Motzkin and Catalan Polynomials

Niccolò Castronuovo

University of Bologna

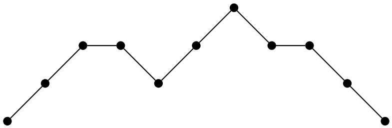
Joint work with M. Barnabei, F. Bonetti and M. Silimbani.

Outline

- Definition and properties of Motzkin Tunnel polynomials.
- Definition and properties of Catalan Tunnel polynomials.
- Connections with full binary trees.

Motzkin paths

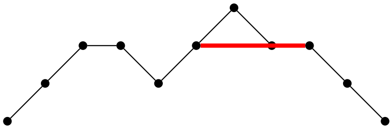
A *Motzkin path* of length n is lattice path in the plane from $(0, 0)$ to $(n, 0)$ consisting of up steps $U = (1, 1)$, down steps $D = (1, -1)$ and horizontal steps $H = (1, 0)$, that never goes below the x-axis.



$\mathcal{M}_n$ = the set of Motzkin paths of length n.

Motzkin polynomials

A *weak tunnel* in a Motzkin path p is a horizontal segment between two lattice points of p lying always weakly below p.

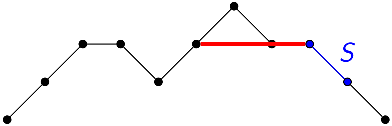


Motzkin polynomials

For every non-horizontal step S of p denote by $t(S)$ the length of the maximal weak tunnel ending at the initial point of S.
Note that

$$t(S) \leq n - 2$$

where n is the length of p.



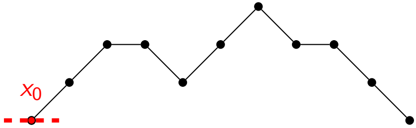
$$t(S) = 3$$

Motzkin polynomials

We assign to every step S of p the weight

- $x_{t(S)}$ if S is an up step,
- $y_{t(S)}$ if S is a down step,
- z if S is a horizontal step.

Then we associate to p the monomial $m(p)$ in the commutative variables $x_0, x_1, \ldots, x_{n-2}, y_0, y_1, \ldots, y_{n-2}, z$ given by the product of the weights of p.



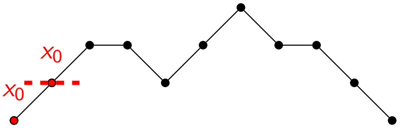
$\longrightarrow x_0$

Motzkin polynomials

We assign to every step S of p the weight

- $x_{t(S)}$ if S is an up step,
- $y_{t(S)}$ if S is a down step,
- z if S is a horizontal step.

Then we associate to p the monomial $m(p)$ in the commutative variables $x_0, x_1, \ldots, x_{n-2}, y_0, y_1, \ldots, y_{n-2}, z$ given by the product of the weights of p.



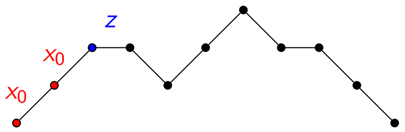
$\longrightarrow x_0^2$

Motzkin polynomials

We assign to every step S of p the weight

- $x_{t(S)}$ if S is an up step,
- $y_{t(S)}$ if S is a down step,
- z if S is a horizontal step.

Then we associate to p the monomial $m(p)$ in the commutative variables $x_0, x_1, \ldots, x_{n-2}, y_0, y_1, \ldots, y_{n-2}, z$ given by the product of the weights of p.



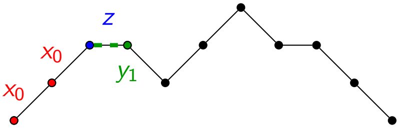
$\longrightarrow x_0^2 z$

Motzkin polynomials

We assign to every step S of p the weight

- $x_{t(S)}$ if S is an up step,
- $y_{t(S)}$ if S is a down step,
- z if S is a horizontal step.

Then we associate to p the monomial $m(p)$ in the commutative variables $x_0, x_1, \ldots, x_{n-2}, y_0, y_1, \ldots, y_{n-2}, z$ given by the product of the weights of p.



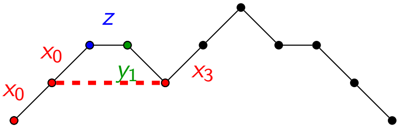
$\longrightarrow x_0^2 z y_1$

Motzkin polynomials

We assign to every step S of p the weight

- $x_{t(S)}$ if S is an up step,
- $y_{t(S)}$ if S is a down step,
- z if S is a horizontal step.

Then we associate to p the monomial $m(p)$ in the commutative variables $x_0, x_1, \ldots, x_{n-2}, y_0, y_1, \ldots, y_{n-2}, z$ given by the product of the weights of p.



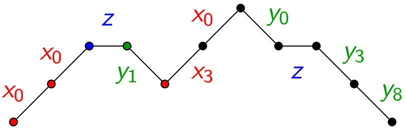
$\longrightarrow x_0^2 z y_1 x_3$

Motzkin polynomials

We assign to every step S of p the weight

- $x_{t(S)}$ if S is an up step,
- $y_{t(S)}$ if S is a down step,
- z if S is a horizontal step.

Then we associate to p the monomial $m(p)$ in the commutative variables $x_0, x_1, \ldots, x_{n-2}, y_0, y_1, \ldots, y_{n-2}, z$ given by the product of the weights of p.



$\longrightarrow x_0^3 x_3 y_0 y_1 y_3 y_8 z^2$

Motzkin polynomials

We define the $n-th$ *Motzkin Tunnel polynomial* as

$$MT_n = MT_n(x_0, x_1, \ldots, x_{n-2}; y_0, y_1, \ldots, y_{n-2}; z) := \sum_{p \in \mathcal{M}_n} m(p).$$

For $0 \leq n \leq 4$,

$$MT_0 = 1 \qquad MT_1 = z \qquad MT_2 = x_0 y_0 + z^2$$

$$MT_3 = x_0 y_0 z + x_1 y_0 z + x_0 y_1 z + z^3$$

$$MT_4 = x_0 x_2 y_0^2 + x_0^2 y_0 y_2 + x_0 y_0 z^2 + x_1 y_0 z^2 + x_2 y_0 z^2 + x_0 y_1 z^2 + x_1 y_1 z^2 + x_0 y_2 z^2 + z^4$$

Motzkin polynomials

We define the $n-th$ *Motzkin Tunnel polynomial* as

$$MT_n = MT_n(x_0, x_1, \ldots, x_{n-2}; y_0, y_1, \ldots, y_{n-2}; z) := \sum_{p \in \mathcal{M}_n} m(p).$$

For $0 \leq n \leq 4$,

$$MT_0 = 1 \qquad MT_1 = z \qquad MT_2 = x_0 y_0 + z^2$$
$$MT_3 = x_0 y_0 z + x_1 y_0 z + x_0 y_1 z + z^3$$
$$MT_4 = x_0 x_2 y_0^2 + x_0^2 y_0 y_2 + x_0 y_0 z^2 + x_1 y_0 z^2 + x_2 y_0 z^2 + x_0 y_1 z^2 + x_1 y_1 z^2 + x_0 y_2 z^2 + z^4$$

A recurrence for the polynomials M_n

Theorem

The polynomials MT_i satisfy the recurrence

$$MT_n = z\,MT_{n-1} + \sum_{i=0}^{n-2} x_i\,y_{n-2-i}\,MT_i\,MT_{n-2-i}, \qquad n \geq 1$$

with initial value

$$MT_0 = 1$$

This is an analogue of the recurrence for Motzkin numbers.

Symmetry of Motzkin polynomials

Let f be a polynomial in the variables $x_0, x_1, \ldots, y_0, y_1, \ldots, z$. Set

$$f^\sigma(x_0, x_1, \ldots; y_0, y_1, \ldots; z) = f(y_0, y_1, \ldots; x_0, x_1, \ldots; z).$$

Theorem

$$MT_n = MT_n^\sigma$$

To show this we define a bijection E over the set $\mathcal{M}_n$ such that

$$m(E(p)) = m^\sigma(p)$$

for all $p \in \mathcal{M}_n$.

Symmetry of Motzkin polynomials

Let f be a polynomial in the variables $x_0, x_1, \dots, y_0, y_1, \dots, z$. Set

$$f^\sigma(x_0, x_1, \dots; y_0, y_1, \dots; z) = f(y_0, y_1, \dots; x_0, x_1, \dots; z).$$

Theorem

$$MT_n = MT_n^\sigma$$

To show this we define a bijection E over the set $\mathcal{M}_n$ such that

$$m(E(p)) = m^\sigma(p)$$

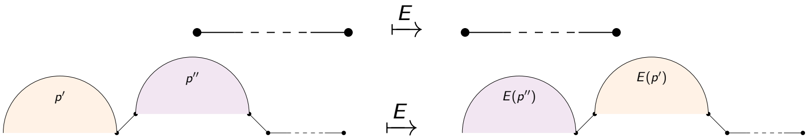
for all $p \in \mathcal{M}_n$.

Symmetry of Motzkin polynomials

Inspired by a map over Dyck paths due to Deutsch, we define the bijection E recursively as follows.

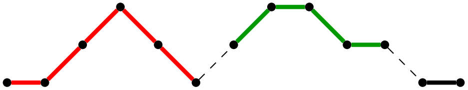
- if $p = H^k$ for $k \geq 0$, $E(H^k) = H^k$;
- if $p = p'\,U\,p''\,D\,H^k$, with $p', p'' \in \mathcal{M}$,

$$E(p'\,U\,p''\,D\,H^k) = E(p'')\,U\,E(p')\,D\,H^k.$$

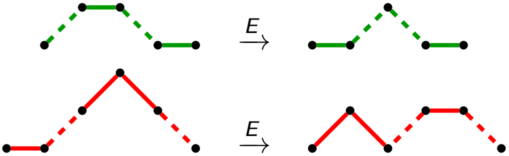


Symmetry of Motzkin polynomials

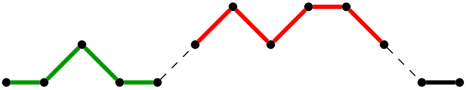
As an example consider the following Motzkin path

$p =$ 

Since

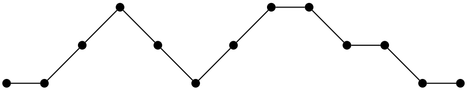


the path $E(p)$ is

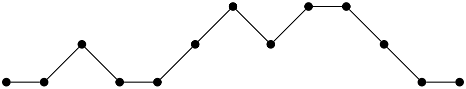
$E(p) =$ 

Symmetry of Motzkin polynomials

The path

$p =$ 

and the path

$E(p) =$ 

have associated monomials

$$m(p) = x_0^2 x_1 x_5 y_0 y_1 y_2 y_4 z^4,$$
$$m(E(p)) = x_0 x_1 x_2 x_4 y_0^2 y_1 y_5 z^4.$$

Trivial specializations

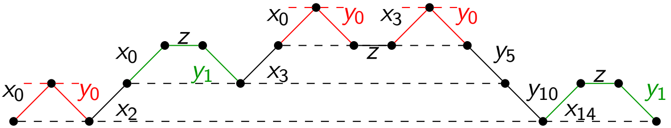
a. $MT_n(1,1,\ldots,1,1,\ldots,1)$ is the n-th Motzkin number.

b. $MT_n(1,1,\ldots;1,1,\ldots;2)$ is the number of Motzkin paths of length n where horizontal steps have two possible colors. This number equals the $(n+1)$-th Catalan number.

c. The coefficient of z^j in $MT_n(1,1,\ldots;1,1,\ldots;z)$ is the number of Motzkin paths of length n with j horizontal steps.

Less trivial specializations

A down step has weight $y_0 \iff$ it is preceded by an up step, that is, it is the down step of a peak.
$\Rightarrow$ the coefficient of y_0^j in $MT_n(1, 1, \ldots ; y_0, 1, \ldots ; 1)$ is the number of Motzkin paths of length n with j peaks.
A down step has weight $y_1 \iff$ it is preceded by a UH.
$\Rightarrow$ the coefficient of y_1^j in $MT_n(1, 1, \ldots ; 1, y_1, 1, \ldots ; 1)$ is the number of Motzkin paths of length n with j occurences of UHD.



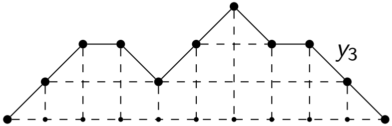
$$m(p) = x_0^3 x_2 x_3^2 x_{14} y_0^3 y_1^2 y_5 y_{10} z^3$$

Less trivial Specializations

The area $A(p)$ of the path p is defined to be the area of the trapezoid under the path and above the x-axis.
The coefficient of q^j in $MT_n(1,1,\ldots;q,q^2,\ldots;1)$ is the number of Motzkin paths of length n of area j.

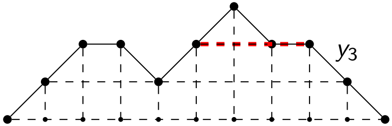
Less trivial Specializations

In fact, the label of every down step D is the area of the trapezoid lying between the maximal weak tunnels ending at the initial and final point of D, minus one.



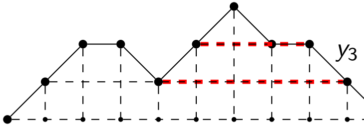
Less trivial Specializations

In fact, the label of every down step D is the area of the trapezoid lying between the maximal weak tunnels ending at the initial and final point of D, minus one.



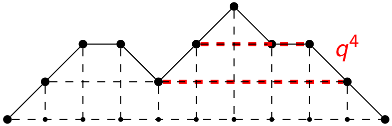
Less trivial Specializations

In fact, the label of every down step D is the area of the trapezoid lying between the maximal weak tunnels ending at the initial and final point of D, minus one.



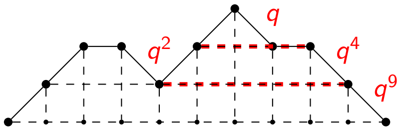
Less trivial Specializations

In fact, the label of every down step D is the area of the trapezoid lying between the maximal weak tunnels ending at the initial and final point of D, minus one.



Less trivial Specializations

In fact, the label of every down step D is the area of the trapezoid lying between the maximal weak tunnels ending at the initial and final point of D, minus one.



The Motzkin path has area $2+1+4+9=16$.

A joint distribution

Consider the polynomials

$$T_n = T_n(q, z, x_0, x_1) = MT_n(x_0 q, x_1 q^2, q^3, q^4, \dots; 1, 1, \dots; z)$$

which take into account the area, the peaks, the occurrences of *UHD*, and the number of horizontal steps.
Let

$$F(w) = F(q, z, w, x_0, x_1) = \sum_{n \geq 0} T_n w^n$$

be the corresponding generating function.

A joint distribution

$F(w)$ admits a continued fraction expansion.

Theorem

$$F(w) = \cfrac{1}{1 + a_0(w)w - \cfrac{b_1 w^2}{1 + a_1(w)w - \cfrac{b_2 w^2}{1 + a_2(w)w - \cdots}}},$$

where

$$a_i(w) = -zq^i + q^{2i+1}w(1 - x_0) + zq^{3i+2}w^2(1 - x_1) \qquad b_i = q^{2i-1}$$

The proof is an immediate consequence of the recurrence relation for the polynomials MT_n.

Catalan Tunnel polynomials

Dyck path $\Longleftrightarrow$ Motzkin path without H steps

$$\Rightarrow MT_n(1,1,...;1,1,...;0) = \begin{cases} c_{\frac{n}{2}} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

c_n = n-th Catalan number.

Catalan Tunnel polynomials

In a Dyck path every weak tunnel has even length.

We assign to every step S of p the weight

- $\lambda_{t(S)/2}$ if S is an up step,
- $\mu_{t(S)/2}$ if S is a down step,

and associate to p the monomial $\hat{m}(p)$ = the product of the weights of p.

We define the n-th *Catalan Tunnel polynomial*

$$CT_n = CT_n(\lambda_0, \lambda_1, \ldots \lambda_{n-1}; \mu_0, \mu_1, \ldots, \mu_{n-1})$$

as

$$CT_n = \sum_{p \in \mathscr{C}_n} \hat{m}(p)$$

$\mathscr{C}_n$ = the set of Dyck paths of semilength n.

A recurrence for the polynomials CT_n

Theorem

The polynomials CT_i satisfy the recurrence

$$CT_n = \sum_{i=0}^{n-1} \lambda_i \, \mu_{n-1-i} \, CT_i \, CT_{n-1-i}, \qquad n \geq 1$$

with initial value

$$CT_0 = 1.$$

This is an analogue of the recurrence for Catalan numbers.

A recurrence for the polynomials CT_n

Theorem

The polynomials CT_i satisfy the recurrence

$$CT_n = \sum_{i=0}^{n-1} \lambda_i \, \mu_{n-1-i} \, CT_i \, CT_{n-1-i}, \qquad n \geq 1$$

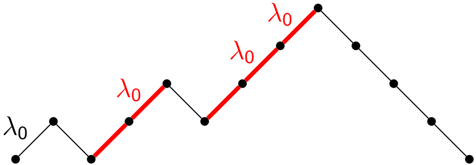
with initial value

$$CT_0 = 1.$$

This is an analogue of the recurrence for Catalan numbers.

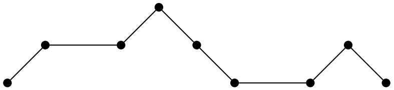
Specializations

- $CT_n(1,1,\ldots;1,1,\ldots)$ is the n-th Catalan number.
- $CT_n(1,1,\ldots;\mu_0,1,1,1,\ldots) = \sum_{k\geq 0} N_{n,k}\mu_0^k$ where $N_{n,k}$ is the $\{n,k\}$-th Narayana number.
- The coefficient of λ_0^i in $CT_n(\lambda_0,1,1,\ldots;1,1,1,\ldots)$ is the number of Dyck paths of semilength n with $i-1$ double rises.



Specializations

- The coefficient of μ_1^i in $CT_n(1, 1, \ldots; 1, \mu_1, 1, 1, 1, \ldots))$ is the number of Dyck paths of semilength n with i occurrences of $UUDD$.
- $CT_n(1, 1, \ldots; 2, 1, 1, \ldots)$ is the n-th large Schröder number. More generally, $CT_n(1, 1, \ldots; k, 1, 1, \ldots)$ counts large Schröder paths from $(0, 0)$ to $(0, 2n)$ where double horizontal steps may have $k - 1$ colors.

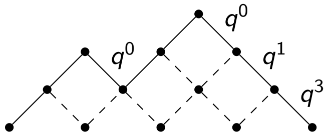


Specializations

The *normalized area* $\tilde{A}(d)$ of a Dyck path d is equal to

$$\tilde{A}(d) = \frac{A(d) - n}{2},$$

where $A(d)$ is the area between the path d and the x-axis.
$\Rightarrow$ the coefficient of q^k in $CT_n(1, 1, \dots; 1, q, q^2, \dots)$ is the number of Dyck paths of semilength n with normalized area equal to k.
$\Rightarrow$ the polynomial $CT_n(1, 1, \dots; 1, q, q^2, \dots)$ is nothing but the *Carlitz-Riordan q-analogue of Catalan numbers*.



$\rightarrow \tilde{A}(d) = 4$

Specializations

Consider the polynomials $T_n(q,\mu_0,\mu_1) = CT_n(1,1,\dots;\mu_0,\mu_1 q, q^2,\dots)$ which take into account the distribution of normalized area, peaks and occurrences of $UUDD$.

Theorem

If

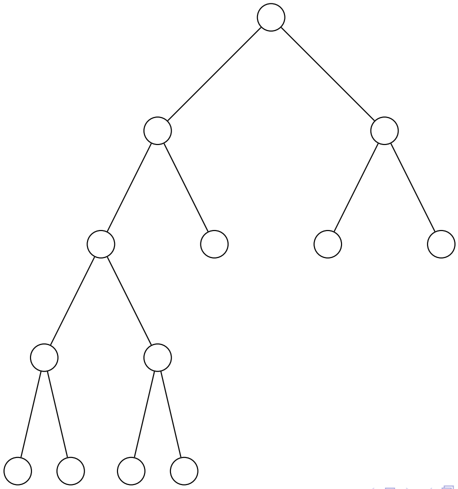
$$G(w) = \sum_{n\geq 0} T_n(q,\mu_0,\mu_1) w^n$$

$$a_i(w) = -\mu_0 q^i + q^i + \mu_0 q^{2i+1} w(1-\mu_1), \qquad b_i = q^{i-1}, \qquad \text{then:}$$

$$G(w) = \cfrac{1}{1 + a_0(w)w - \cfrac{b_1 w}{1 + a_1(w)w - \cfrac{b_2 w}{1 + a_2(w)w - \cfrac{b_3 w}{\dots}}}}$$

Full binary trees

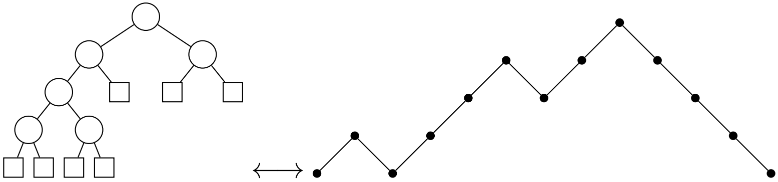
A *full binary tree* is a tree each of whose node has exactly zero or two children.



Full binary trees

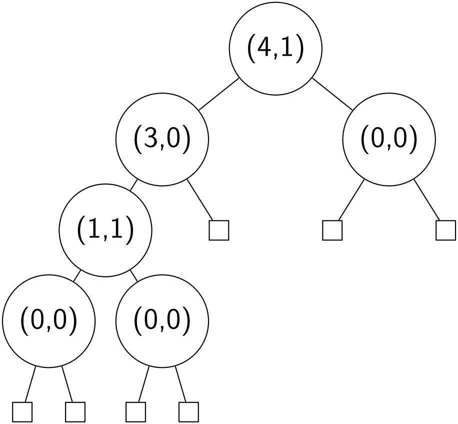
A bijection Φ between Dyck paths of semilength n and full binary trees with n internal nodes can be defined as follows. Let t be a full binary tree and let t_l and t_r be the left and right subtree of the root. Then the image of the tree t is defined recursively as

$$\Phi(t) = \begin{cases} \text{the empty path} & \text{if t is the empty tree} \\ \Phi(t_r)U\Phi(t_l)D & \text{otherwise} \end{cases}$$



Full binary trees

Associate to each internal node v a pair of integers $(a_l(v), a_r(v))$ where $a_l(v)$ is the number of internal nodes in the left subtree of v an $a_r(v)$ is the number of internal node in the right subtree of v.

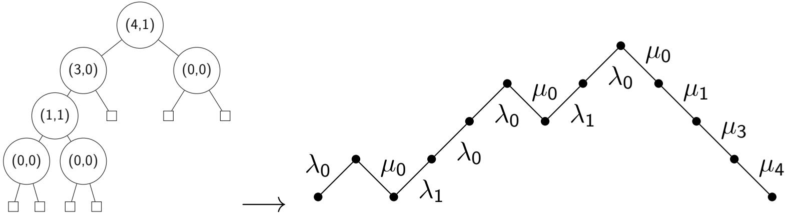


Full binary trees

Theorem

Let t be a full binary tree t. The monomial associated to the path $\Phi(t)$ is

$$\prod_{v \in IntNodes(t)} \lambda_{a_r(v)} \mu_{a_l(v)}.$$



Full binary trees

For every full binary tree t, the (internal) path length $l(t)$ is defined to be the sum of the length of the paths from the root to each internal node.

For every full binary tree t,

$$l(t) = \sum_{v \in IntNodes(t)} a_l(v) + a_r(v)$$

Theorem

The coefficient of q^k in $CT_n(1, q, q^2, \ldots, 1, q, q^2, \ldots)$ is the number of full binary trees with n internal nodes and path length k.

Full binary trees

For every full binary tree t, the (internal) path length $l(t)$ is defined to be the sum of the length of the paths from the root to each internal node.
For every full binary tree t,

$$l(t) = \sum_{v \in IntNodes(t)} a_l(v) + a_r(v)$$

Theorem

The coefficient of q^k in $CT_n(1, q, q^2, \ldots, 1, q, q^2, \ldots)$ is the number of full binary trees with n internal nodes and path length k.

Full binary trees

For every full binary tree t, the (internal) path length $l(t)$ is defined to be the sum of the length of the paths from the root to each internal node.
For every full binary tree t,

$$l(t) = \sum_{v \in IntNodes(t)} a_l(v) + a_r(v)$$

Theorem

The coefficient of q^k in $CT_n(1, q, q^2, \ldots, 1, q, q^2, \ldots)$ is the number of full binary trees with n internal nodes and path length k.

THANK YOU!