The sandpile model on complete bipartite graphs

(joint work with Yvan Le Borgne)

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Université Libre de Bruxelles

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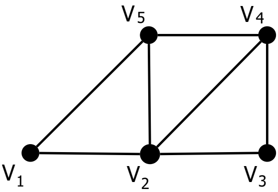

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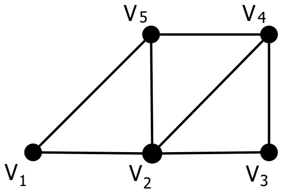

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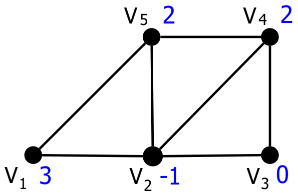

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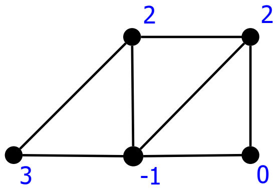

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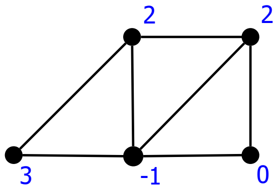

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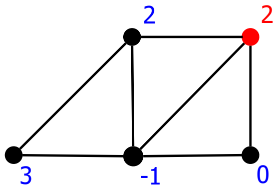

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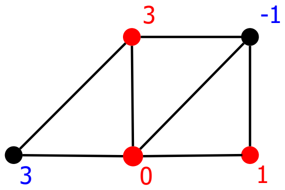

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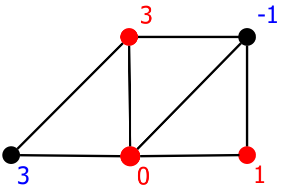

Observe that $\sum_i \Delta_i = 0 \in \mathbb{Z}^V$.

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Let $\Delta := \langle \Delta_1, \Delta_2, \ldots \rangle \leq \mathbb{Z}^V$. The group $\mathbb{Z}^V/\Delta$ is called the sandpile group of the graph $\mathcal{G}$.

Two configurations $u, u' \in \mathbb{Z}^V$ are toppling equivalent, denoted $u \sim u'$, if $u - u' \in \Delta$.

Fix a vertex $q \in V$ which we call the sink.

A configuration $u \in \mathbb{Z}^V$ is stable (w.r.t. the sink q) if $0 \leq u(v_i) < d_i$ for all $v_i \in V \setminus \{q\}$, where d_i is the degree of v_i.

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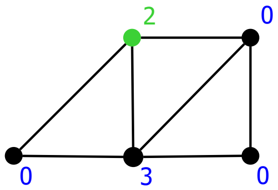

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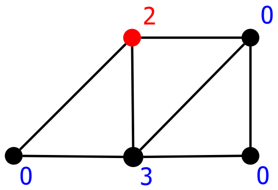

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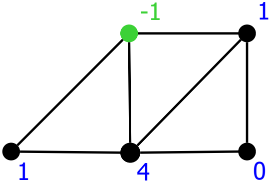

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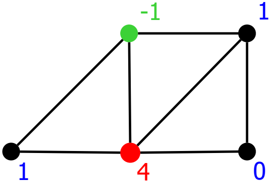

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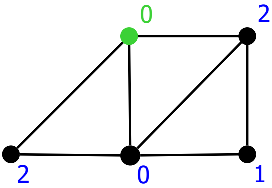

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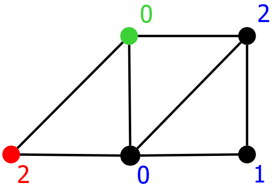

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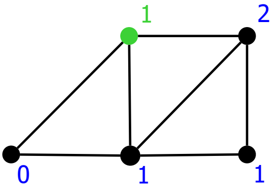

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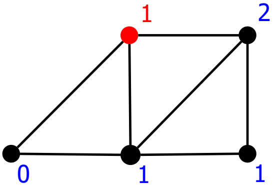

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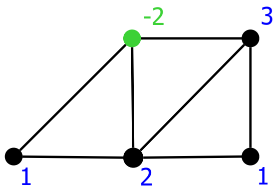

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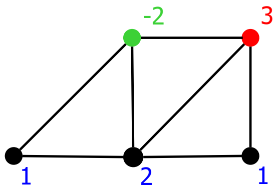

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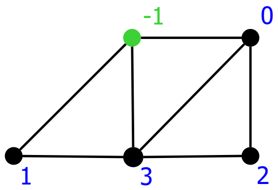

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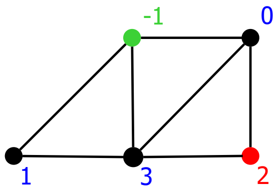

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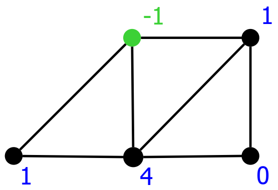

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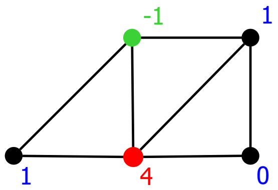

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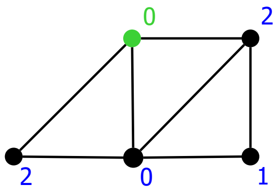

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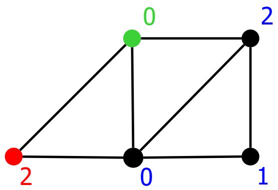

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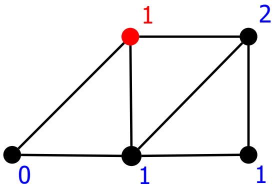
A stable $u \in \mathbb{Z}^V$ is recurrent (w.r.t. the sink q) if for every $\emptyset \neq A \subseteq V \setminus \{q\}$ the configuration $u + \sum_{v_i \in A} \Delta_i$ is not stable.

Let $\mathcal{G} := (V, E)$, where $V := \{v_1, v_2, v_3, v_4, v_5\}$, and let v_5 be the sink. We start with the stable $u = (0, 3, 0, 0, 2)$. And now we got the stable $u' = (0, 1, 1, 2, 1)$. Back to u'!



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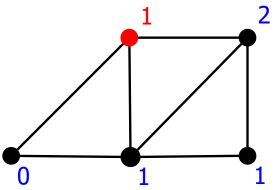

Theorem (Dhar)

Every configuration is equivalent to a unique recurrent configuration.

Sandpile model IV: parking configurations

A stable $u \in \mathbb{Z}^V$ is parking (w.r.t. the sink q) if for every $\emptyset \neq A \subseteq V \setminus \{q\}$ the configuration $u - \sum_{v_i \in A} \Delta_i$ is not stable.

Theorem

Every configuration is equivalent to a unique parking configuration.

The degree of a configuration $u \in \mathbb{Z}^V$ is simply $\sum_{v \in V} u(v) \in \mathbb{Z}$. Notice that $\text{degree}(\Delta_i) = 0$.

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The number of recurrent (parking) configurations of a given degree on $\mathcal{G}$ is equal to the number of spanning trees of $\mathcal{G}$.

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Sandpile on $K_{m,n}$: sorted configurations

Let $K_{m,n} = (V, E)$ with $V := A \sqcup B$, $A = \{a_1, a_2, \ldots, a_m\}$, $B = \{b_1, b_2, \ldots, b_n\}$, E all possible edges between A and B, a_m be the sink.

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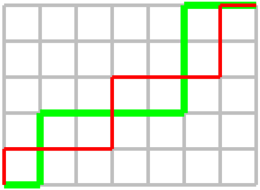
The diagram of the stable sorted configuration $u = \binom{0,0,0,2,2,2;*}{0,0,4,4,4}$ of $K_{7,5}$.



Sandpile on $K_{m,n}$: recurrent and parking configurations

Theorem (Dukes-Le Borgne 2013)

The recurrent sorted configurations of $K_{m,n}$ are the parallelogram polyominoes with bounding box $m \times n$.

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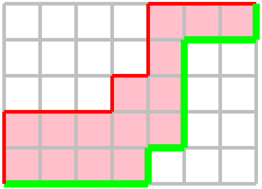

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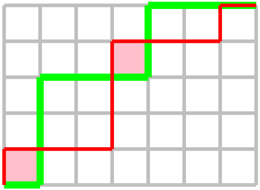

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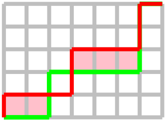

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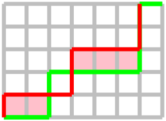

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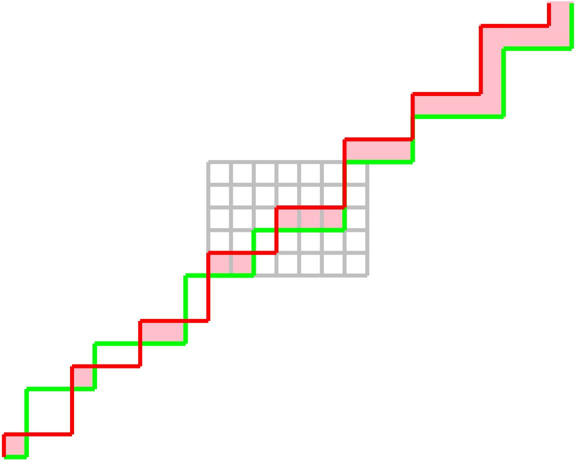

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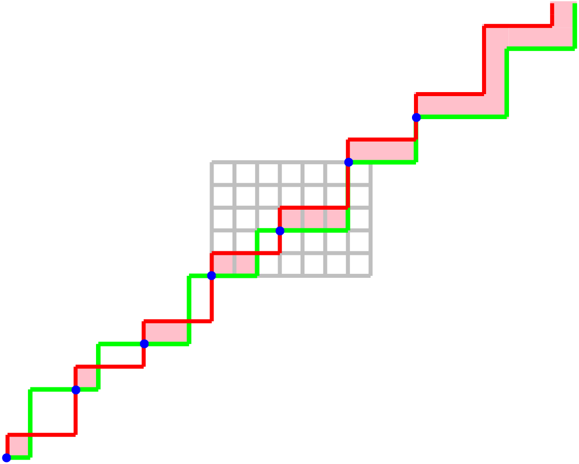

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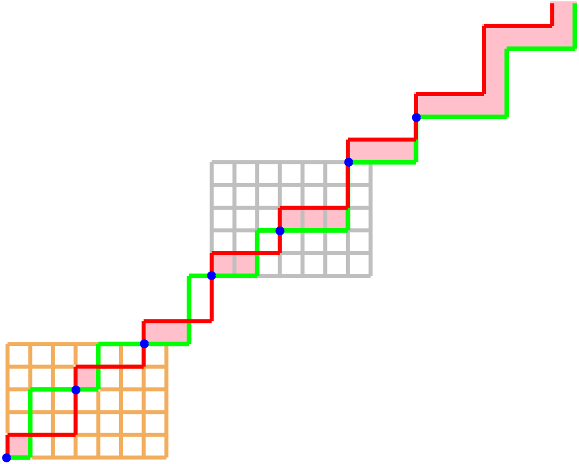

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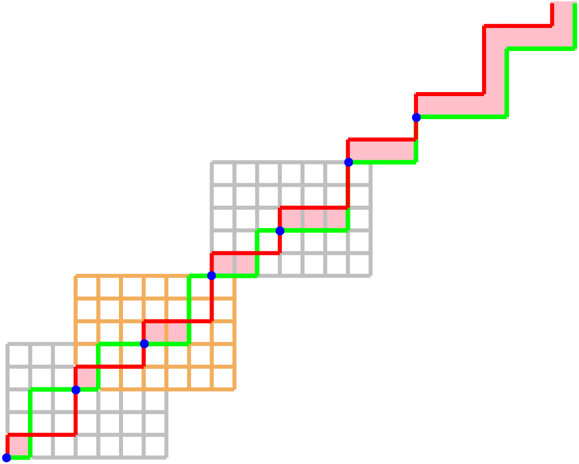

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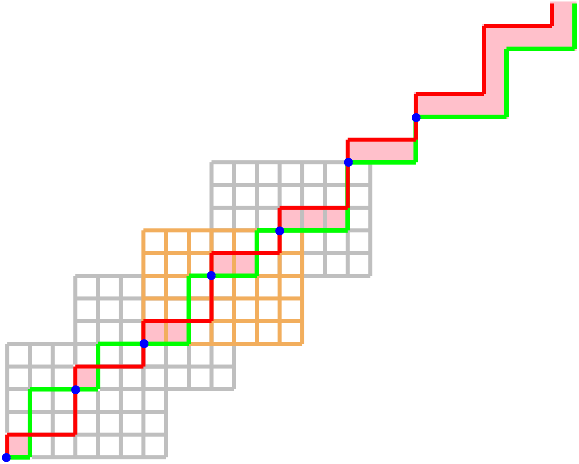

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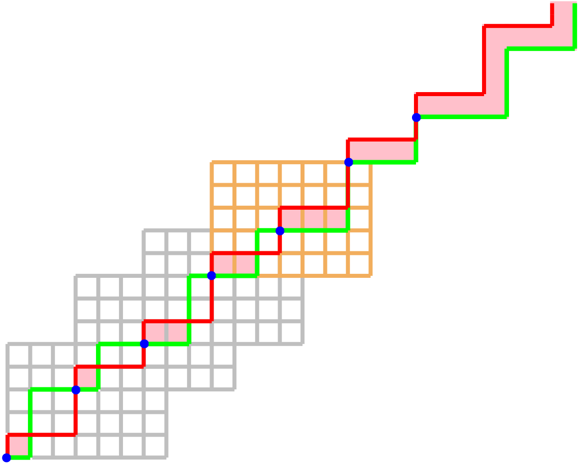

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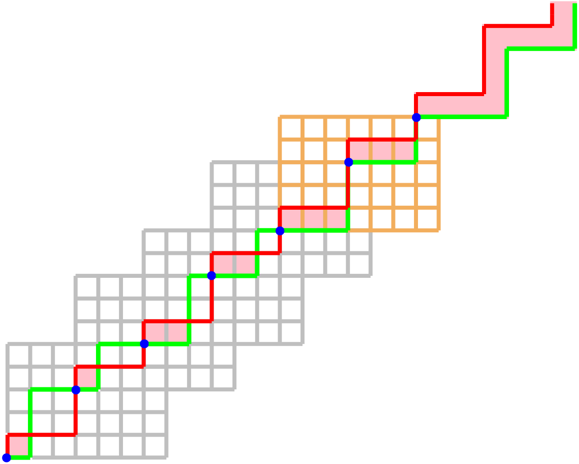

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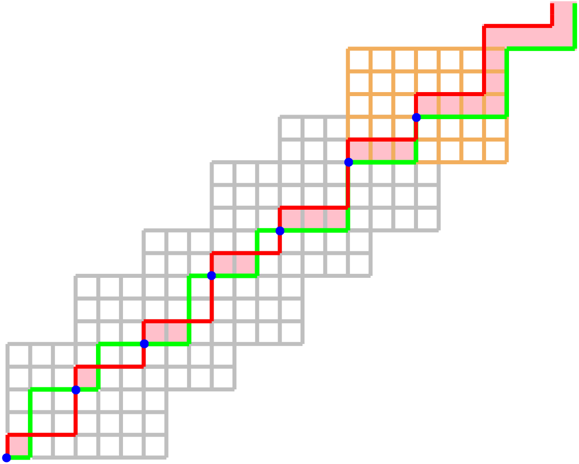

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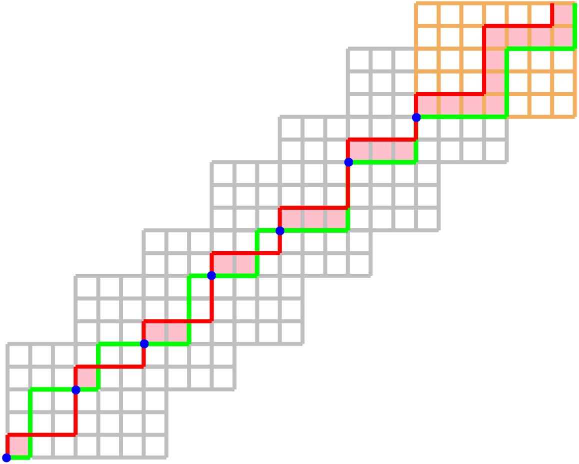

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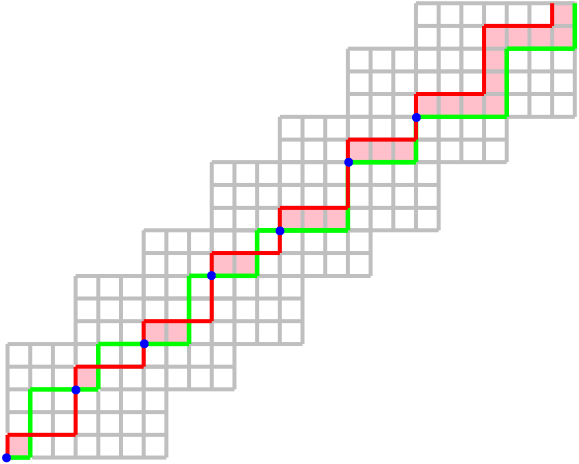

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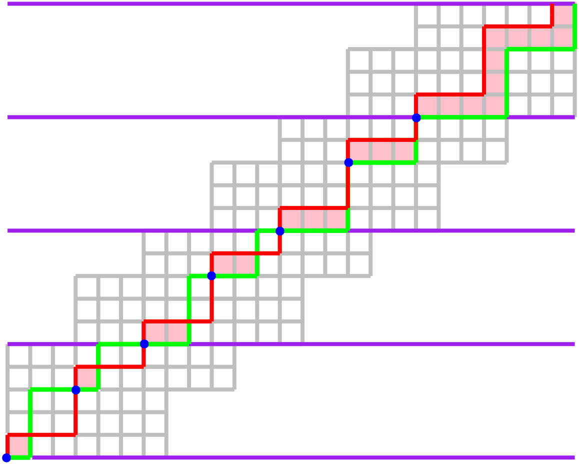

Sandpile model V: rank of configurations

A configuration u is non-negative ($u \geq 0$) if $u(v) \geq 0$ for all $v \in V$.
A configuration u is effective if u is toppling equivalent to a non-negative configuration.
The rank of a configuration u is defined as
$\text{rank}(u) := -1 + \min\{\text{degree}(f) \mid f \geq 0 \text{ and } u - f \text{ is non-effective}\}$.
Observe that $u \sim u'$ implies $\text{rank}(u) = \text{rank}(u')$.
The motivation is a Riemann-Roch theorem:

Theorem (Baker-Norine 2007)

For any configuration u on a graph $\mathcal{G} = (V, E)$ we have

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Theorem (Cori-Le Borgne 2016)

There is an algorithm to compute the rank on K_n of linear complexity in n.

Rank on $K_{m,n}$: algorithm

Theorem (D-Le Borgne)

The following algorithm computes the rank of a configuration u on $K_{m,n}$.

```
def compute_rank(u):
  u = park(u)
  rank = -1
  f = 0 # f is the 0 configuration
  while u(a_m) >= 0:
    let i be such that u(b_i) = 0
    u = park(u - b_i)
    f = f + b_i
    rank = rank + 1
  return (rank, f)
```

Rank on $K_{m,n}$: algorithm

Theorem (D-Le Borgne)

The following algorithm computes the rank of a configuration u on $K_{m,n}$.

```
def compute_rank(u):
  u = park(u)
  rank = -1
  f = 0 # f is the 0 configuration
  while u(a_m) >= 0:
    let i be such that u(b_i) = 0
    u = park(u - b_i)
    f = f + b_i
    rank = rank + 1
  return (rank, f)
```

Rank on $K_{m,n}$: cylindric diagram

Consider the parking sorted configuration $u = \binom{0,0,0,3,3,3;21}{0,0,0,3,3}$.

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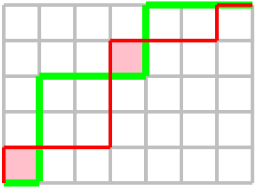

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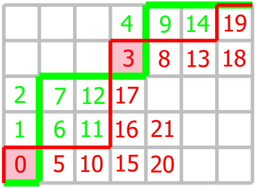

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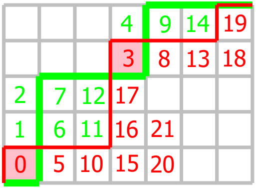

Theorem (D-Le Borgne)

The rank of a parking sorted configuration u on $K_{m,n}$ is equal to -1 plus the number of red labels in its cylindric diagram.

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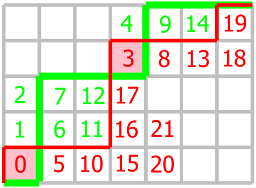

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Theorem (D-Le Borgne)

There is an algorithm to compute the rank on $K_{m,n}$ of linear complexity in $n + m$.

Sandpile on $K_{m,n}$: enumeration

Consider the generating function
$\widetilde{K}_{m,n}(d,r) := \sum_{u \text{ parking sorted on } K_{m,n}} d^{\mathrm{degree}(u)} r^{\mathrm{rank}(u)}$.

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This is a partial table of the coefficients of $\widetilde{K}_{5,3}(d,r)$

	d^{-3}	d^{-2}	d^{-1}	d^{0}	d^{1}	d^{2}	d^{3}	d^{4}	d^{5}	d^{6}	d^{7}	d^{8}	d^{9}	d^{10}	d^{11}	d^{12}	d^{13}	d^{14}	d^{15}	d^{16}	d^{17}
r^{9}																					105
r^{8}																				105	
r^{7}																		1	105		
r^{6}																	3	104			
r^{5}															1	8	102				
r^{4}														3	15	97					
r^{3}												1	9	27	89						
r^{2}										1	6	20	39	75							
r^{1}							1	3	9	20	36	49	57								
r^{0}				1	3	8	15	27	39	49	48	35									
r^{-1}	105	105	105	104	102	97	89	75	57	35	15										

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This suggests the change of variables
$\text{xpara}(u) := (m-1)(n-1) + \text{rank}(u) - \text{degree}(u)$
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So we consider instead

$$K_{m,n}(x,y) := \sum_{u \text{ parking sorted on } K_{m,n}} x^{\text{xpara}(u)} y^{\text{ypara}(u)}$$
$$= x^{(m-1)(n-1)} y \widetilde{K}_{m,n}(x^{-1}, xy).$$

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Theorem (D-Le Borgne)

We have $K_{m,n}(x,y) = K_{m,n}(y,x)$.

Sandpile on $K_{m,n}$: enumeration

Consider the generating function
$\mathcal{F}(x, y, w, h) := \sum_{n\geq 1, m\geq 1} K_{m,n}(x, y) w^m h^n$.

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Theorem (D-Le Borgne)

We have $\mathcal{F}(x,y,w,h) = \frac{(1-xy)(hw-P(x;w,h)P(y;w,h))}{(1-x)(1-y)(1-h-w-P(x;w,h)-P(y;w,h))}$

$$\textit{where} \quad P(q;w,h) := qwh\frac{L(qw,qh)}{L(w,h)},$$

$$L(w,h) := \sum_{n\geq 0, m\geq 0} \frac{(-1)^{m+n} h^n w^m q^{\binom{m+n+1}{2}}}{(q)_n (q)_m}$$

$$\textit{and} \quad (a)_n := \prod_{i=0}^{n-1} (1 - q^i a).$$

THE END

References

1. J.-C. Aval, M. D'Adderio, M- Dukes, Y. Le Borgne, *Two operators on sandpile configurations, the sandpile model on the complete bipartite graph, and a cyclic lemma*, Adv. in Appl. Math. **73** (2016), 59–98.
2. M. D'Adderio, Y. Le Borgne, *The sandpile model on $K_{m,n}$ and the rank of its configurations*, arXiv:1608.01521

THE END

THANKS!

References

1. J.-C. Aval, M. D'Adderio, M- Dukes, Y. Le Borgne, *Two operators on sandpile configurations, the sandpile model on the complete bipartite graph, and a cyclic lemma*, Adv. in Appl. Math. **73** (2016), 59–98.
2. M. D'Adderio, Y. Le Borgne, *The sandpile model on $K_{m,n}$ and the rank of its configurations*, arXiv:1608.01521