Enumerating traceless matrices

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(joint work with S. Shechter and C. Voll)

- Motivation: study of Dirichlet series encoding sizes of images of matrices with certain properties

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- Enumeration of (traceless) matrices over finite fields and quotients of B_n
- Some nice combinatorial identities (between $4-$variate polynomials over B_n and S_n)

Matrices over finite fields

$$
\begin{aligned}
Mat_n^{n-i}(\mathbb{F}_q) &= n \times n \text{ matrices of rank } n-i \\
Alt_{2n+\delta}^{2(n-i)}(\mathbb{F}_q) &= (2n+\delta) \times (2n+\delta) \text{ antisymmetric matrices of rank } 2(n-i) \\
Sym_n^{n-i}(\mathbb{F}_q) &= n \times n \text{ symmetric matrices of rank } n-i
\end{aligned}
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|Mat_n^{n-i}(\mathbb{F}_q)| &= q^{n^2-i^2}\binom{n}{i}_{q^{-1}}\prod_{j=i+1}^{n}(1-q^{-j}) \\
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|Sym_n^{n-i}(\mathbb{F}_q)| &= q^{\binom{n+1}{2}-\binom{i+1}{2}}\prod_{j=0}^{n-i}(1-q^{-1-i-j})\left(\prod_{j=0}^{\lfloor\frac{n-i}{2}\rfloor}(1-q^{-2-2j})\right)^{-1}
\end{aligned}
$$

Hyperoctahedral groups, quotients and some statistics

The hyperoctahedral group B_n is the group of signed permutations, or permutations w of the set $[-n, n]$ such that $w(j) = -w(-j)$. We use the window notation $[w(1), \ldots, w(n)]$.

The Coxeter generating set of B_n is $S = \{s_0, s_1, \ldots, s_{n-1}\}$, where $s_0 = [-1, 2, 3, \ldots, n]$ and $s_1, \ldots, s_{n-1}$ are the simple transpositions.

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$\mathrm{Des}(w) = \{i \in [0, n-1] \mid w(i) > w(i+1)\}$ and $\mathrm{des}(w) = |\,\mathrm{Des}(w)|$

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For $i \in [0, n-1]$ the quotient

$$B_n^{\{i\}^c} = \{w \in B_n \mid \mathrm{Des}(w) \subset \{i\}\}$$

Hyperoctahedral groups, quotients and some statistics

Negative set of $w \in B_n$ is

$\mathrm{Neg}(w) = \{i \in [n] \mid w(i) < 0\}$ and $\mathrm{neg}(w) = |\mathrm{Neg}(w)|$

The Coxeter length ℓ on B_n satisfies

$$\ell = \mathrm{inv} + \mathrm{neg} + \mathrm{nsp}$$

where

$\mathrm{inv}(w) = |\{(i,j) \in [n]^2 \mid i < j,\ w(i) > w(j)\}|$

and

$\mathrm{nsp}(w) = |\{(i,j) \in [n]^2 \mid i < j,\ w(i) + w(j) < 0\}|$

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The odd length L on B_n satisfies

$$L = \mathrm{oinv} + \mathrm{oneg} + \mathrm{onsp}$$

where

$\mathrm{oinv}(w) = |\{(i,j) \in [n]^2 \mid i < j,\, w(i) > w(j),\, i \not\equiv j \pmod 2\}|$

and

$\mathrm{onsp}(w) = |\{(i,j) \in [n]^2 \mid i < j,\, w(i) + w(j) < 0,\, i \not\equiv j \pmod 2\}|$

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Proposition

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Matrices over finite fields and permutation statistics

Proposition (Reiner, '00; Stasinski - Voll, '14; Brenti - C., '17)

$$
\begin{aligned}
|Mat_n^{n-i}(\mathbb{F}_q)| &= q^{n^2-i^2} \sum_{w \in B_n^{\{i\}^c}} (-1)^{\mathrm{neg}(w)} q^{-\ell(w)} \\
|Alt_{2n+\delta}^{2(n-i)}(\mathbb{F}_q)| &= q^{\binom{2n+\delta}{2}-\binom{2i+\delta}{2}} \sum_{w \in B_n^{\{i\}^c}} (-1)^{\mathrm{neg}(w)} q^{-(2\ell+(2\delta-1)\,\mathrm{neg})(w)} \\
|Sym_n^{n-i}(\mathbb{F}_q)| &= q^{\binom{n+1}{2}-\binom{i+1}{2}} \sum_{w \in B_n^{\{i\}^c}} (-1)^{\ell(w)} q^{-L(w)}
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$B_n^{n-i}(\mathbb{F}_q) = n \times n$ traceless matrices of rank $n - i$

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Proposition (Bender, '74)

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$$|B_n^{n-i}(\mathbb{F}_q)| = q^{n^2-i^2-1} \sum_{w \in B_n^{\{i\}^c}} (-1)^{?} q^{?}$$

$$|Mat_n^{n-i}(\mathbb{F}_q)| = q^{n^2-i^2} \sum_{w \in B_n^{\{i\}^c}} (-1)^{\mathrm{neg}(w)} q^{-\ell(w)}$$

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where

$$\varepsilon_n(w) = \chi_{\{w(n)<0\}}$$

(note that on $B_n^{\{i\}^c}$, $w(n) < 0$ iff $w(j) < 0$, $j = i+1 \dots n$)

Image zeta function

The image zeta function is the Dirichlet series encoding the size of images of matrices viewed as endomorphisms. In our case:

$$\mathcal{P}_n(s) = \sum_{N=0}^{+\infty} \sum_{x \in \mathfrak{sl}_n(\mathfrak{o}_N)^*} |\operatorname{im}(x)|^{-s}$$

elementary divisors + our count for (traceless) matrices of fixed rank $\rightsquigarrow$

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Theorem (C. - Shechter - Voll, '17)

$$\mathcal{P}_n(s) = \sum_{w \in B_n} \left((-1)^{\mathrm{neg}(w)} q^{(-\ell+\varepsilon_n)(w)} \prod_{j \in \mathrm{Des}(w)} \frac{q^{n^2-j^2-1-s(n-j)}}{1-q^{n^2-j^2-1-s(n-j)}} \right)$$

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Need some statistics to describe these numerators..

More statistics on S_n and B_n

Recall for $w \in S_n$

$$\mathrm{Des}(w) = \{i \in [n-1] \mid w(i) > w(i+1)\}$$

Define

$$\mathrm{rmaj}(w) = \sum_{i \in \mathrm{Des}(w)} (n-i) \qquad \sigma_A(w) = \sum_{i \in \mathrm{Des}(w)} i(n-i)$$

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Similarly for $w \in B_n$

$$\mathrm{Des}(w) = \{i \in [0, n-1] \mid w(i) > w(i+1)\}$$

Define

$$\mathrm{rmaj}(w) = \sum_{i \in \mathrm{Des}(w)} (n-i) \qquad \sigma_B(w) = \sum_{i \in \mathrm{Des}(w)} (n^2 - i^2)$$

“Forgotten” statistics and a nice generating function

W finite Weyl group

Root system Φ, (positive roots Φ^+, simple roots $\Delta = \{\alpha_1, \ldots, \alpha_n\}$)

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Descents and length:

$$\mathrm{Des}(w) = \{i \mid w^{-1}\alpha_i < 0\}$$

$$\ell(w) = |\{\alpha \in \Phi^+ \mid w^{-1}\alpha < 0\}|$$

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$$\rho = \sum_{\alpha \in \Phi^+} \alpha \quad \rightsquigarrow \quad \rho = b_1\alpha_1 + \cdots + b_n\alpha_n$$

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$$\sigma(w) = \sum_{i \in \mathrm{Des}(w)} b_i \qquad (W = S_n,\, b_i = i(n-i);\;\; W = B_n,\, b_i = n^2 - i^2)$$

“Forgotten” statistics and a nice generating function

Theorem (Stembridge-Waugh, ’98)

W finite Weyl group

$$\sum_{w \in W} q^{\sigma(w) - \ell(w)} = f \cdot \prod_{i=1}^{n} \frac{1 - q^{b_i}}{1 - q^{e_i}}$$

where $e_1, \ldots, e_n$ are the exponents of W and f is the determinant of Cartan matrix.

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Type A and B:

$$\sum_{w\in S_n} q^{\sigma_A(w)-\ell(w)} = n \cdot \prod_{i=1}^{n-1} \frac{1-q^{i(n-i)}}{1-q^{i}},$$

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Type A and B:

$$\sum_{w\in S_n} q^{\sigma_A(w)-\ell(w)} = n \cdot \prod_{i=1}^{n-1} \frac{1-q^{i(n-i)}}{1-q^{i}}, \qquad \sum_{w\in B_n} q^{\sigma_B(w)-\ell(w)} = 2 \cdot \prod_{i=1}^{n} \frac{1-q^{n^2-i^2}}{1-q^{2i-1}}$$

4-variate distribution and factorisation

Proposition (C. - Shechter - Voll, '17)

$$\sum_{w\in B_n} W^{(\mathrm{des}-\varepsilon_n)(w)} X^{(\sigma_B-\ell)(w)} Y^{\mathrm{neg}(w)} Z^{\mathrm{rmaj}(w)} =$$

$$\left(\sum_{w\in S_n} W^{\mathrm{des}(w)} X^{(\sigma_A-\ell)(w)} (X^n Z)^{\mathrm{rmaj}(w)}\right) \prod_{j=0}^{n-1}(1+X^j Y Z).$$

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Proof

- Decompose B_n according to *inverse negative set*

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Proof

- Decompose B_n according to *inverse negative set*
- Perform an *alphabet replacement*

4-variate distribution and factorisation

- Recall:
  $\mathrm{Neg}(w^{-1})$ =values which appear with negative sign in the window notation of w.
  (e.g. $w = 1\bar{4}\bar{2}5\bar{3} \rightsquigarrow \mathrm{Neg}(w^{-1}) = \{2, 3, 4\}$)

$$B_n = \bigcup_{J \subseteq [n]} \left\{ w \in B_n \mid \mathrm{Neg}(w^{-1}) = J \right\}$$

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- for $J \subset [n]$ and $\max J < j \leqslant n$

$$\begin{array}{rcl} \left\{ w \in B_n | \mathrm{Neg}(w^{-1}) = J \right\} & \xrightarrow{\varphi_j} & \left\{ w \in B_n | \mathrm{Neg}(w^{-1}) = J \cup \{j\} \right\} \\ J & \longmapsto & J \\ n & \longmapsto & \bar{\jmath} \\ [n] \setminus (J \cup \{n\}) & \longmapsto & [n] \setminus (J \cup \{j\}) \end{array}$$

4-variate distribution and factorisation

Example: $\mathrm{Neg}(w^{-1}) = J = \{1, 4\}, \quad j = 3$

$\bar{4}\bar{1}2356 \quad \rightsquigarrow \quad \bar{4}\bar{1}256\bar{3}$

$536\bar{1}2\bar{4} \quad \rightsquigarrow \quad 65\bar{3}\bar{1}2\bar{4}$

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$$\mathrm{Des}(\varphi_j(w)) = \big(\mathrm{Des}(w) \setminus \{w^{-1}(n)\}\big) \cup \{w^{-1}(n) - 1\}$$

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$$\mathrm{Des}(\varphi_j(w)) = \big(\mathrm{Des}(w) \setminus \{w^{-1}(n)\}\big) \cup \{w^{-1}(n) - 1\}$$

$$\rightsquigarrow \sum_{w \in B_n} W^{(\mathrm{des} - \varepsilon_n)(w)} X^{(\sigma_B - \ell)(w)} Y^{\mathrm{neg}(w)} Z^{\mathrm{rmaj}(w)} =$$

$$\Big(\sum_{w \in S_n} W^{\mathrm{des}(w)} X^{(\sigma_A - \ell)(w)} (X^n Z)^{\mathrm{rmaj}(w)} \Big) \prod_{j=0}^{n-1} (1 + X^j Y Z).$$

The end