Enumeration of polyominoes with fixed perimeter defect

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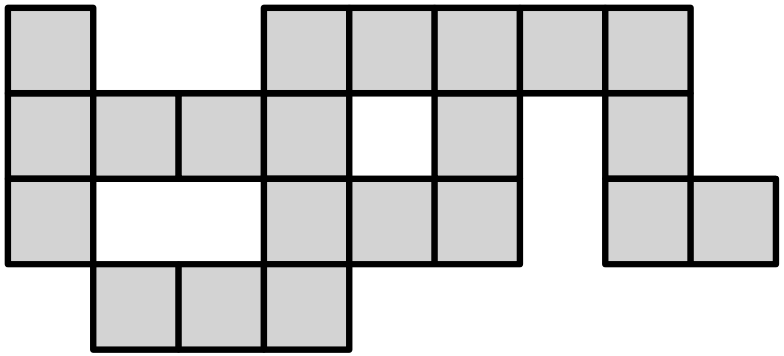
A polyomino is a set of unit cells in the grid, connected via edges.



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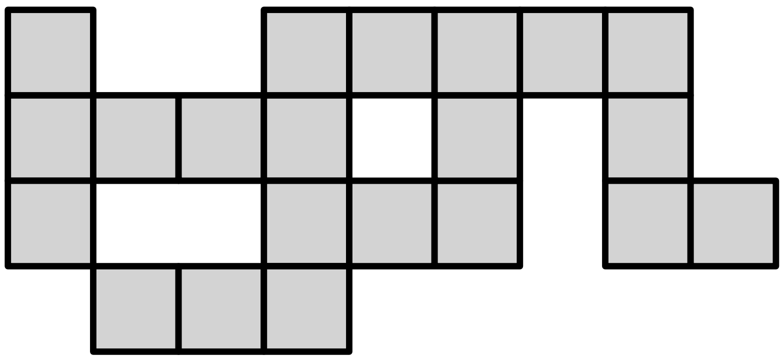

The size of a polyomino is the number of cells.

Enumeration of polyominoes with respect to their area is an open problem, widely believed to be difficult.

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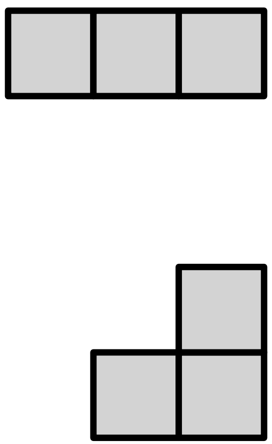
$n = 3$



$\times 2$

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Only 56 numbers are known (Jensen);
The nature of the generating function is not known;
The limit $\lambda := \lim A(n+1)/A(n)$ exists (Madras);
Bounds: $\lambda > 4.0025$ (Barequet, Rote, Shalah),
$\lambda < 4.6496$ (Klarner, Rivest);
Conjectured: $\lambda \approx 4.0626$.

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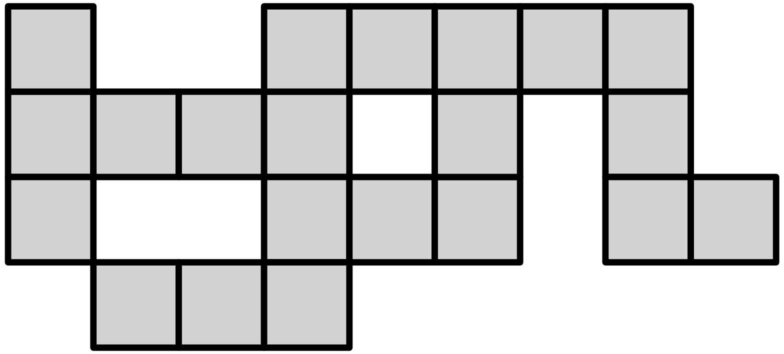

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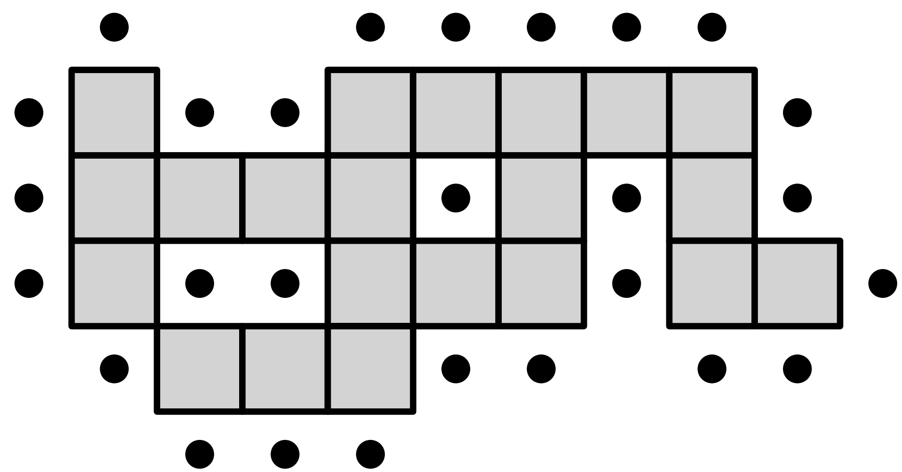

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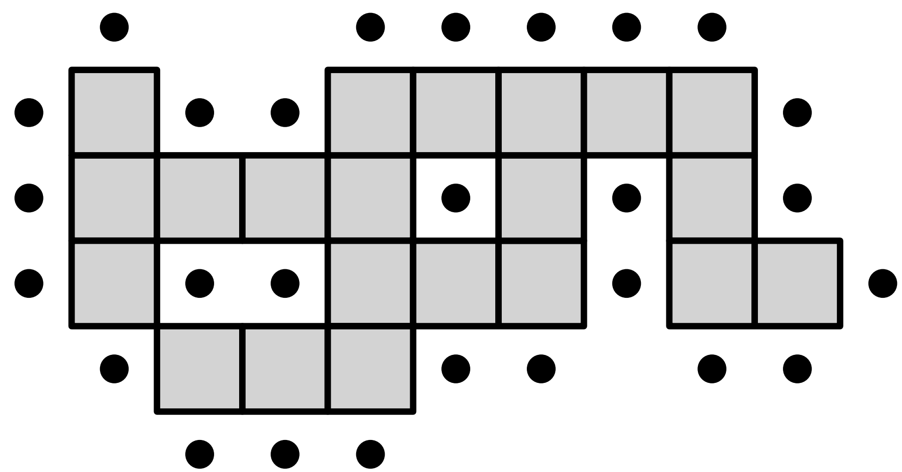
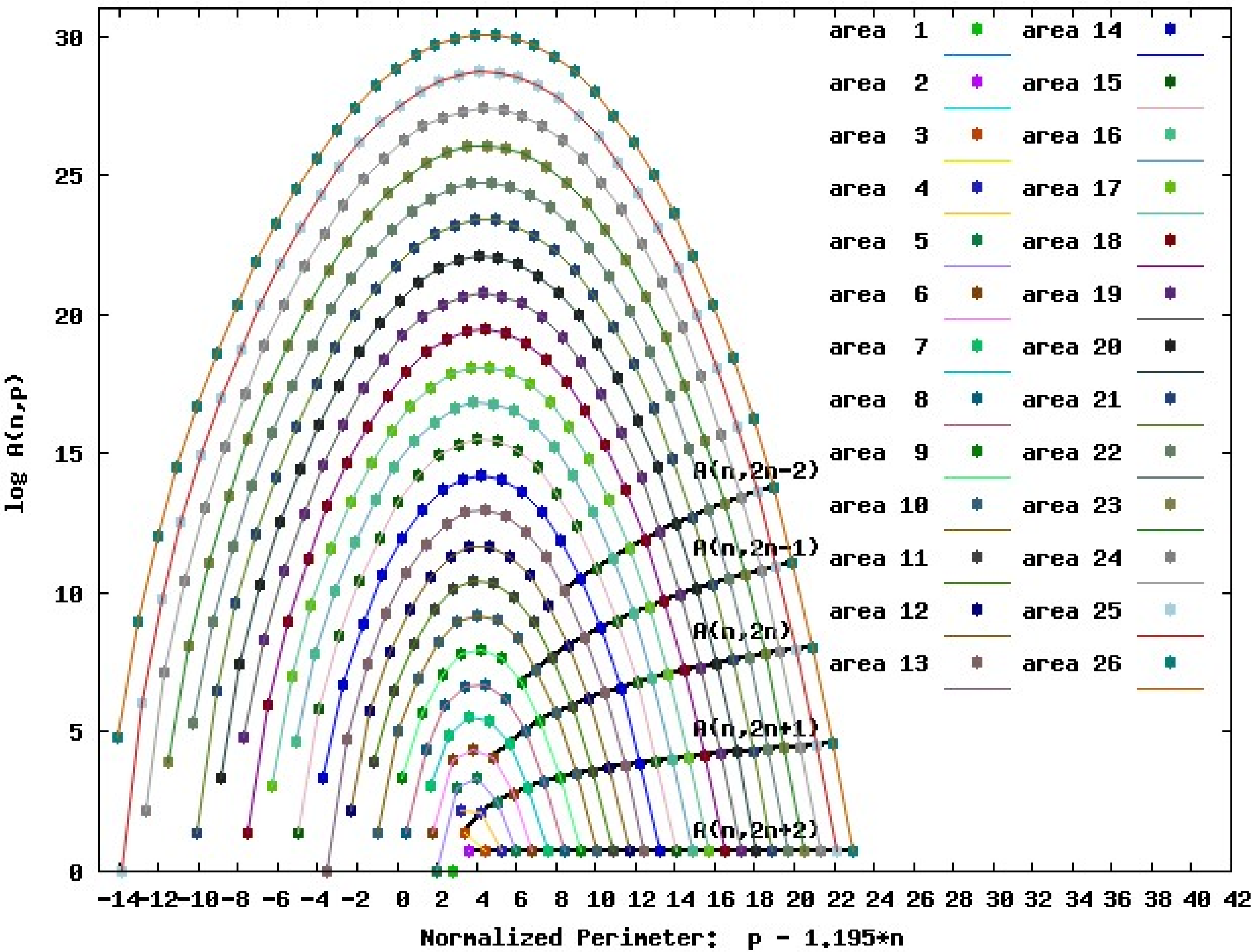


We denote by $A(n, p)$ the number of polyominoes with area n and perimeter p.



log A(n,p)
Normalized Perimeter: p - 1.195*n
A(n,2n-2)
A(n,2n-1)
A(n,2n)
A(n,2n+1)
A(n,2n+2)
area 1
area 2
area 3
area 4
area 5
area 6
area 7
area 8
area 9
area 10
area 11
area 12
area 13
area 14
area 15
area 16
area 17
area 18
area 19
area 20
area 21
area 22
area 23
area 24
area 25
area 26
0
5
10
15
20
25
30
-14 -12 -10 -8 -6 -4 -2 0 2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40 42

Observation:

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Definition:
The *(perimeter) defect* of P is $k := 2n + 2 - p$.

A *perimeter cell*: a free cell that has an occupied neighbor.

An *excess* of a perimeter cell is the number of occupied neighbors minus 1.

The *total excess* of a polyomino is the sum of excesses over all perimeter cells.

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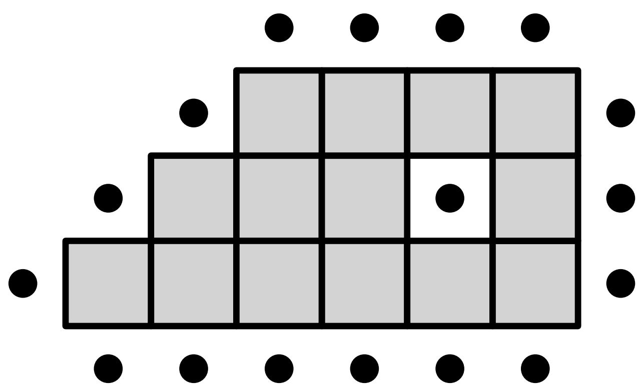

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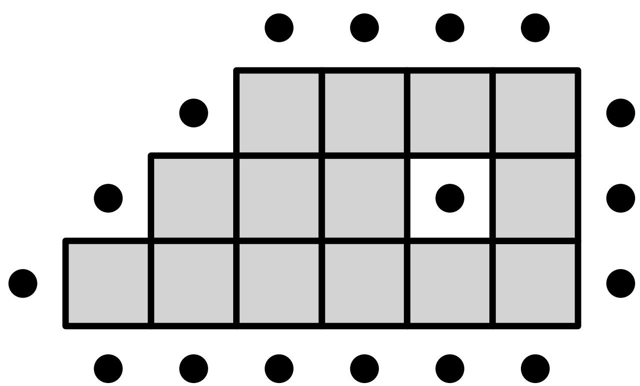

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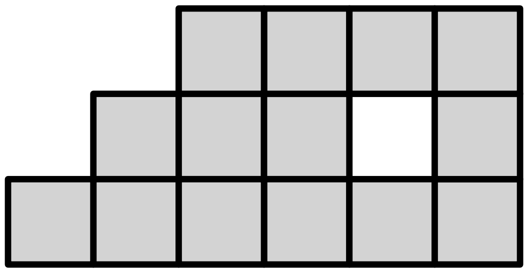

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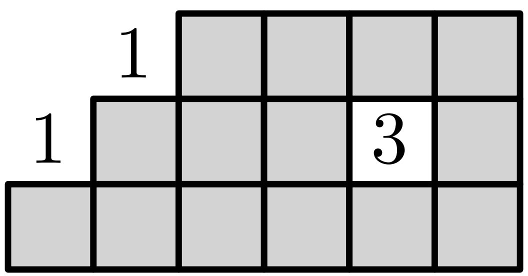

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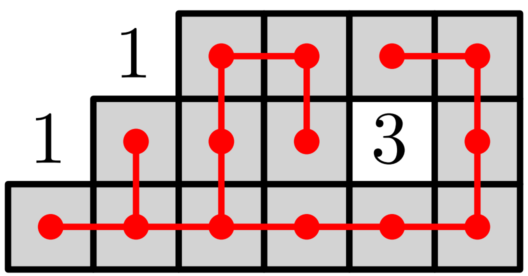

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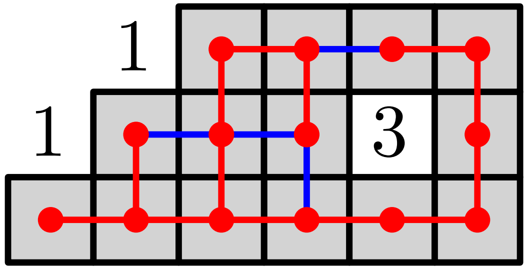

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$k = 1 \Rightarrow$
$e = 1, f = 0 \Rightarrow$
the polyomino is a "one-bend path" $\Rightarrow$
for $n \geq 3$, we have $A(n, 2n + 1) = 4(n - 2)$.

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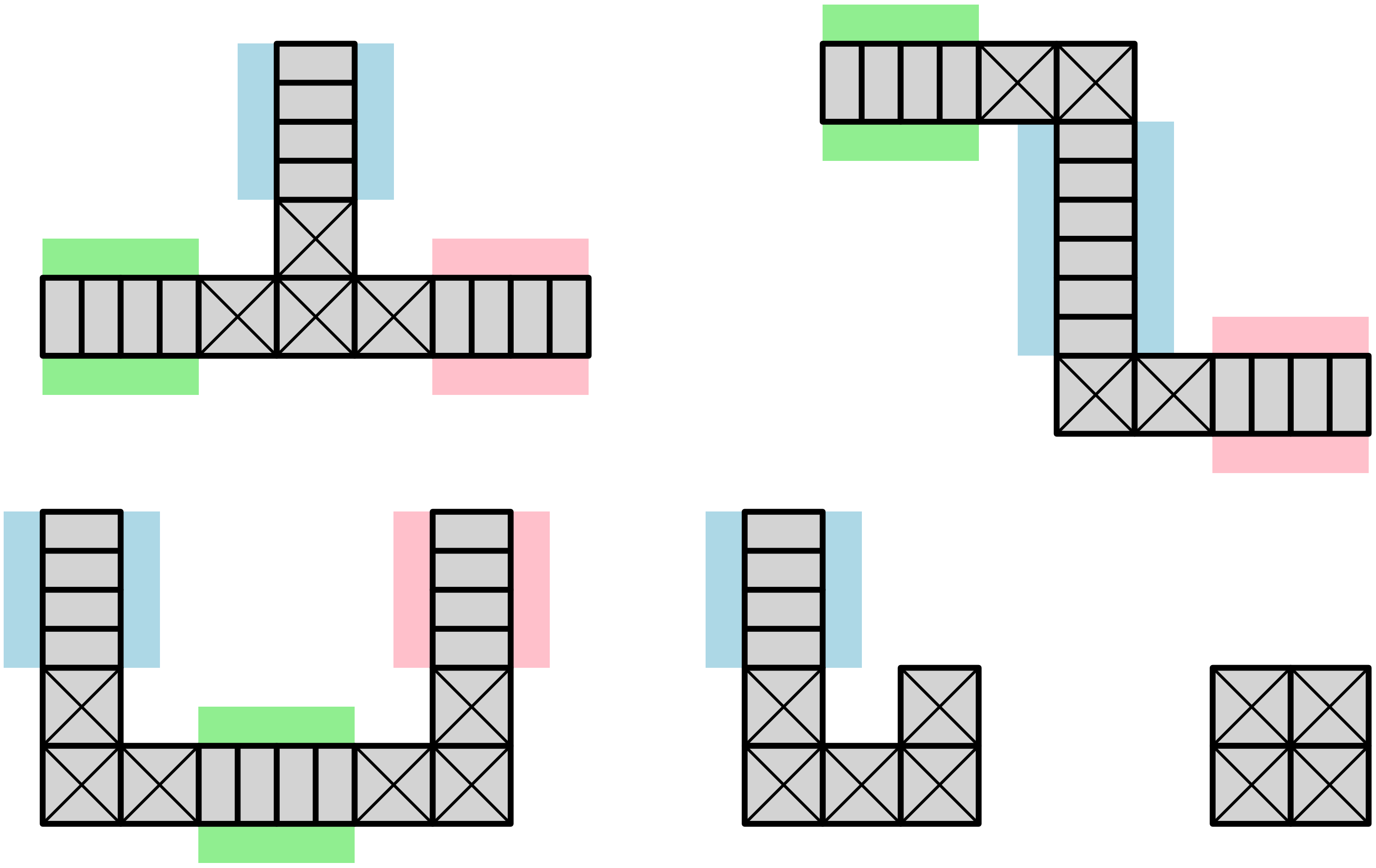

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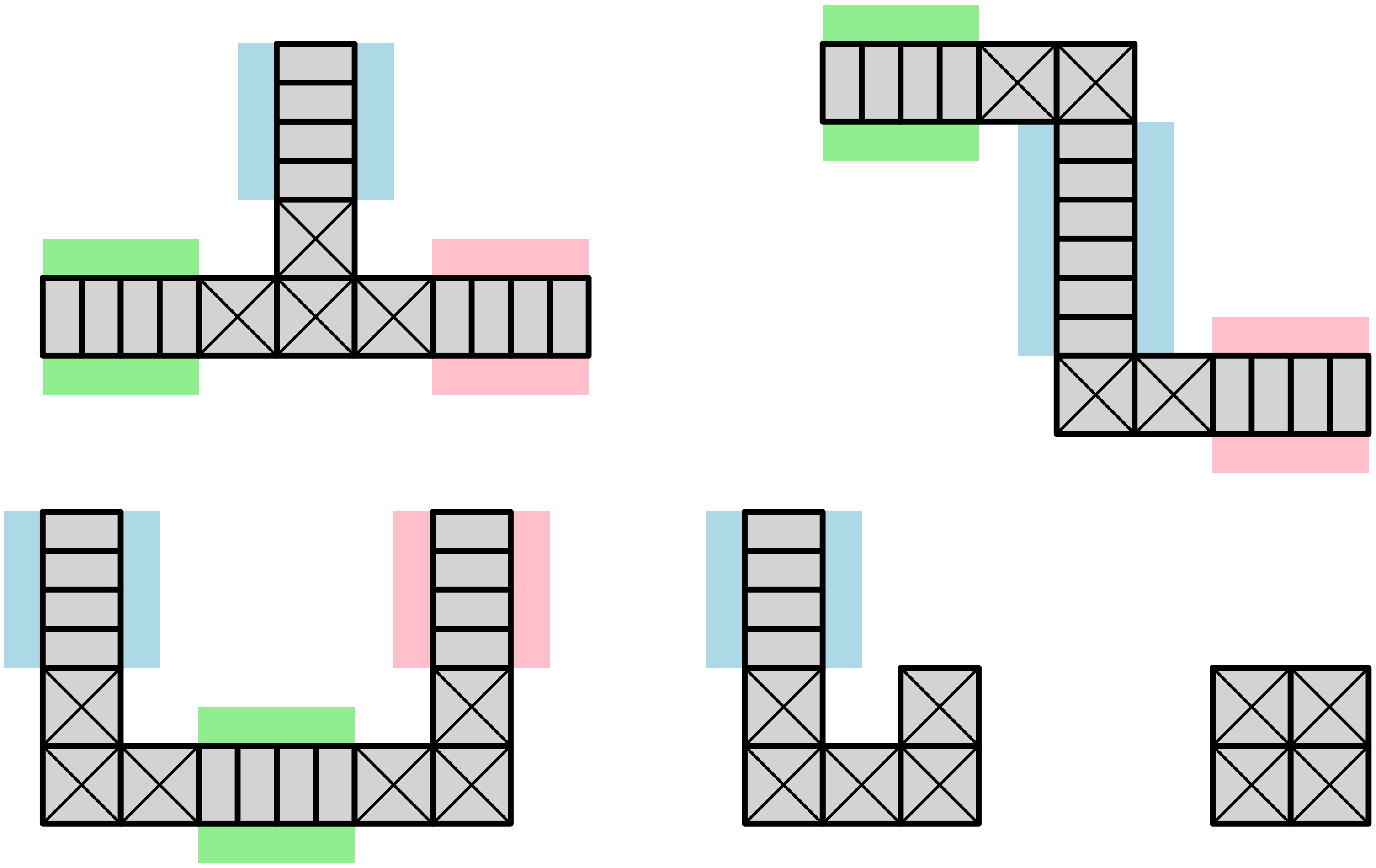

$A(n, 2n) =$ a quadratic polynomial.

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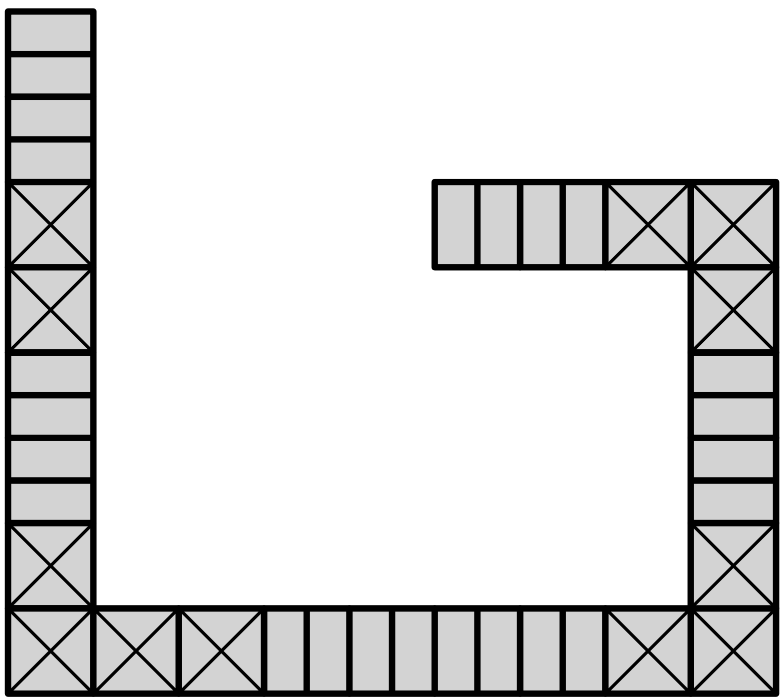

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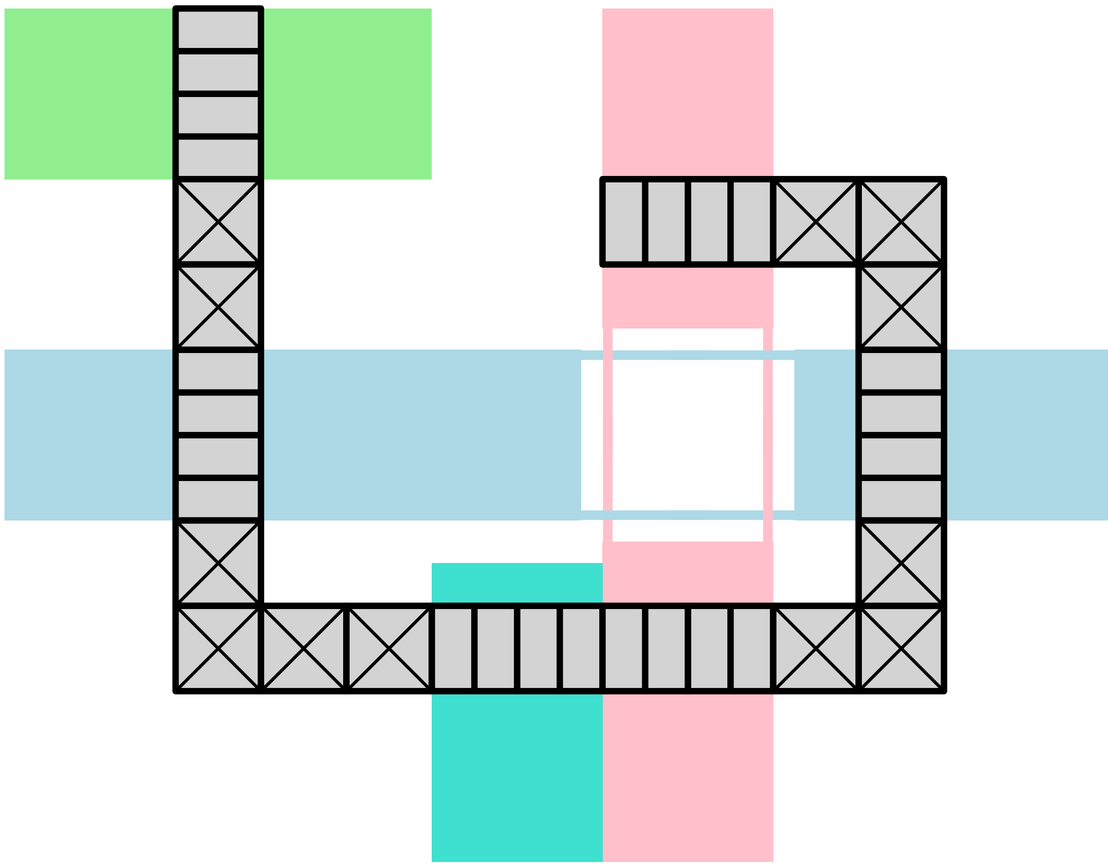

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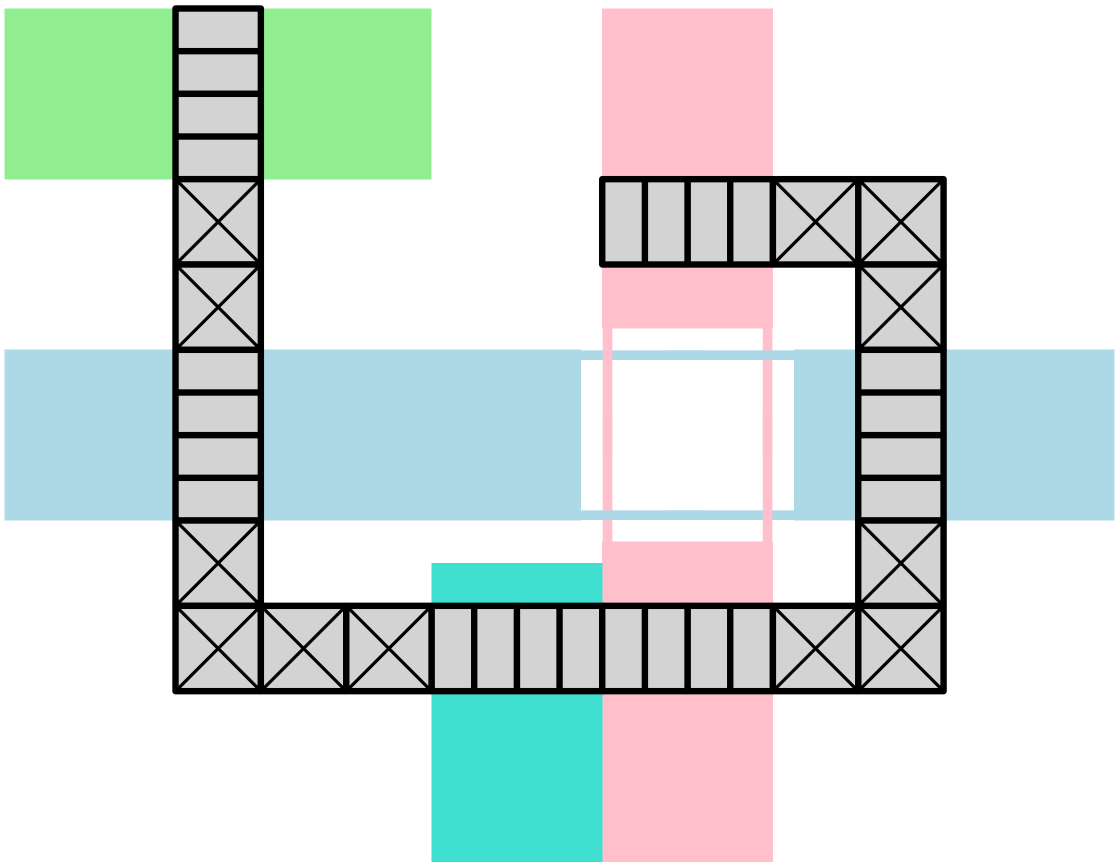

$$\frac{x^{12}}{(1-x)^2(1-x^2)^2}$$

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$$A(n, 2n-1) =$$

$$-\frac{2107}{2} - \frac{21}{2}(-1)^n + \left(\frac{1947}{4} + \frac{5}{4}(-1)^n\right)n - 89n^2 + \frac{13}{2}n^3$$

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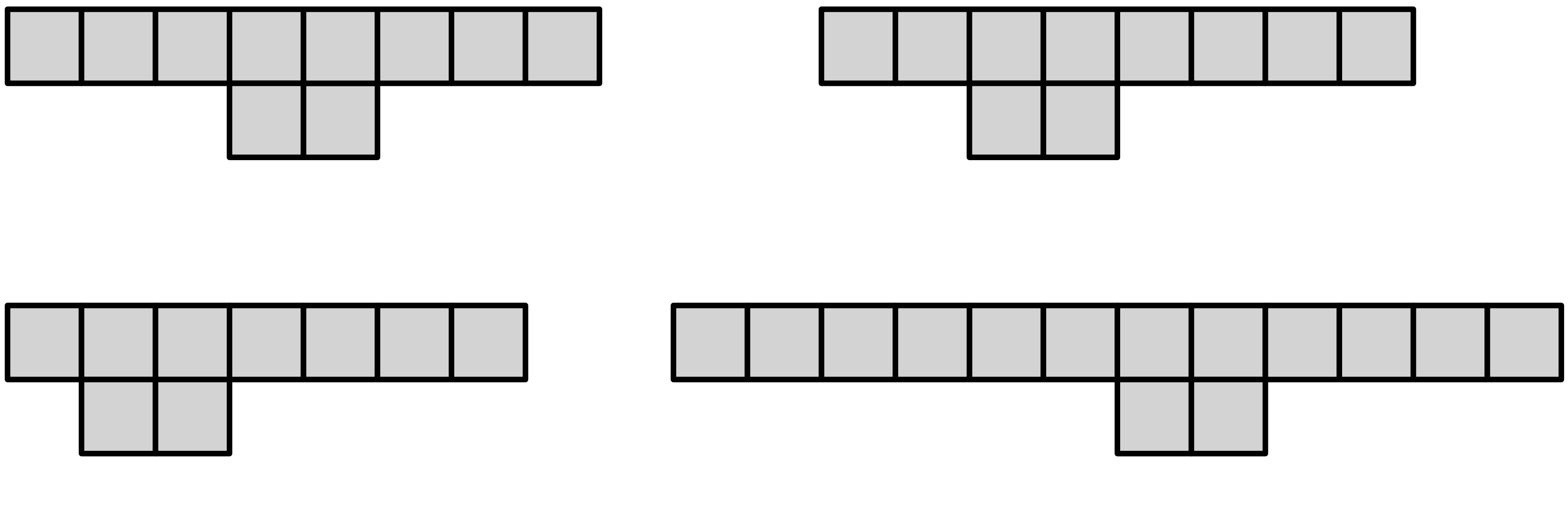
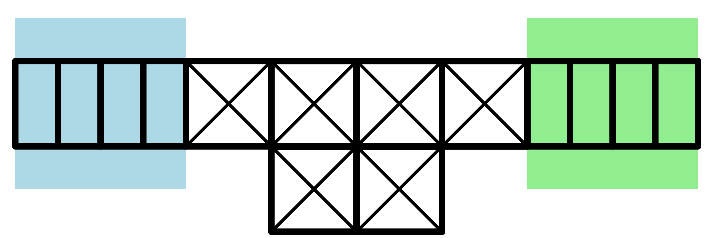
$\longrightarrow$ A “correct” notion of pattern (to ensure partition).



We expect that these
will be in the same class.



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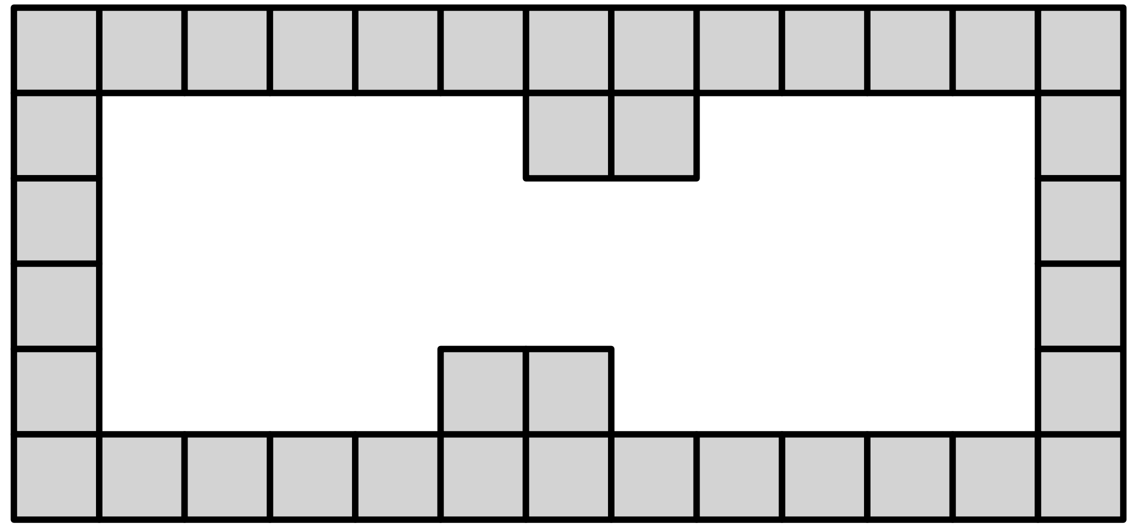

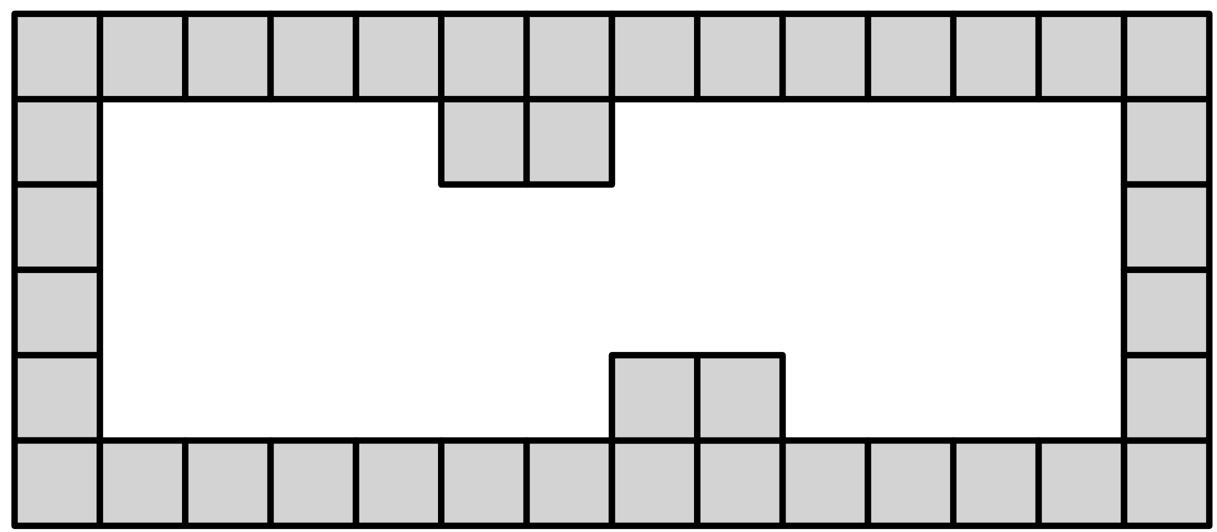

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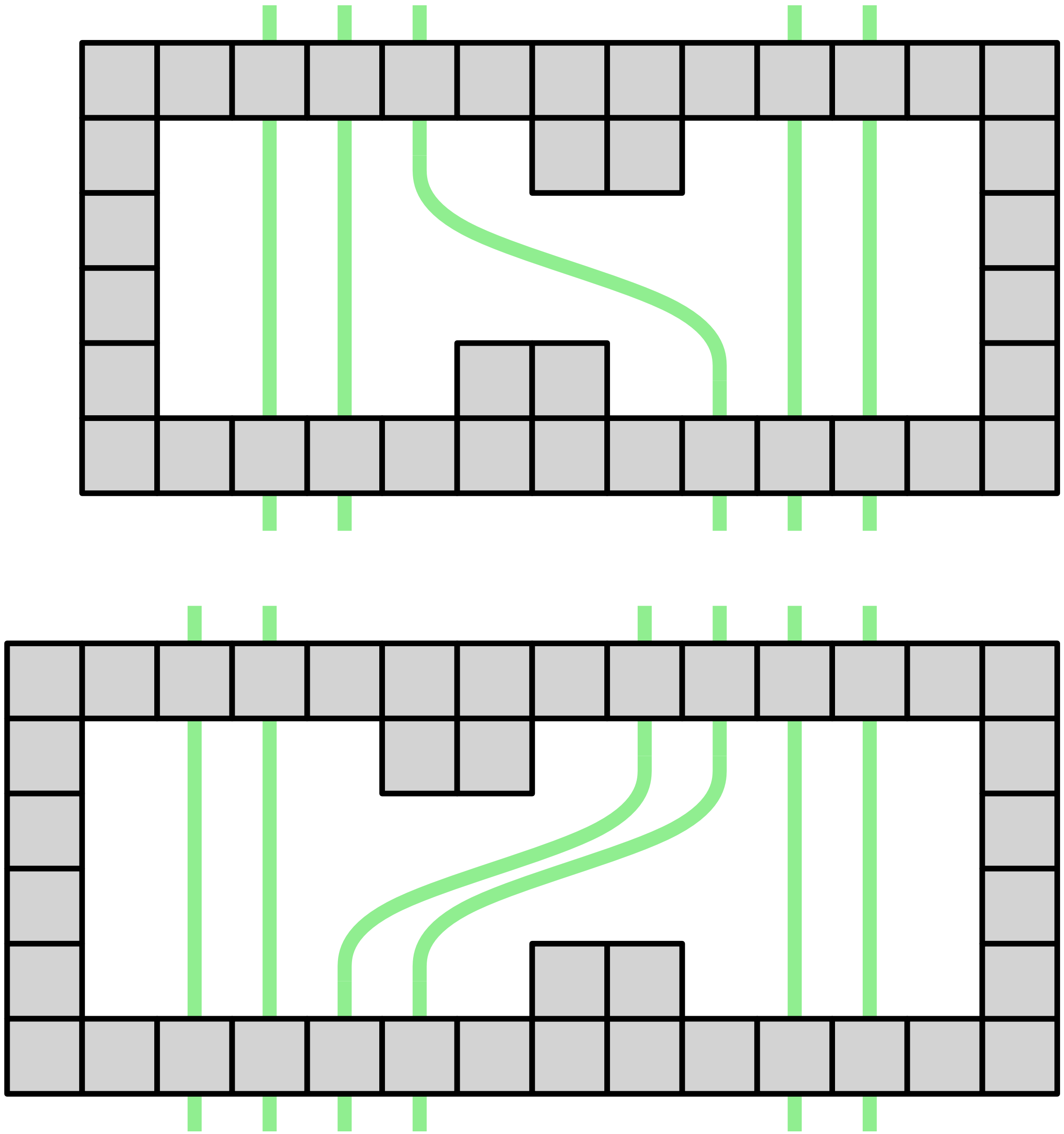

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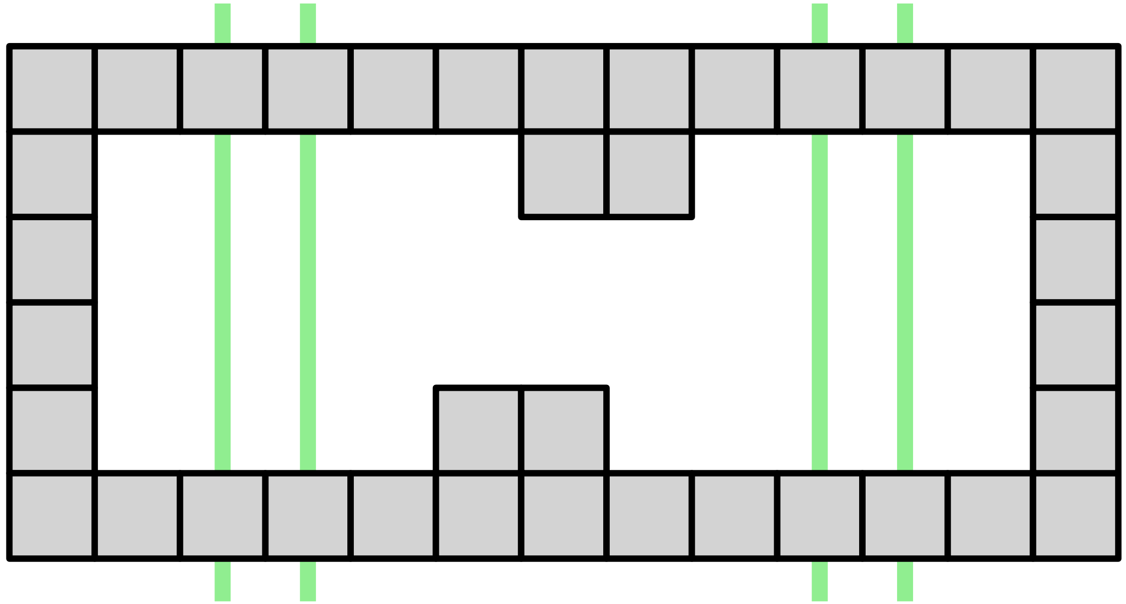

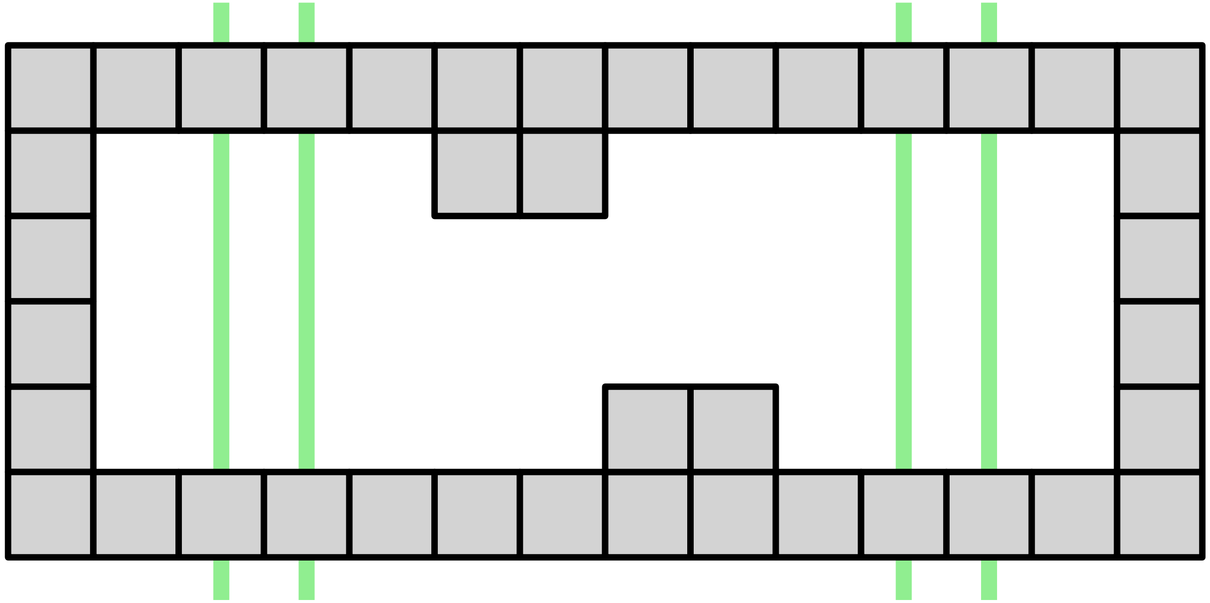

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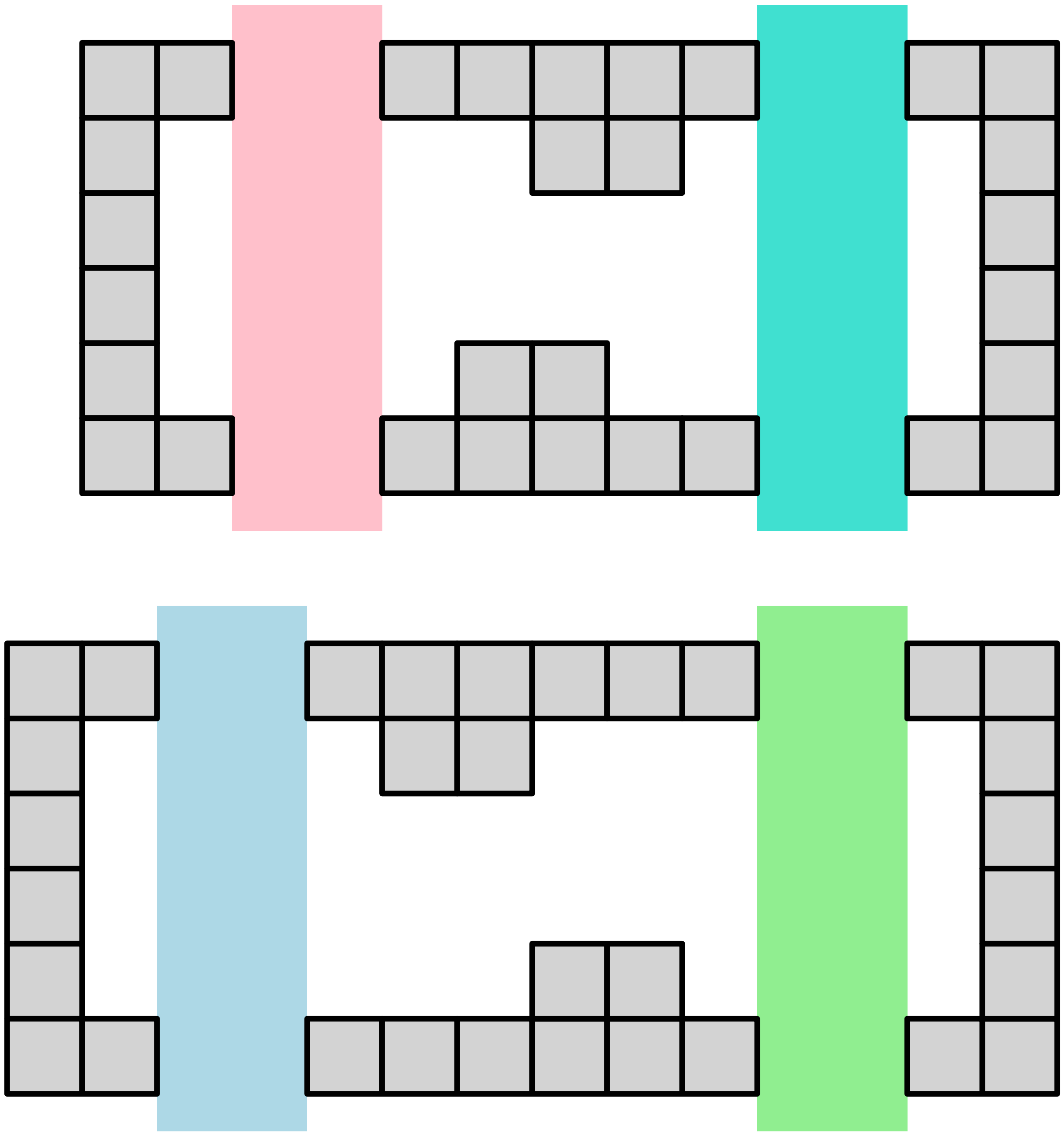

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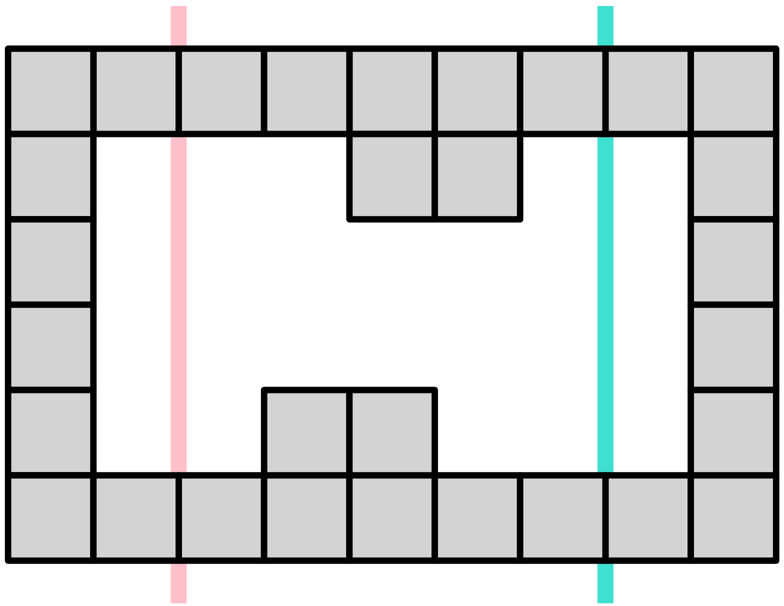
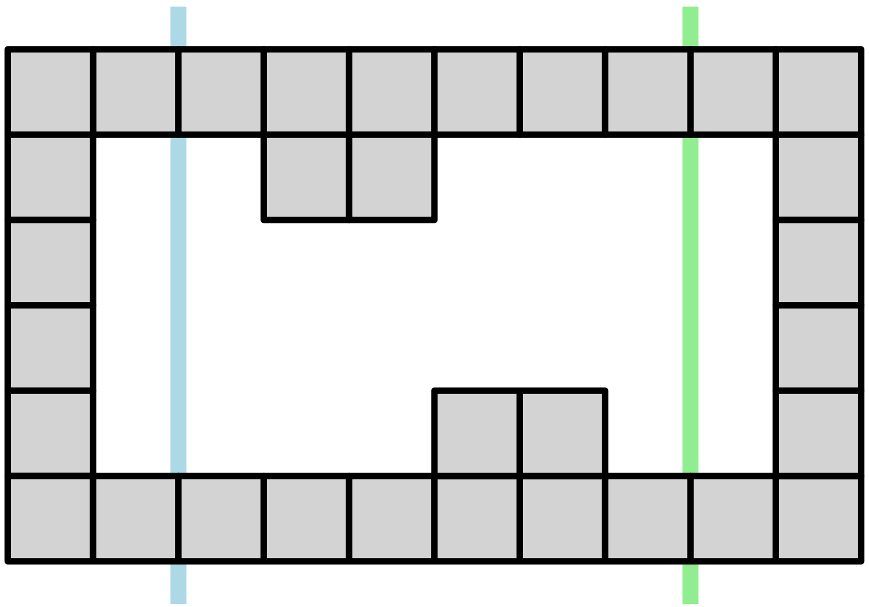
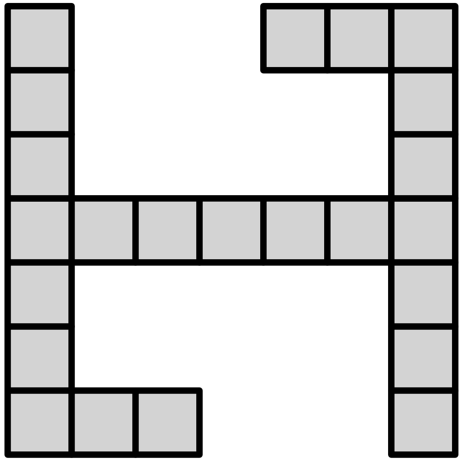
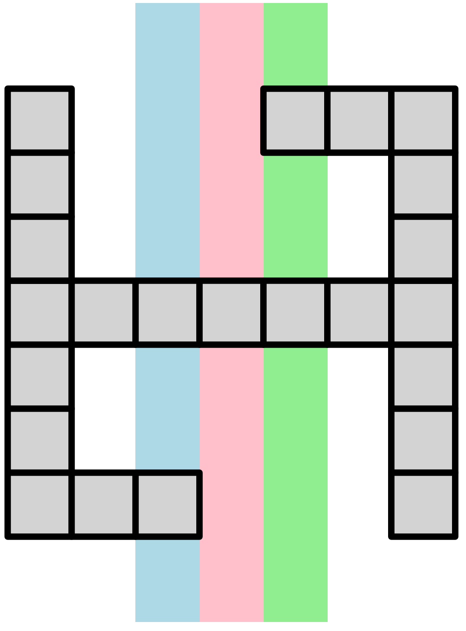
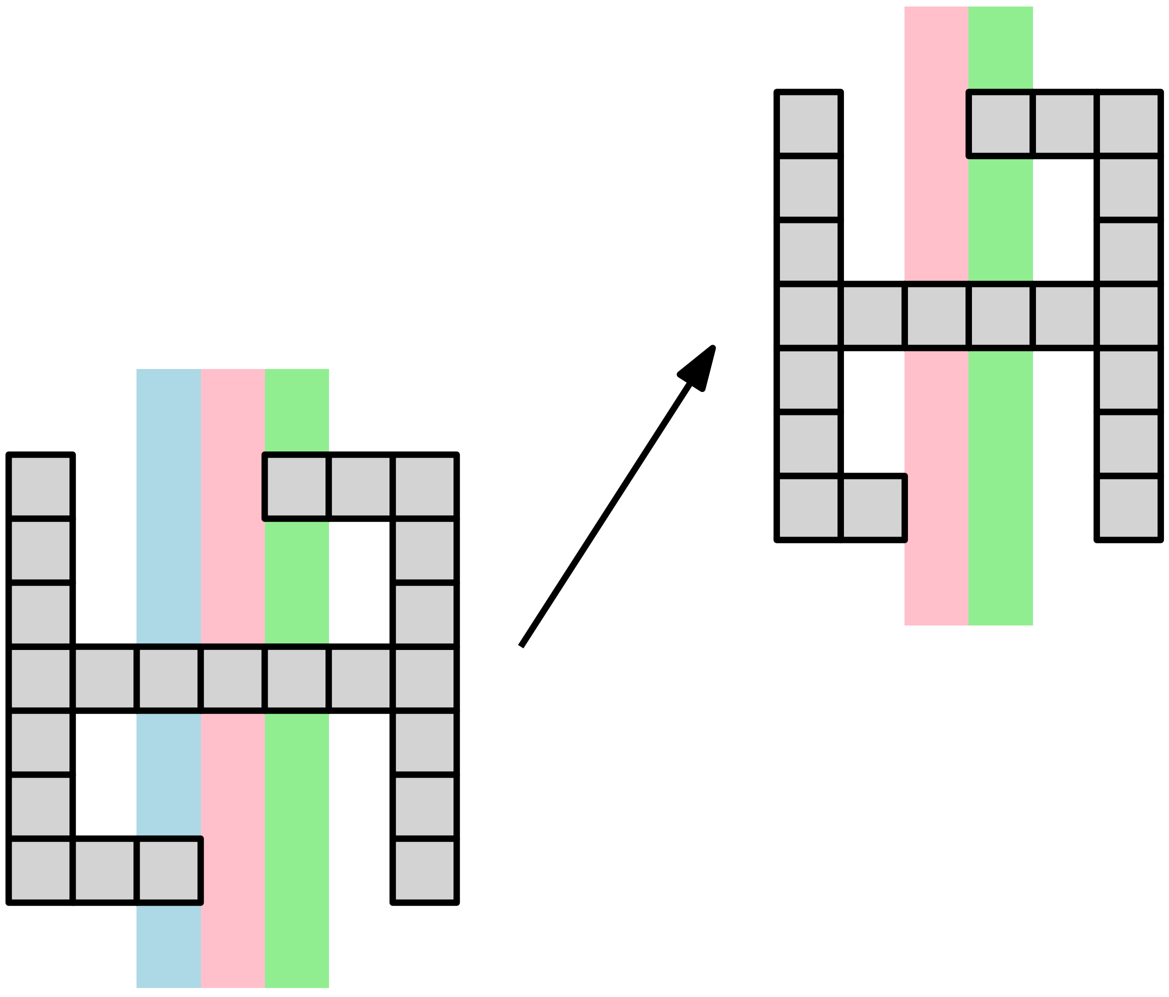
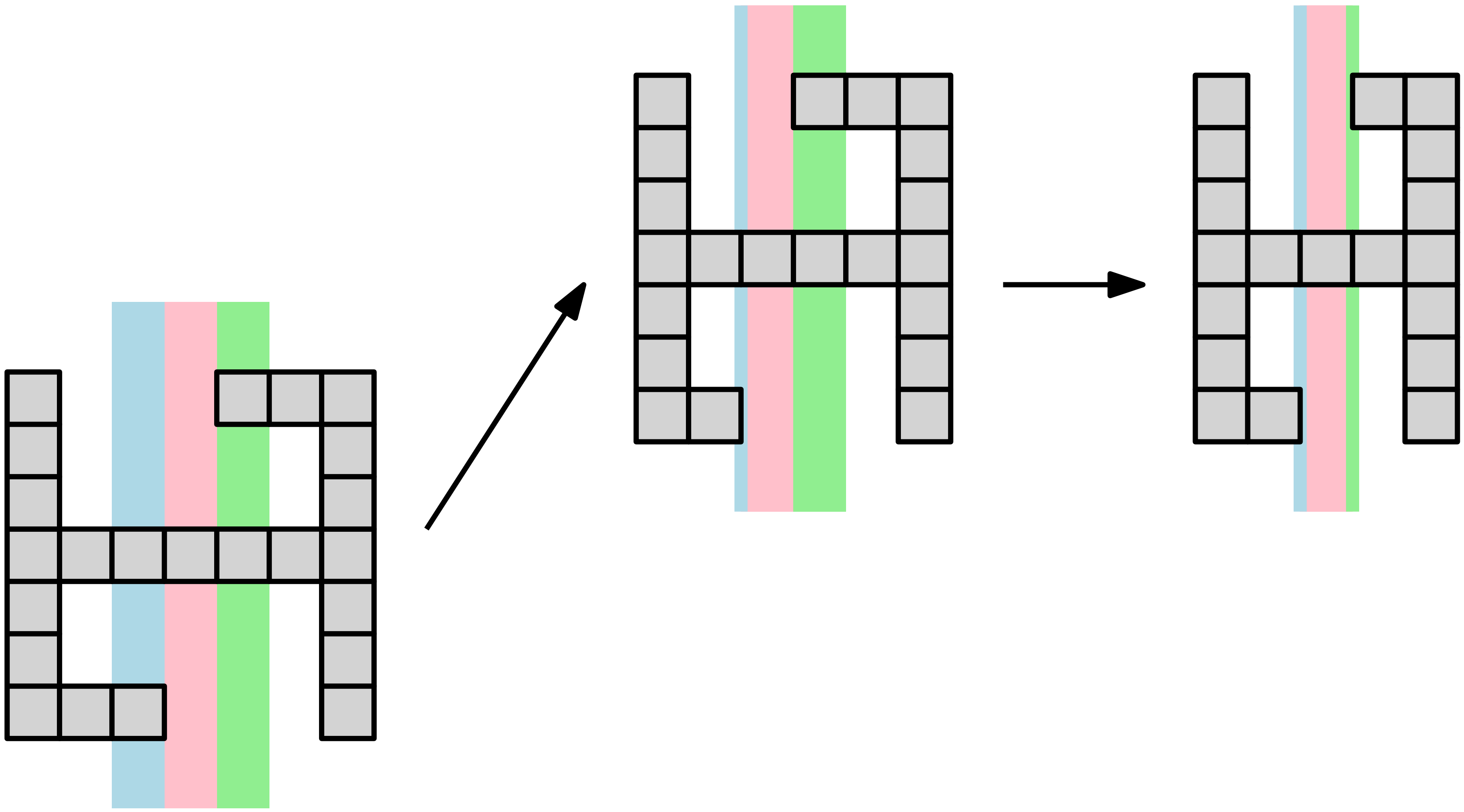


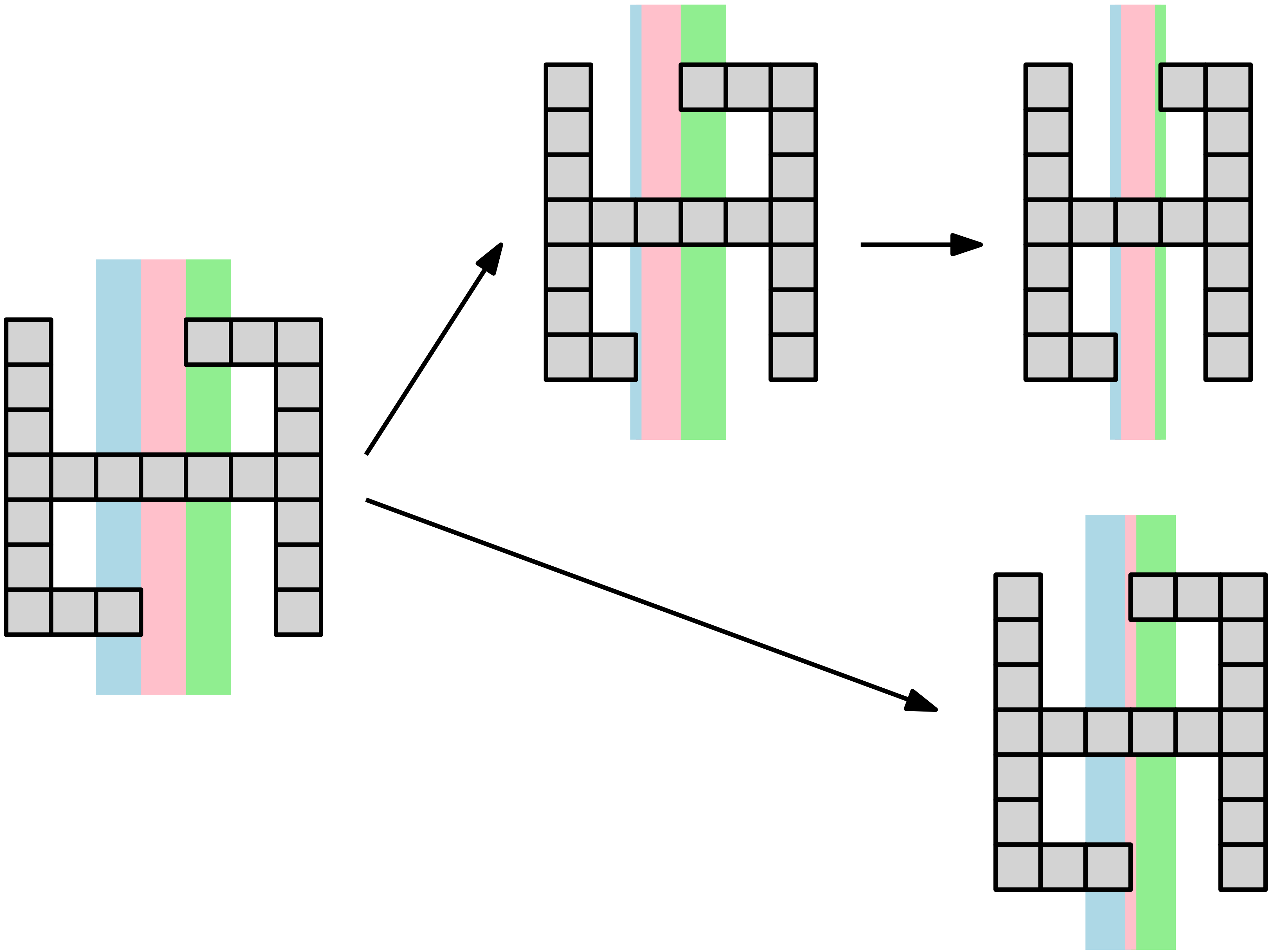


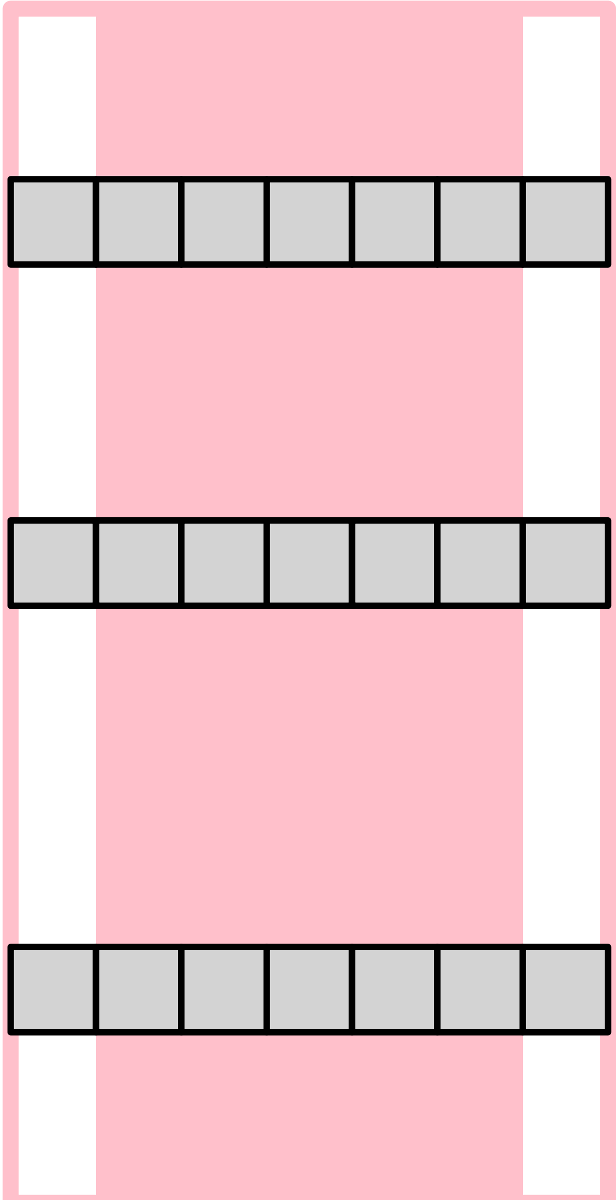


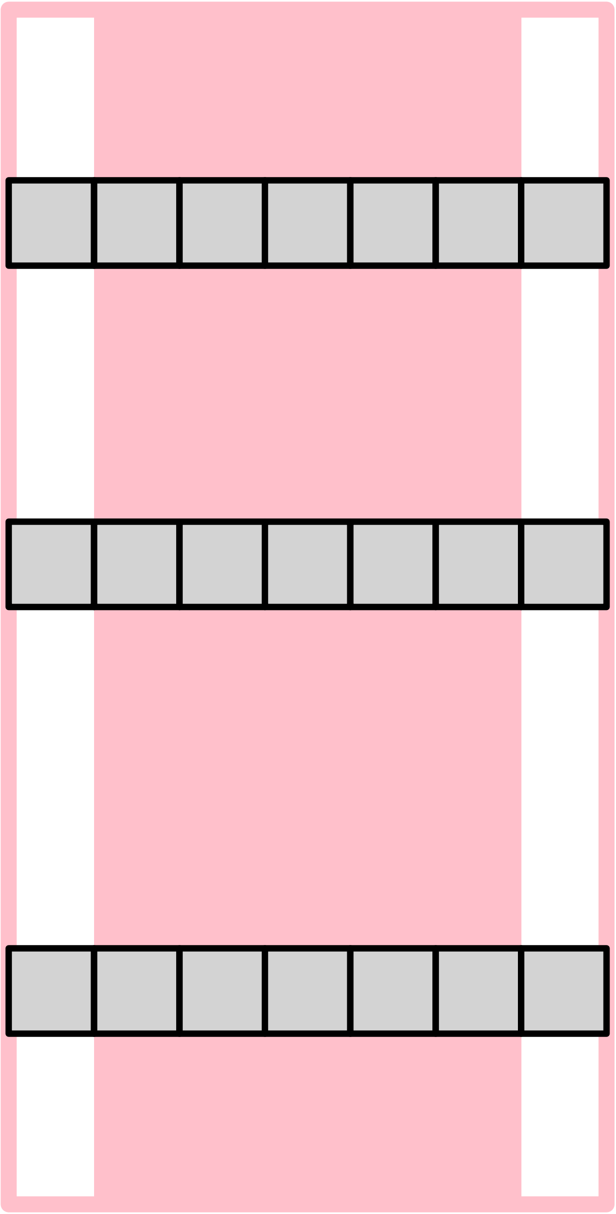




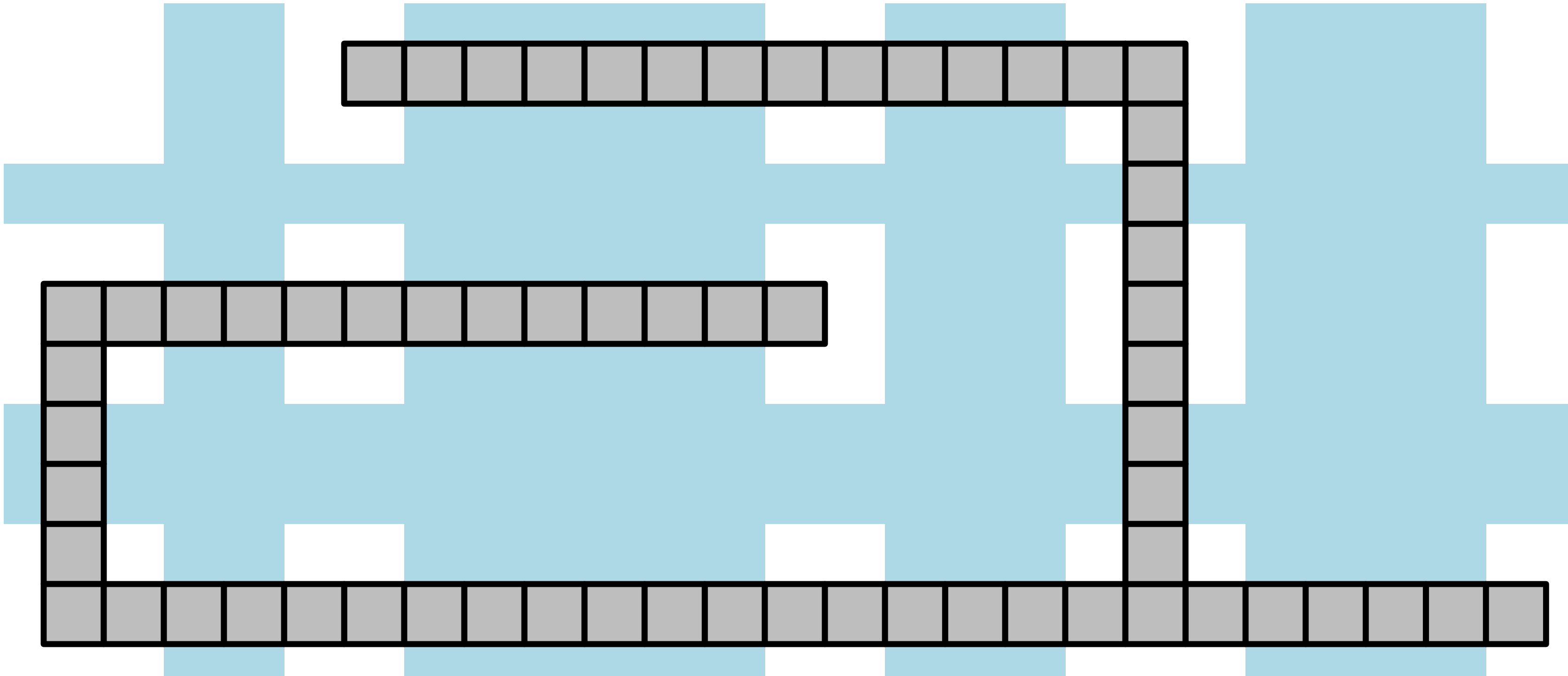


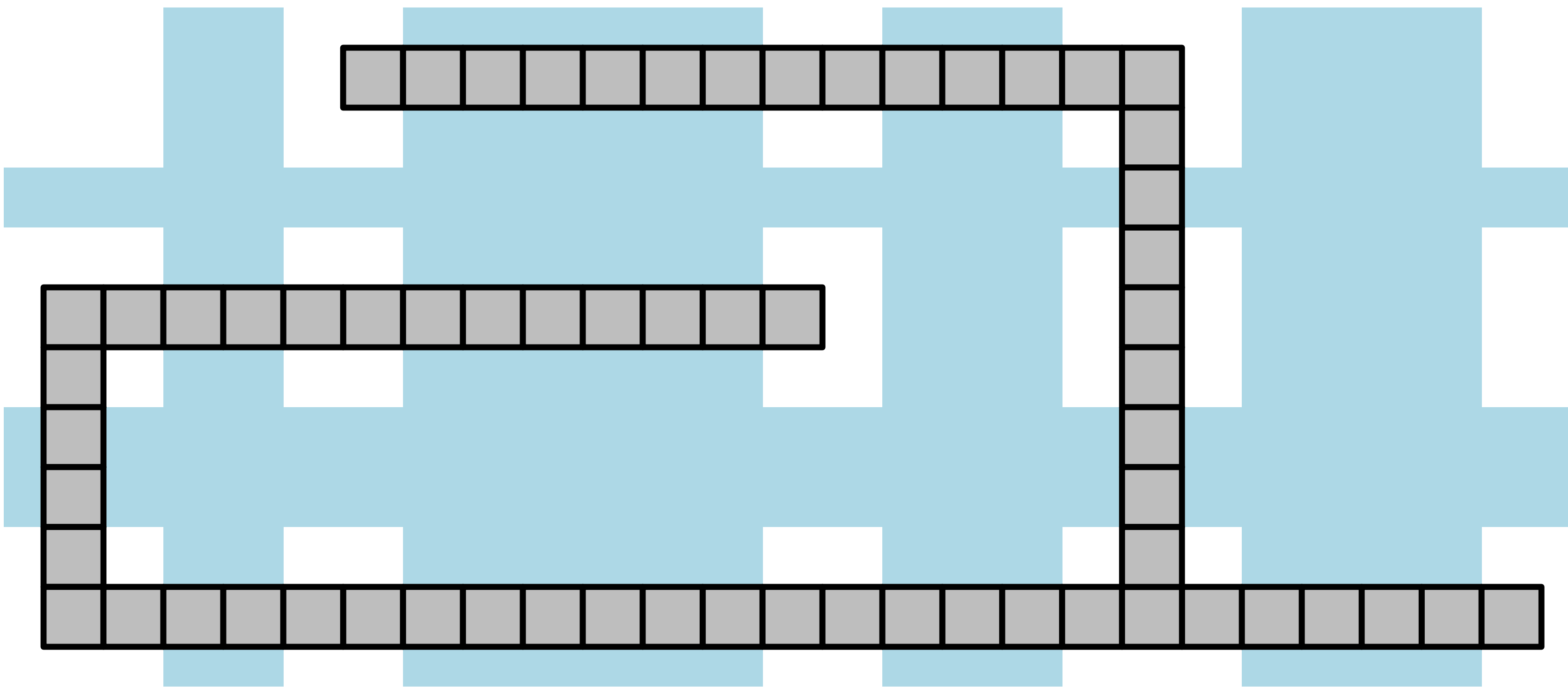


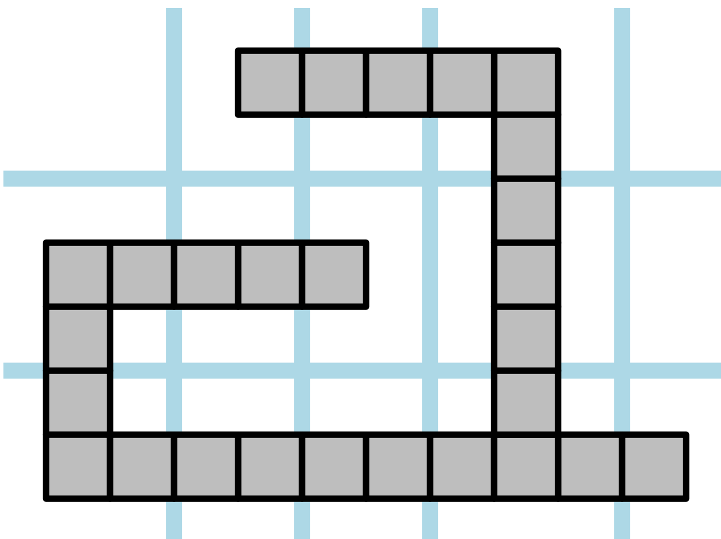


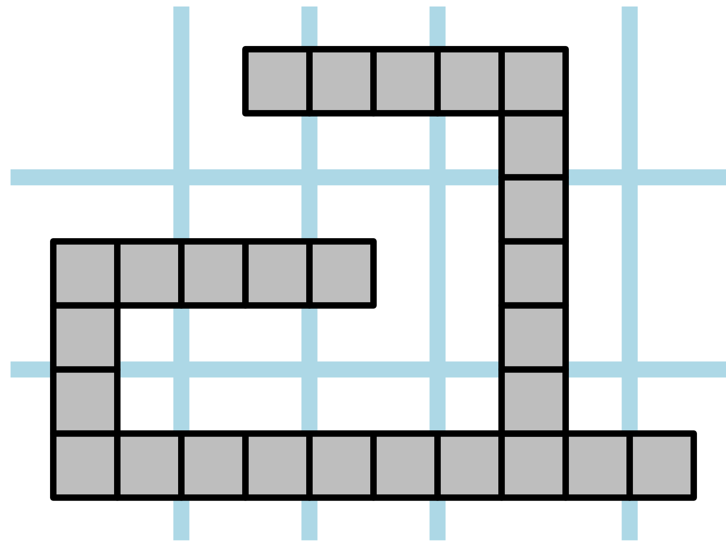


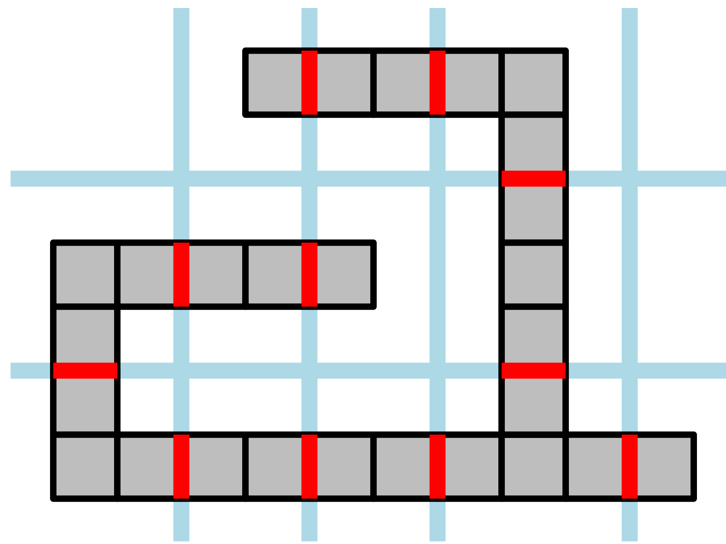
Vertical cut $\mathcal{C}$:
A maximal set of consecutive columns, $C_i, \ldots, C_j$, such that for $\mathcal{C}^* = \{C_{i-1}, \ldots, C_{j+1}\}$ we have
$P \cap \mathcal{C}^* = \{i-1, \ldots, j+1\} \times A$,
where $A \neq \emptyset$ and
$a, b \in A$, $a \neq b \Rightarrow |a-b| \geq 3$.

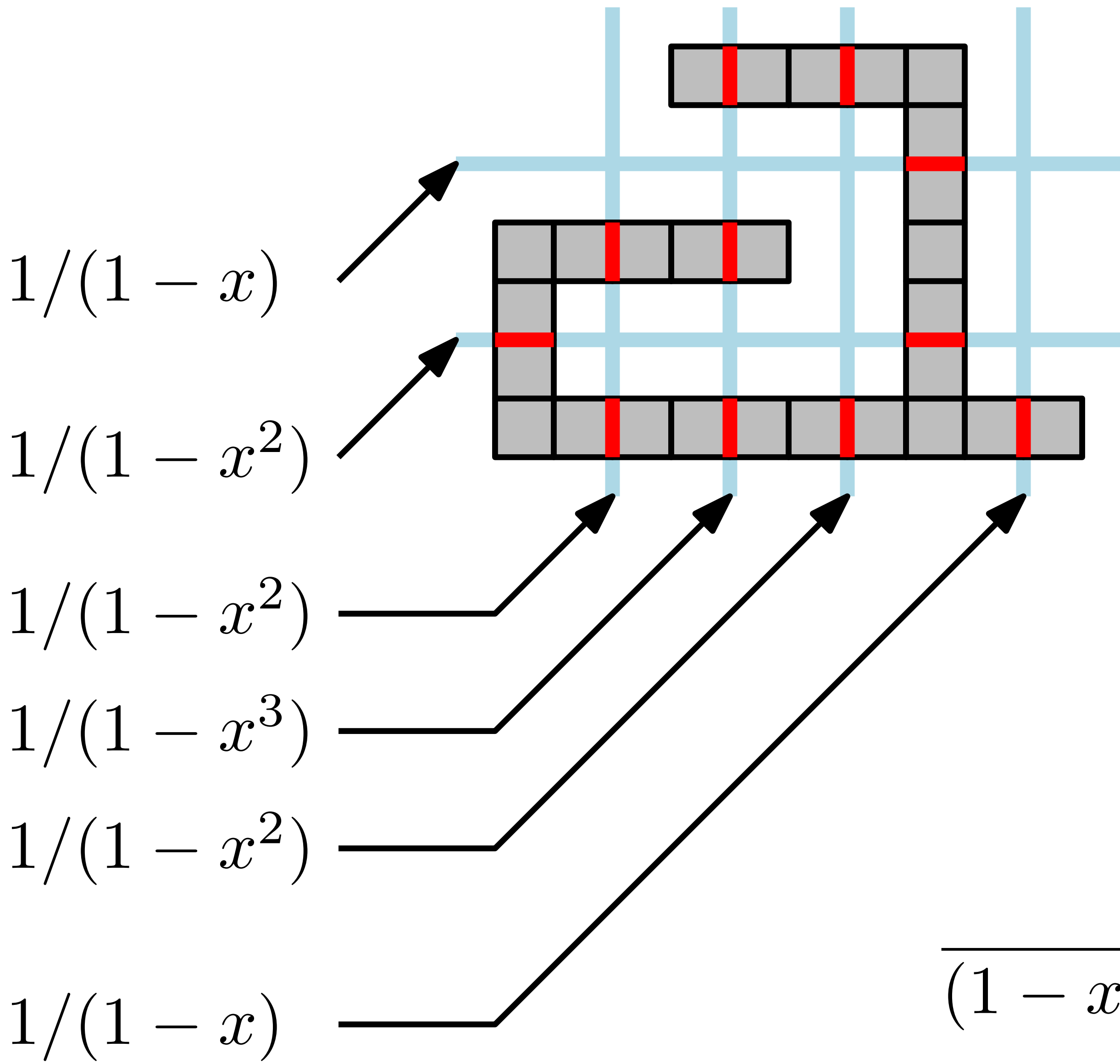












$$\frac{x^{27}}{(1-x)^2(1-x^2)^3(1-x^3)}$$

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For the proof, we define three kinds of special cells:
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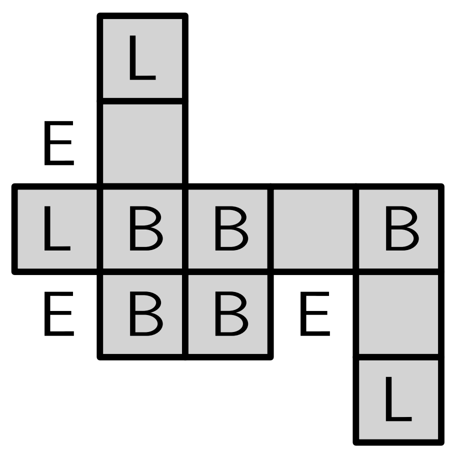

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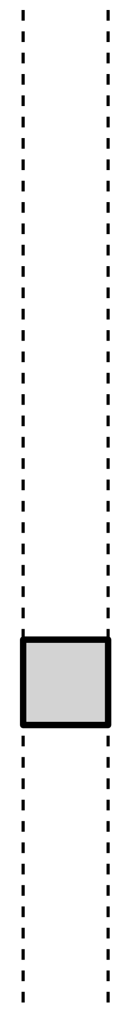

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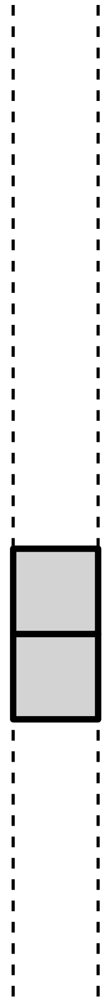

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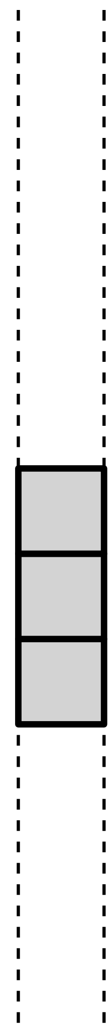

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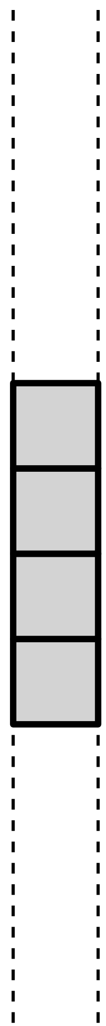

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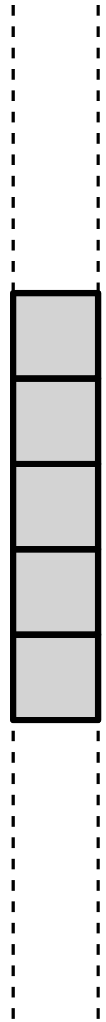

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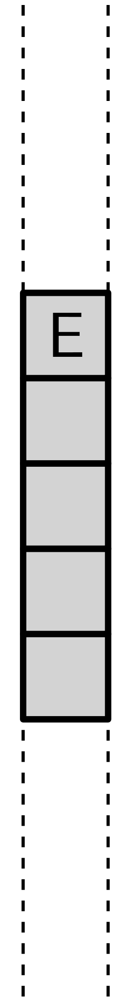

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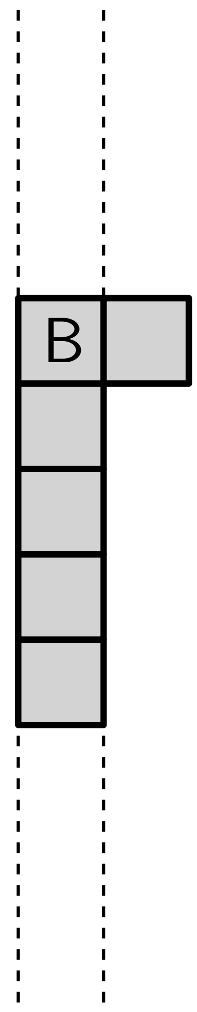

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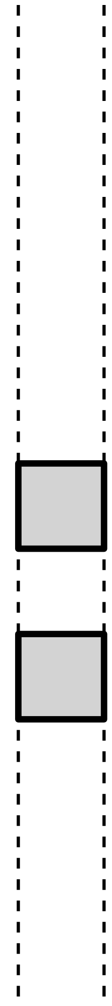

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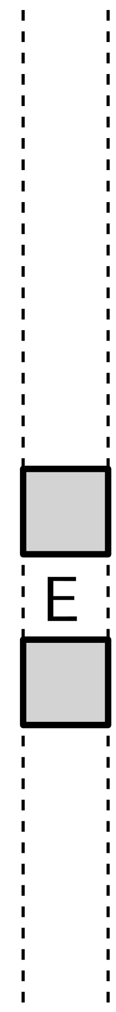

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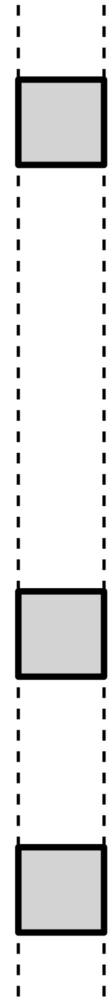

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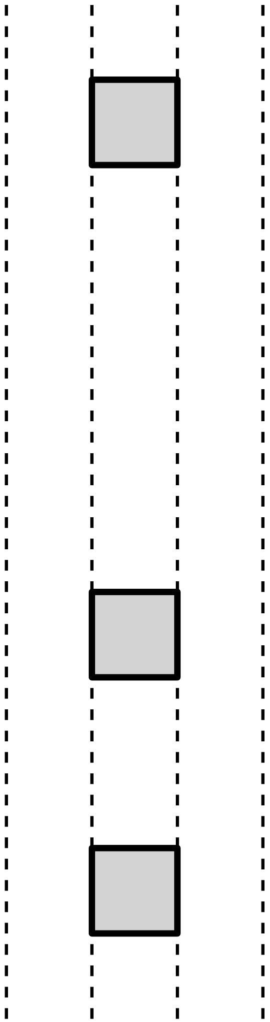

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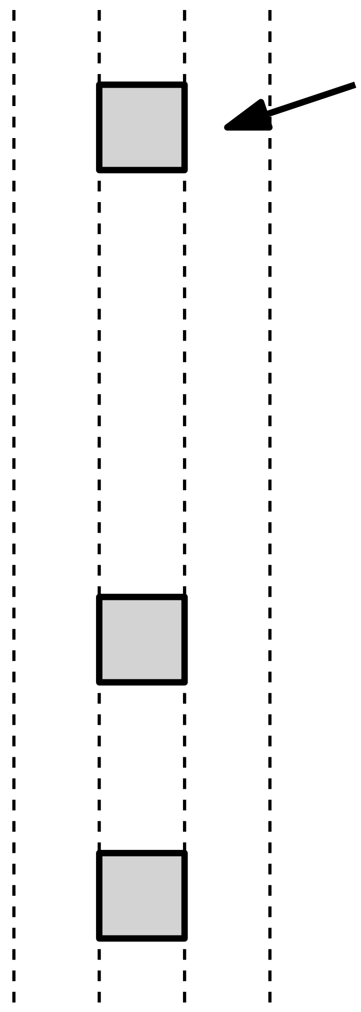

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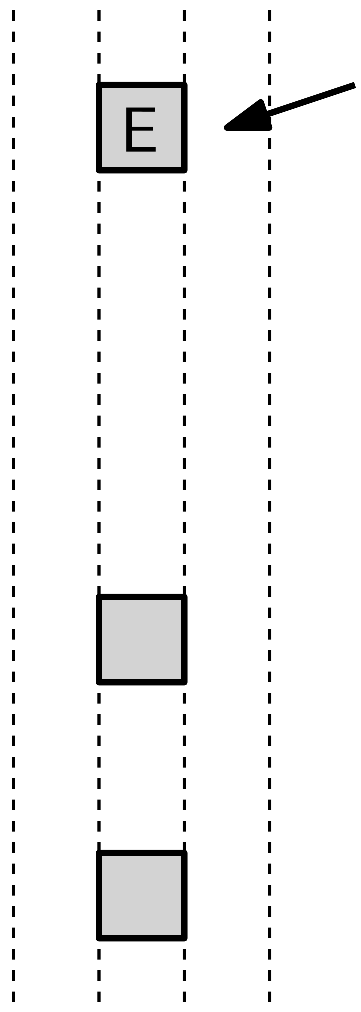

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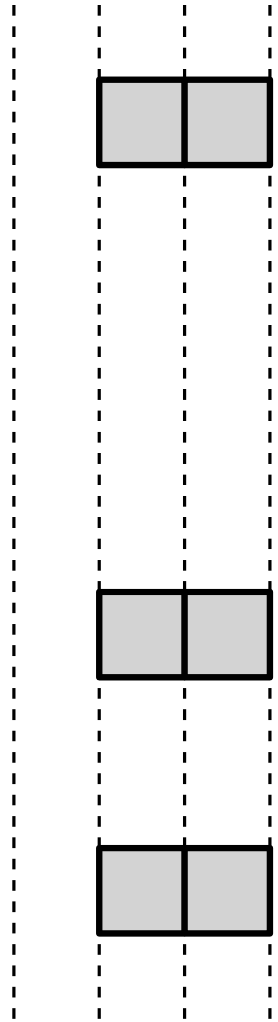

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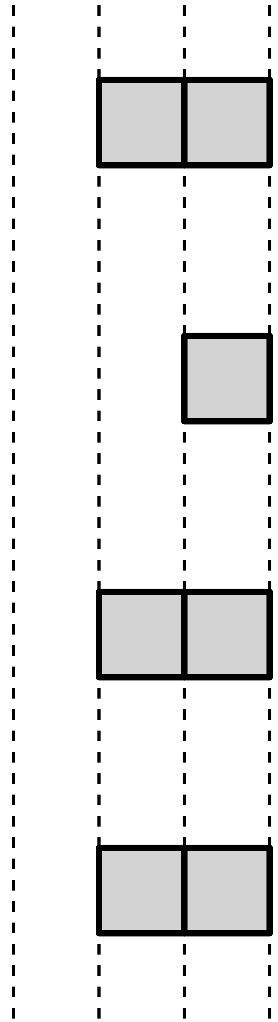

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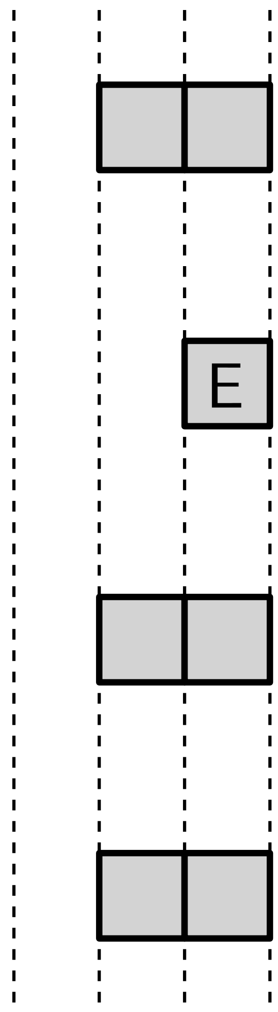

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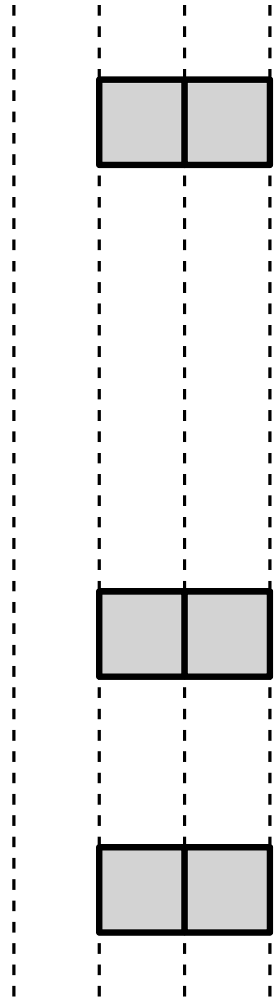

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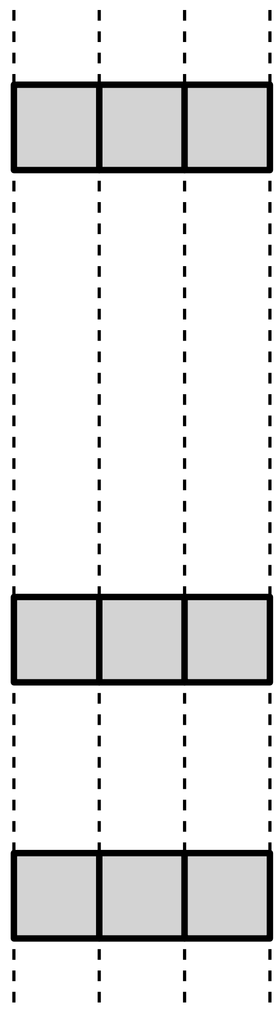

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$$k = e + 2f$$

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1. Each excess cells contributes directly to e.
2. Each bend cell contributes directly to e or to f.
3. $|V_1| \leq |V_3| + 2|V_4| + 2$, and each cell of degree 3 or 4 is a bend cell.

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Claim 2: The number of special cells is bounded by a constant.

$\Rightarrow$ The number of columns that do not belong to a cut, is bounded.
$\Rightarrow$ All the reduced polyominoes lie in a square of a bounded size.
$\Rightarrow$ There is a finite number of reduced polyominoes.
$\Rightarrow$ There is a finite number of patterns classes.

1. Each pattern produces a rational generating function (more precisely: its denominator is a product of cyclotomic polynomials).

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For each fiked k, the generating function of $(A(n, 2n+2-k))_{n \geq k}$ is rational.

The multidimensional case (d-polycubes): The main result is the same.

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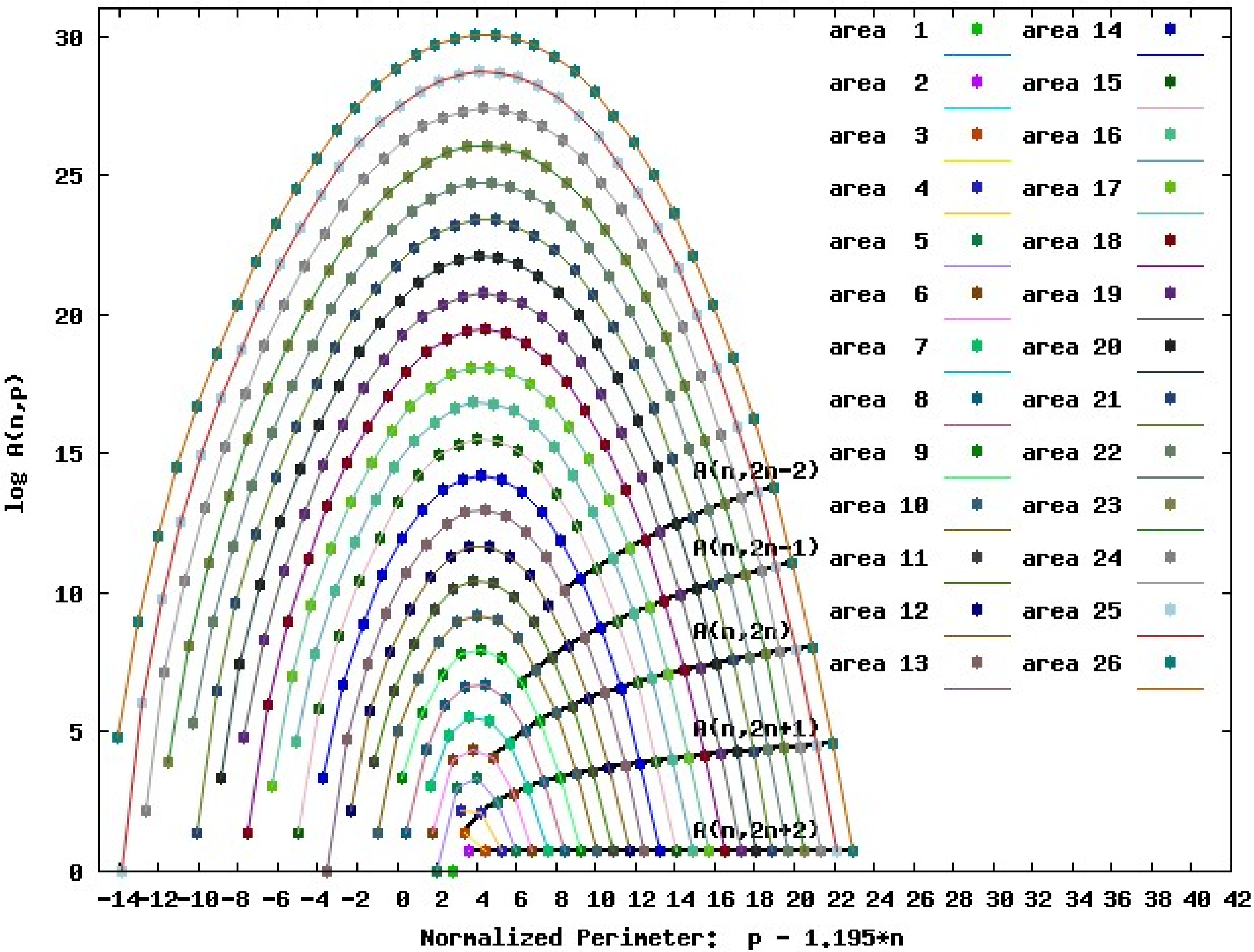
(The maximum perimeter is $2n(d-1)+2$.
In the formula $k = e + 2f$, f is the "circuit rank" – the number of edges that must be removed from the graph in order to obtain a tree. It is equal to $|E| - |V| + 1$.)

k	CP
0	$x-1$
1	$(x-1)^2$
2	$(x-1)^3$
3	$(x-1)^4(x+1)^2$
4	$(x-1)^5(x+1)^3$
5	$(x-1)^6(x+1)^4(x^2+x+1)$



log A(n,p)
Normalized Perimeter: p - 1.195*n
A(n,2n-2)
A(n,2n-1)
A(n,2n)
A(n,2n+1)
A(n,2n+2)
area 1
area 2
area 3
area 4
area 5
area 6
area 7
area 8
area 9
area 10
area 11
area 12
area 13
area 14
area 15
area 16
area 17
area 18
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area 21
area 22
area 23
area 24
area 25
area 26