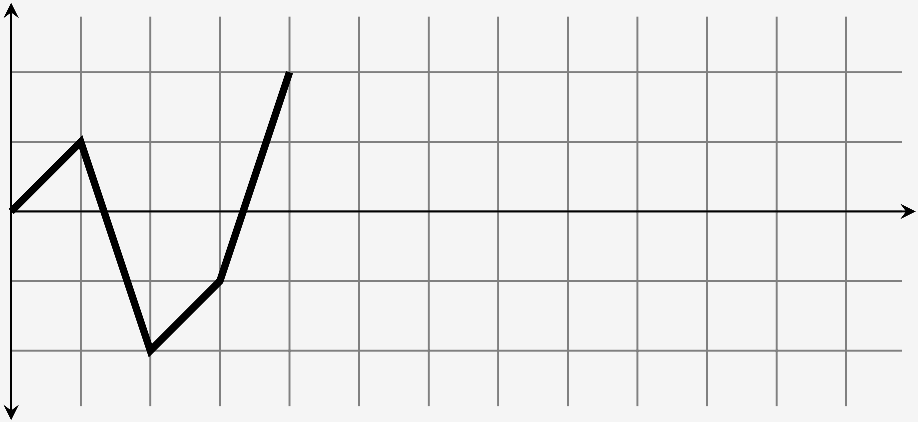
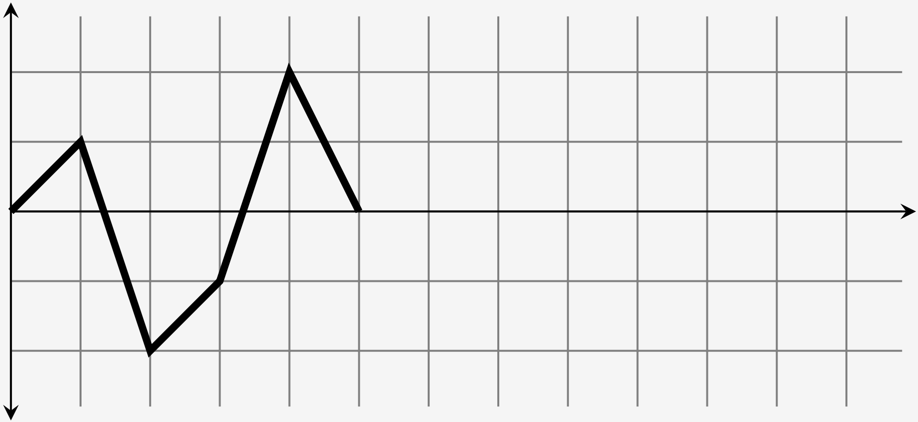


Lattice paths



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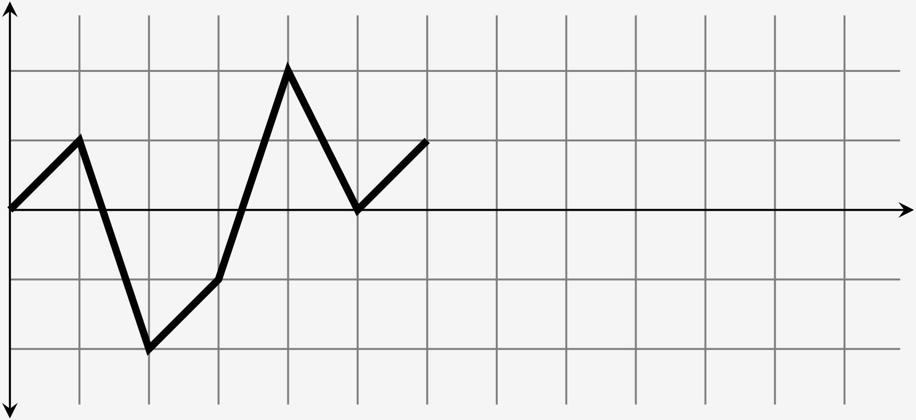

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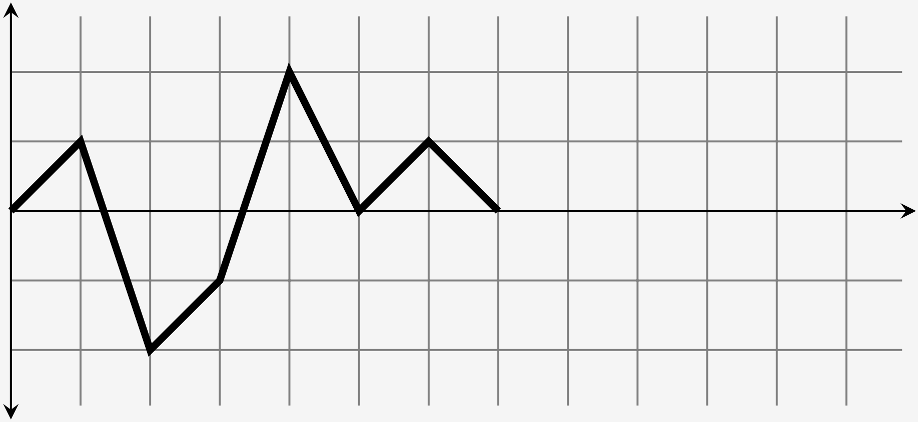

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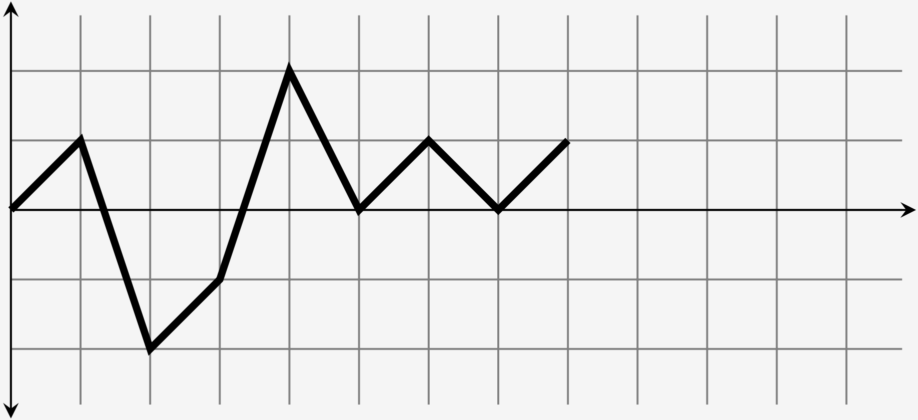


Lattice paths

Lattice paths

Lattice paths

Lattice paths

What is a lattice path?

Definition

- **Step set:** $\mathcal{S} = \{(1, b_1), \ldots, (1, b_m)\} \subset \mathbb{Z}^2$
- ***n*-step lattice path:** Sequence of vectors $(v_1, \ldots, v_n) \in \mathcal{S}^n$

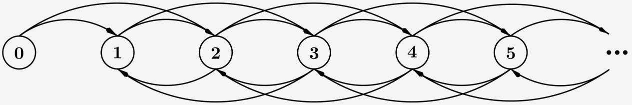
Weights

- For $\mathcal{S} = \{-c, \ldots, d\}$ define $\Pi = \{p_{-c}, \ldots, p_d\}$
- **Jump polynomial:** $P(u) = \sum_{i=-c}^{d} p_i u^i$
- **Drift:** $\delta = P'(1)$

Examples

- **Dyck path/Random walk:** $P(u) = p_{-1} u^{-1} + p_1 u^1$
- **Motzkin walk:** $P(u) = p_{-1} u^{-1} + p_0 + p_1 u^1$

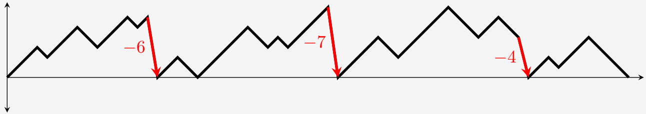
Lattice paths with Catastrophes



[Chang & Krinik & Swift: Birth-multiple catastrophe processes, 2007]
[Krinik & Rubino: The Single Server Restart Queueing Model, 2013]

Catastrophe

A *catastrophe* is a jump $j \notin \mathcal{S}$ to altitude 0.



Motivation

Questions from queuing theorists

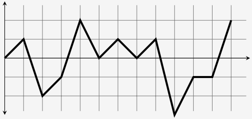
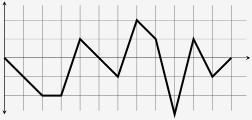
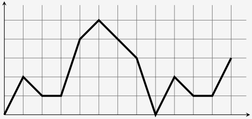
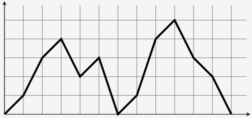
1. Can you do exact enumeration for the Bernoulli walk, for which one also allows at any time some *catastrophe* (=unbounded jump from anywhere directly to 0)?
2. What are typical properties of such walks, distribution of patterns?
3. How to generate them?

Caveat: The limiting object is not a Brownian motion (infinite negative drift!).

Applications

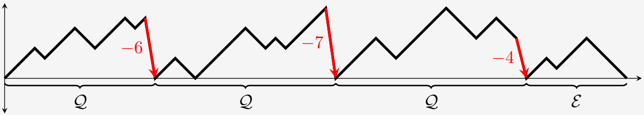
- financial mathematics (catastrophe = bankrupt)
- evolution of the queue of a printer (catastrophe = reset)
- population genetics (species extinctions by pandemic)

Terminology of directed paths

	ending anywhere	ending at 0
unconstrained (on $\mathbb{Z}$)	 walk/path $W(z,u) = \sum_{n,k} w_{n,k} z^n u^k$	 bridge $B(z) = \sum_n b_n z^n$
constrained (on $\mathbb{Z}_+$)	 meander $M(z,u) = \sum_{n,k} m_{n,k} z^n u^k$	 excursion $M_0(z) = \sum_n m_{n,0} z^n$

Known algebraic objects: [Banderier–Flajolet02]

Generating functions



Following results stated for Dyck paths with catastrophes.

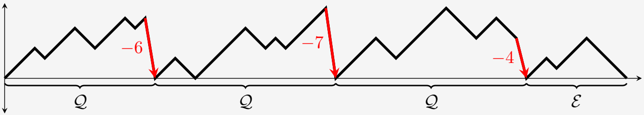
Theorem (Generating functions for lattice paths with catastrophes)

Let $f_{n,k}$ be the number of catastrophe-walks of length n from altitude 0 to altitude k, then $F(z,u) = \sum_{k\geq 0} F_k(z)u^k = \sum_{n,k\geq 0} f_{n,k}u^k z^n$ is algebraic and

$$F(z,u) = D(z)M(z,u) \qquad F_k(z) = D(z)M_k(z) \quad \text{for } k \geq 0,$$

where $D(z) = \frac{1}{1-Q(z)}$ is the generating function of excursions ending with a catastrophe, $Q(z) = zq\left(M(z) - E(z) - M_1(z)\right)$

Generating functions



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where $D(z) = \frac{1}{1-Q(z)}$ is the generating function of excursions ending with a catastrophe, $Q(z) = zq\left(M(z) - E(z) - M_1(z)\right)$

Proof: Walk = Sequence(Arches ending with a catastrophe) $\times$ Meander.
Arches ending with cat = meander ending at > 1, followed by a catastrophe:
$Q(z) = zq(M(z) - E(z) - M_1(z))$. □

Dyck paths with catastrophes

Jumps and weights: $P(u) = u^{-1} + u^{1}$, and $q = 1$.

Corollary (Generating functions for Dyck paths with catastrophes)

1. $m_n :=$ # *Dyck meanders with catastrophes of length n starting from* 0.

$$F(z,1) = \sum_{n\geq 0} m_n z^n = \frac{z(u_1(z)-1)}{z^2 + (z^2+z-1)u_1(z)} = 1 + z + 2z^2 + 4z^3 + O(z^4),$$

where $u_1(z) = \frac{1-\sqrt{1-4z^2}}{2z}$.

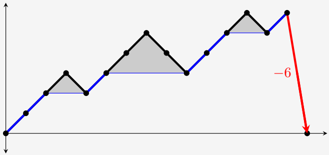
2. $e_n :=$ # *Dyck excursions with catastrophes of length n ending at* 0.

$$F_0(z) = \sum_{n\geq 0} e_n z^n = \frac{(2z-1)u_1(z)}{z^2 + (z^2+z-1)u_1(z)} = 1 + z^2 + z^3 + 3z^4 + O(z^5).$$

Moreover, e_{2n} *is also the number of Dumont permutations of the first kind of length* $2n$ *avoiding the patterns 1423 and 4132. [Burstein05].*

Bijection with Motzkin paths

1. **Dyck paths with catastrophes** are Dyck paths with the additional option of jumping to the x-axis from any altitude $h > 0$; and
2. **1-horizontal Dyck paths** are Dyck paths with the additional allowed horizontal step $(1, 0)$ at altitude 1.



Dyck arch with catastrophe

(solving conjectures by Alois P. Heinz, R. J. Mathar, and other contributors in the On-Line-Encyclopedia of Integer Sequences)

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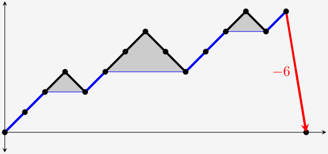

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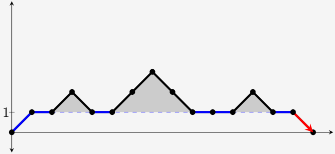

1-horizontal Dyck arch

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Excursions

Theorem (Bijection for Dyck paths with catastrophes)

The number e_n of Dyck paths with catastrophes of length n is equal to the number h_n of 1-horizontal Dyck paths of length n.

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Proof (Generating functions):
A first proof consists in using the continued fraction point of view (each level $k+1$ of the continued fraction encodes the jumps allowed at altitude k). Then,

$$H(z) = \sum_{n\geq 0} h_n z^n = \cfrac{1}{1 - \cfrac{z^2}{1 - z - \cfrac{z^2}{1 - \cfrac{z^2}{1 - \ddots}}}}$$

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Where $C(z) = \frac{1}{1-zC(z)} = \frac{1-\sqrt{1-4z^2}}{2z^2}$.

Excursions

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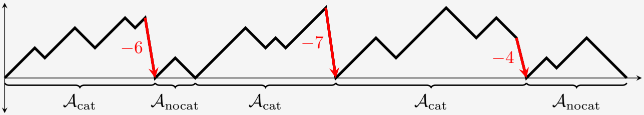
Where $C(z) = \frac{1}{1-zC(z)} = \frac{1-\sqrt{1-4z^2}}{2z^2}$. $\square$

Excursions

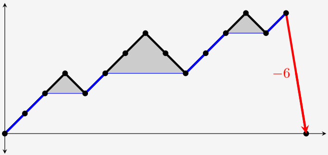
Theorem (Bijection for Dyck paths with catastrophes)

The number e_n of Dyck paths with catastrophes of length n is equal to the number h_n of 1-horizontal Dyck paths of length n.

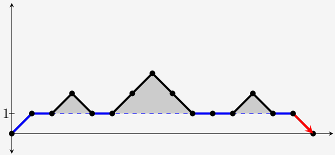
Proof (Bijection): Decomposition into a sequence of arches:



Bijection between Dyck arches with catastrophes and 1-horizontal Dyck arches:



Dyck arch with catastrophe



1-horizontal Dyck arch

□

Asymptotics and limit laws

Proposition (Asymptotics of Dyck paths with catastrophes)

The number of Dyck paths with catastrophes e_n, and Dyck meanders with catastrophes m_n is asymptotically equal to

$$e_n = C_e \rho^{-n} \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \qquad m_n = C_m \rho^{-n} \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right),$$

where $\rho \approx 0.46557$ is the unique positive root of $\rho^3 + 2\rho^2 + \rho - 1$,
$C_e \approx 0.10381$ is the unique positive root of $31C_e^3 - 62C_e^2 + 35C_e - 3$,
$C_m \approx 0.32679$ is the unique positive root of $31C_m^3 - 31C_m^2 + 16C_m - 3$.

Proof: Singularity analysis: simple pole at $\rho = \frac{1}{6}\left(116 + 12\sqrt{93}\right)^{1/3} + \frac{2}{3}\left(116 + 12\sqrt{93}\right)^{-1/3} - \frac{2}{3} \approx 0.46557$ which is strictly smaller than $1/2$ which is the dominant singularity of $u_1(z)$:

$$F_0(z) = \frac{C}{1 - z/\rho} + \mathcal{O}(1), \qquad \text{for } z \to \rho. \qquad \square$$

Supercritical composition

Variant of the supercritical composition scheme [Proposition IX.6 Flajolet–Sedgewick09], where a perturbation function $q(z)$ is added.

Proposition (Perturbed supercritical composition)

If $\boldsymbol{F(z,u) = q(z)g(uh(z))}$ *where* $g(z)$ *and* $h(z)$ *satisfy the supercriticality condition* $\boldsymbol{h(\rho_h) > \rho_g}$*, that* g *is analytic in* $|z| < R$ *for some* $R > \rho_g$*, with a unique dominant singularity at* ρ_g*, which is a simple pole, and that* h *is aperiodic. Furthermore, let* $q(z)$ *be analytic for* $|z| < \rho_h$*. Then the number* χ *of* $\mathcal{H}$*-components in a random* $\mathcal{F}_n$*-structure, corresponding to the probability distribution* $[u^k z^n]F(z,u)/[z^n]F(z,1)$ *has a mean and variance that are asymptotically proportional to* n*; after standardization, the parameter* χ *satisfies a limiting* **Gaussian distribution***, with speed of convergence* $\mathcal{O}(1/\sqrt{n})$*.*

Proof: As $q(z)$ is analytic at the dominant singularity, it contributes only a constant factor.
+Hwang's quasi-powers theorem on $F(z,u) = g(uh(z))$. □

Supercritical sequences

Proposition (Perturbed supercritical sequences)

For a schema $F(z,u) = \frac{q(z)}{1-uh(z)}$ such that $h(\rho_h) > 1$,
(with $q(z)$ analytic for $|z| < \rho$, where ρ is the positive root of $h(\rho) = 1$),
the number X_n of $\mathcal{H}$-components in a random $\mathcal{F}_n$-structure of large size n is, asymptotically Gaussian with

$$\mathbb{E}(X_n) \sim \frac{n}{\rho h'(\rho)}, \qquad \mathbb{V}(X_n) \sim n\frac{\rho h''(\rho) + h'(\rho) - \rho h'(\rho)^2}{\rho^2 h'(\rho)^3}.$$

Proof: previous Prop with $g(z) = (1-z)^{-1}$ and ρ_g replaced by 1. The second part results from the bivariate generating function

$$F(z,u) = \frac{q(z)}{1-(u-1)h_m z^m - h(z)},$$

and from the fact, that u close to 1 induces a smooth perturbation of the pole of $F(z,1)$ at ρ, corresponding to $u = 1$. □

Analytic properties

Generating function of excursions ending with a catastrophe

$$D(z) = \frac{1}{1 - Q(z)}, \qquad Q(z) = zq\left(M(z) - E(z) - M_1(z)\right).$$

Lemma

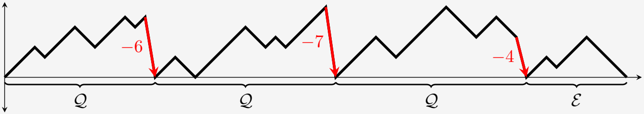
The equation $1 - Q(z) = 0$ has at most one solution $\rho_0 > 0$ for $|z| \leq \rho$. For $\delta \geq 0$ this solution always exists and $\rho_0 < \rho$. For $\delta < 0$ it depends on the value $Q(\rho)$:

$$\begin{cases} \rho_0 < \rho, & \text{for } Q(\rho) > 1, \\ \rho_0 = \rho, & \text{for } Q(\rho) = 1, \\ \nexists \rho_0, & \text{for } Q(\rho) < 1. \end{cases}$$

And $Q(z)$ satisfies the expansion for $z \to \rho$ with $\eta > 0$

$$Q(z) = Q(\rho) - \eta\sqrt{1 - z/\rho} + \mathcal{O}(1 - z/\rho).$$

Number of catastrophes



Let $d_{n,k}$ be the number of excursions ending with a catastrophe of length n with k catastrophes, then

$$D(z,v) := \sum_{n,k\geq 0} d_{n,k} z^n v^k = \frac{1}{1 - vQ(z)}.$$

Let $c_{n,k}$ be the number of excursions with k catastrophes. Then, we get

$$C(z,v) := \sum_{n,k\geq 0} c_{n,k} z^n v^k = \frac{1}{1 - vQ(z)} E(z).$$

Let X_n be the random variable, representing paths of length n consisting of k catastrophes. In other words the probability is defined as

$$\mathbb{P}(X_n = k) = \frac{[z^n v^k] C(z,v)}{[z^n] C(z,1)}.$$

Average number of catastrophes

Theorem

1. In the case of $\rho_0 < \rho$ the standardized random variable

$$\frac{X_n - \mu n}{\sigma\sqrt{n}}, \qquad \mu = \frac{1}{\rho_0 Q'(\rho_0)}, \qquad \sigma^2 = \frac{\rho_0 Q''(\rho_0) + Q'(\rho_0) - \rho_0 Q'(\rho_0)^2}{\rho_0^2 Q'(\rho_0)^3},$$

converges in law to a **Gaussian variable** $\mathcal{N}(0,1)$.

2. In the case of $\rho_0 = \rho$ the normalized random variable

$$\frac{X_n}{\vartheta\sqrt{n}}, \qquad \vartheta = \frac{\sqrt{2}}{\eta},$$

converges in law to a **Rayleigh distribution** (density: $xe^{-x^2/2}$).

3. In the case that ρ_0 does not exist, the limit distribution is a discrete one:

$$\mathbb{P}(X_n = k) = \frac{(n\eta/\lambda + C/\tau)\,\lambda^n}{\eta D(\rho)^2 + C/\tau D(\rho)}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \qquad \lambda = Q(\rho),$$

Average number of catastrophes

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2. In the case of $\rho_0 = \rho$ the normalized random variable $\frac{X_n}{\vartheta\sqrt{n}}, \vartheta = \frac{\sqrt{2}}{\eta}$, converges in law to a **Rayleigh distribution** (density: $xe^{-x^2/2}$).

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$$\mathbb{P}(X_n = k) = \frac{(n\eta/\lambda + C/\tau)\,\lambda^n}{\eta D(\rho)^2 + C/\tau D(\rho)}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \qquad \lambda = Q(\rho),$$

with $C = \sqrt{2\frac{P(\tau)}{P''(\tau)}}$, and $\tau > 0$ the unique positive real root of $P'(u) = 0$. In particular X_n converges to the random variable given by the law of $\boldsymbol{\eta}$ **NegBinom**$(\mathbf{2}, \boldsymbol{\lambda}) + \frac{C}{\tau}$ **NegBinom**$(\mathbf{1}, \boldsymbol{\lambda})$.

Average number of catastrophes – Dyck

Corollary

The **number of catastrophes** *of a random Dyck path with catastrophes of length n is normally distributed. The standardized version of X_n,*

$$\frac{X_n - \mu n}{\sigma\sqrt{n}}, \qquad \mu \approx 0.0708358118, \qquad \sigma^2 \approx 0.05078979113,$$

where μ is the unique positive real root of $31\mu^3 + 31\mu^2 + 40\mu - 3$, and σ^2 is the unique positive real root of $29791\sigma^6 - 59582\sigma^4 + 60579\sigma^2 - 2927$, converges in law to a Gaussian variable $\mathcal{N}(0,1)$:

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{X_n - \mu n}{\sigma\sqrt{n}} \leq x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy.$$

Number of returns to zero

Definition

- An *arch* is an excursion of size > 0 whose only contact with the x-axis is at its end points.
- A *return to zero* is a vertex of a path of altitude 0 whose abscissa is positive.

Figure: An excursion with 3 returns to zero

Generating function

$$A(z) = 1 - \frac{1}{F_0(z)},$$
$$G(z,v) = \frac{1}{1 - vA(z)}.$$

Excursion of length n having k returns to zero

$$\mathbb{P}(Y_n = k) = \mathbb{P}(\text{size} = n,\ \#\text{returns to zero} = k) = \frac{[z^n]A(z)^k}{[z^n]E(z)}$$

Average number of returns to zero

Theorem

1. In the case of $\rho_0 < \rho$ the standardized random variable

$$\frac{Y_n - \mu n}{\sigma\sqrt{n}}, \qquad \mu = \frac{1}{\rho_0 A'(\rho_0)}, \qquad \sigma^2 = \frac{\rho_0 A''(\rho_0) + A'(\rho_0) - \rho_0 A'(\rho_0)^2}{\rho_0^2 A'(\rho_0)^3},$$

converges in law to a **Gaussian variable** $\mathcal{N}(0,1)$.

2. In the case of $\rho_0 = \rho$ the normalized random variable

$$\frac{Y_n}{\vartheta\sqrt{n}}, \qquad \vartheta = \sqrt{2}\frac{E(\rho)}{\eta},$$

converges in law to a **Rayleigh distribution** defined by the density $xe^{-x^2/2}$.

3. In the case of ρ_0 does not exist, the limit distribution is **NegBinom(2, λ)**:

$$\mathbb{P}(Y_n = k) = \frac{n\lambda^n}{(1-A(\rho))^2}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \qquad \lambda = A(\rho).$$

Average number of returns to zero – Dyck

Corollary

The **number of returns to zero** *of a random Dyck path with catastrophes of length n is normally distributed. The standardized version of Y_n,*

$$\frac{Y_n - \mu n}{\sigma\sqrt{n}}, \qquad \mu \approx 0.1038149281, \qquad \sigma^2 \approx 0.1198688826,$$

where μ is the unique positive root of $31\mu^3 - 62\mu^2 + 35\mu - 3$, and σ^2 is the unique positive root of $29791\sigma^6 + 231\sigma^2 - 79$, converges in law to $\mathcal{N}(0,1)$.

Compare

The **number of catastrophes** of a random Dyck path with catastrophes of length n is normally distributed.

$$\frac{X_n - \mu n}{\sigma\sqrt{n}}, \qquad \mu \approx 0.0708358118, \qquad \sigma^2 \approx 0.05078979113.$$

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Final altitude limit law

Theorem

The final altitude of a random lattice path with catastrophes of length n admits a **discrete limit distribution**:

$$\lim_{n\to\infty} \mathbb{P}(Z_n = k) = [u^k]\omega(u), \quad \text{where } \omega(u) = \begin{cases} \frac{1-v_1(\rho_0)}{u-v_1(\rho_0)}, & \text{for } \rho_0 \leq \rho, \\ \frac{\eta D(\rho)+\frac{C}{\tau-u}}{\eta D(\rho)+\frac{C}{\tau-1}} \frac{1-v_1(\rho)}{u-v_1(\rho)}, & \text{for } \nexists \rho_0. \end{cases}$$

Corollary

The final altitude of a random Dyck path with catastrophes of length n admits a geometric limit distribution with parameter $\lambda = v_1(\rho)^{-1} \approx 0.6823278$:

$$\mathbb{P}(Z_n = k) \sim (1-\lambda)\,\lambda^k.$$

Final altitude

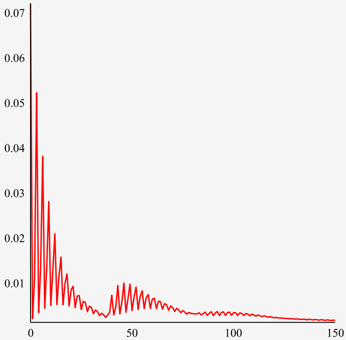
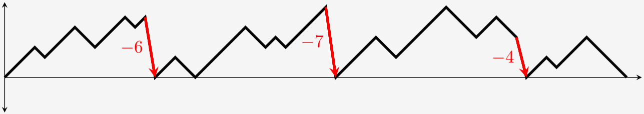


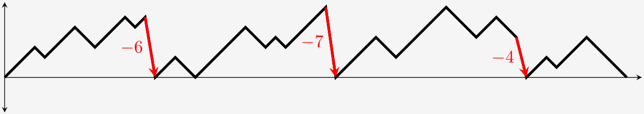
Figure: The limit law for the final altitude in the case of a jump polynomial $P(u) = u^{40} + 10u^3 + 2u^{-1}$. The picture shows a period 40 behavior, which is explained by a sum of 40 geometric-like basic limit laws.

Conclusion



- Generalized Dyck paths with unbounded jumps can be exactly enumerated and asymptotically analyzed.
- Universality of the Gaussian limit law.
- Not Brownian limit objects: some more tricky "fractal periodic geometrically amortized" limit laws (and also Gaussian laws).
- Uniform random generation algorithm.

Conclusion



- Generalized Dyck paths with unbounded jumps can be exactly enumerated and asymptotically analyzed.
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Thank you for your attention!

Backup

Backup slides

Final altitude limit law (proof)

Let us now fix $u \in (0,1)$ and treat is henceforth as a parameter. The probability generating function of X_n is

$$p_n(u) = \frac{[z^n]D(z)M(z,u)}{[z^n]D(z)M(z,1)}.$$

By [Banderier–Flajolet02], $M(z,u)$ and $M(z,1)$ are singular at $z = 1/2 > \rho$ (ρ is the singularity of $D(z)$). By [Flajolet–Sedgewick09]:

$$p_n(u) \sim \frac{M(\rho,u)[z^n]D(z)}{M(\rho,1)[z^n]D(z)} = \frac{M(\rho,u)}{M(\rho,1)}.$$

The branches allow us to factor the kernel equation into $u(1 - zP(u)) = -zp_1(u - u_1(z))(u - v_1(z))$. Thus,

$$M(\rho,u) = \frac{1}{\rho p_{-1}(v_1(\rho) - u)},$$

the limit probability generating function of a geometric distribution. □

Uniform random generation

Generalized Dyck paths (meanders and excursions) can be generated by pushdown-automata/context-free grammars.

- dynamic programming approach, $O(n^2)$ time and $O(n^3)$ bits in memory.
- [Hickey and Cohen83]: context-free grammars.
- [Flajolet–Zimmermann–Van Cutsem94]: the recursive method, a wide generalization to combinatorial structures, so such paths of length n can be generated in $O(n \ln n)$ average-time.
- [Goldwurm95] proved that this can be done with the same time-complexity, with only $O(n)$ memory.
- [Duchon–Flajolet–Louchard–Schaeffer04] : Boltzmann method. Linear average-time random generator for paths of length $[(1-\epsilon)n, (1+\epsilon)n]$.
- [Banderier-Wallner16] :
  generating trees+holonomy theory $\rightarrow O(n^{3/2})$ time, $O(1)$ memory.

Uniform random generation (generating tree+holonomy)

Each transition is computed via

$$\mathbb{P}\left(\begin{cases} \text{jump } j \text{ when at altitude } k \text{, and length } m, \\ \text{ending at 0 at length } n \end{cases} \right) = \frac{f^0_{m,k} f^{k+j}_{n-(m+1),0}}{f^0_{n,0}}.$$

Then, for each pair (i, k), theory of D-finite functions applied to our algebraic functions gives the recurrence for f_m (computable in $O(\sqrt{m})$ via an algorithm of [Chudnovsky & Chudnovsky 86] for P-recursive sequence). Possible win on the space complexity and bit complexity: computing the f_m's in floating point arithmetic, instead of rational numbers (although all the f_m are integers, it is often the case that the leading term of the P-recursive recurrence is not 1, and thus it then implies rational number computations, and time loss in gcd computations). Global cost $\sum_{m=1}^{n} O(\sqrt{m}) O(\sqrt{n-m}) = O(n^2)$ & $O(1)$ memory is enough to output the n jumps of the lattice path, step after step, as a stream. □