Shifted symmetric functions I: the vanishing property, skew Young diagrams and symmetric group characters

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Séminaire Lotharingien de Combinatoire
Bertinoro, Italy, Sept. 11th-12th-13th

Content of the lectures

Main topic: shifted symmetric functions, an analogue of symmetric functions.

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  - interesting (and powerful) vanishing results;
  - link with representation theory;
  - new kind of expansions with nice combinatorics (e.g. in multirectangular coordinates, lecture 2).
- nice extension with Jack or Macdonald parameters with many open problems (lecture 3).

Shifted symmetric function: definition

Definition

A polynomial $f(x_1, \dots, x_N)$ is shifted symmetric if it is symmetric in $x_1 - 1$, $x_2 - 2$, ..., $x_N - N$.

Example: $p^\star_k(x_1, \dots, x_N) = \sum_{i=1}^N (x_i - i)^k$.

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Shifted symmetric function: sequence $f_N(x_1, \dots, x_N)$ of shifted symmetric polynomials with

$$f_{N+1}(x_1, \dots, x_N, 0) = f_N(x_1, \dots, x_N).$$

Example: $p_k^\star = \sum_{i \geq 1} \left[(x_i - i)^k - (-i)^k\right]$.

Shifted Schur functions (Okounkov, Olshanski, '98)

Notation: $\mu = (\mu_1 \geq \cdots \geq \mu_\ell)$ partition.
$(x \downharpoonleft k) := x(x-1)\dots(x-k+1)$;

Definition (Shifted Schur function $s^\star_\mu$)

$$s^\star_\mu(x_1,\dots,x_N) = \frac{\det(x_i + N - i \downharpoonleft \mu_j + N - j)}{\det(x_i + N - i \downharpoonleft N - j)}$$

Example:

$$s_{(2,1)}(x_1,x_2,x_3) = x_1^2\,x_2 + x_1^2\,x_3 + x_1\,x_2^2 + 2\,x_1\,x_2\,x_3 + x_1\,x_3^2 + x_2^2\,x_3 + x_2\,x_3^2 \\ - x_1\,x_2 - x_1\,x_3 + x_2^2 - x_2\,x_3 + 2\,x_3^2 - 2\,x_2 - 6\,x_3$$

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- Top degree term of $s^\star_\mu$ is the standard Schur function s_μ.
- $s^\star_\mu$ is our first favorite basis of the shifted symmetric function ring $\Lambda^\star$.

Transition

The vanishing theorem
and some applications

The vanishing characterization

If λ is a partition (or Young diagram) of length ℓ and F a shifted symmetric function, we denote

$$F(\lambda) := F(\lambda_1, \dots, \lambda_\ell).$$

Easy: a shifted symmetric function is determined by its values on Young diagrams.
$\Lambda^\star$: subalgebra of $\mathcal{F}(\mathcal{Y}, \mathbb{C})$ (functions on Young diagrams).

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Theorem (Vanishing properties of $s^\star_\mu$ (OO '98))

Vanishing characterization $s^\star_\mu$ *is the* *unique* *shifted symmetric function of degree at most* $|\mu|$ *such that* $s^\star_\mu(\lambda) = \delta_{\lambda,\mu} H(\lambda)$*, where* $H(\lambda)$ *is the hook product of* λ*.*

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Extra vanishing property *Moreover, $s^\star_\mu(\lambda) = 0$, unless $\lambda \supseteq \mu$.*

The vanishing characterization

Proof of the extra-vanishing property.

By definition, $s^\star_\mu(\lambda) = \frac{\det(\lambda_i+N-i \downharpoonright \mu_j+N-j)}{\det(\lambda_i+N-i \downharpoonright N-j)}$.

Call $M_{i,j} = (\lambda_i + N - i \downharpoonright \mu_j + N - j)$.
If $\lambda_j < \mu_j$ for some j, then $M_{j,j} = 0$,

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If $\lambda_j < \mu_j$ for some j, then $M_{j,j} = 0$,
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Therefore $s^\star_\mu(\lambda) = 0$ as soon as $\lambda \not\supseteq \mu$. □

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To compute $s^\star_\mu(\mu)$, we get a triangular matrix, the determinant is the product of diagonal entries and we recognize the hook product. (Exercise!)

The vanishing characterization

Proof of uniqueness.

Let F be a shifted symmetric function of degree at most $|\mu|$.
Assume that for each λ of size at most μ,

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Write $G := F - s^\star_\mu$ as linear combination of $s^\star_\nu$:

$$G = \sum_{\nu : |\nu| \le |\mu|} c_\nu \, s^\star_\nu. \tag{1}$$

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Assume $G \neq 0$, and choose ρ minimal for inclusion such that $c_\rho \neq 0$.
We evaluate (1) in ρ:

$$0 = G(\rho) = \sum_{\nu : |\nu| \le |\mu|} c_\nu \, s^\star_\nu(\rho) = c_\rho \, s^\star_\rho(\rho) \neq 0.$$

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Contradiction $\Rightarrow$ $G = 0$, i.e. $F = s^\star_\mu$. □

Application 1: Pieri rule for shifted Schur functions

Proposition (OO '98)

$$s^\star_\mu(x_1,\dots,x_N)\,(x_1+\dots+x_N-|\mu|)=\sum_{\nu:\,\nu\nwarrow\mu} s^\star_\nu(x_1,\dots,x_N),$$

where $\nu \nwarrow \mu$ *means* $\nu \supset \mu$ *and* $|\nu| = |\mu|+1$.

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Sketch of proof.

Since the LHS is shifted symmetric of degree $|\mu|+1$, we have

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for some constants c_ν.

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- LHS vanishes for $x_i=\lambda_i$ and $|\lambda|\le|\mu| \Rightarrow c_\nu = 0$ if $|\nu|\le|\mu|$. (Same argument as to prove uniqueness.)

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  (Same argument as to prove uniqueness.)
- Look at top-degree term (and use Pieri rule for usual Schur functions):
  $\Rightarrow$ for $|\nu|=|\mu|+1$, we have $c_\nu=\delta_{\nu\nwarrow\mu}$. □

Application 2: a combinatorial formula for $s^\star_\mu$

Theorem (OO'98)

$$s^\star_\mu(x_1,\dots,x_N) = \sum_T \prod_{\square\in T} (x_{T(\square)} - c(\square)).$$

where the sum runs over reverse[a] semi-std Young tableaux T, and if $\square = (i,j)$, then $c(\square) = j - i$ (called content).

[a] filling with decreasing columns and weakly decreasing rows

Example:

$$s^\star_{(2,1)}(x_1,x_2) \;=\; \underset{\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 & \\ \cline{1-1}\end{array}}{x_2\,(x_2-1)\,(x_1+1)} \;+\; \underset{\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 & \\ \cline{1-1}\end{array}}{x_2\,(x_1-1)\,(x_1+1)}$$

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[a] filling with decreasing columns and weakly decreasing rows

- extends the classical combinatorial interpretation of Schur function (that we recover by taking top degree terms);
- completely independent proof, via the vanishing theorem (see next slide).

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To prove: $$s^\star_\mu(x_1,\dots,x_N) = \sum_T \prod_{\square\in T} (x_{T(\square)} - c(\square)).$$

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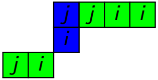
   it is sufficient to check that it is symmetric in $x_i - i$ and $x_{i+1} - i - 1$. Thus we can focus on the boxes containing i and $j := i+1$ in the tableau and reduce the general case to $\mu = (1,1)$ and $\mu = (k)$. Then it's easy.



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   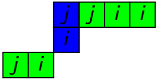

   The compatibility $\mathrm{RHS}(x_1,\dots,x_N,0) = \mathrm{RHS}(x_1,\dots,x_N)$ is straigthforward.

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1. RHS is shifted symmetric: OK.
2. $\mathrm{RHS}\big|_{x_i := \lambda_i} = 0$ if $\lambda \not\supseteq \mu$.

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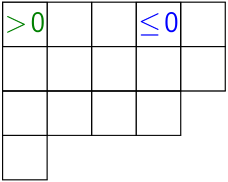

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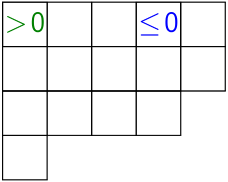

3. Normalization: check the coefficients of $x_1^{\lambda_1} \dots x_N^{\lambda_N}$. $\square$

Transition

Skew tableaux
and characters

Skew standard tableaux

Definition

Let λ and μ be Young diagrams with $\lambda \subset \mu$. A skew standard tableau of shape λ/μ is a filling of λ/μ with integers from 1 to $r = |\lambda| - |\mu|$ with increasing rows and columns.

Example

$\lambda = (3, 3, 1) \supset \mu = (2, 1)$

		2
	1	4
3		

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Alternatively, it is a sequence $\mu \nearrow \mu^{(1)} \nearrow \cdots \nearrow \mu^{(r)} = \lambda$.
The number of skew standard tableau of shape λ/μ is denoted $f^{\lambda/\mu}$.

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Skew standard tableaux and shifted Schur functions

Proposition (OO '98)

If $\lambda \supseteq \mu$, *then*

$$s^\star_\mu(\lambda) = \frac{H(\lambda)}{(|\lambda| - |\mu|)!}\, f^{\lambda/\mu}.$$

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Proof.

Set $r = |\lambda| - |\mu|$ We iterate r times the Pieri rule

$$s^\star_\mu(x_1, \dots, x_N)(x_1 + \dots + x_N - |\mu|) \cdots (x_1 + \dots + x_N - |\mu| - r + 1)$$
$$= \sum_{\substack{\nu^{(1)}, \dots, \nu^{(r)}: \\ \mu \nearrow \nu^{(1)} \nearrow \dots \nearrow \nu^{(r)}}} s^\star_{\nu^{(r)}}(x_1, \dots, x_N)$$

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We evaluate at $x_i = \lambda_i$. The only surviving term corresponds to $\nu = \lambda$. $\square$

Symmetric group characters

Facts from representation theory:

- Irreducible representation ρ^λ of the symmetric groups are indexed by Young diagrams λ;
- We are interested in computing the character $\chi^\lambda(\mu)$ of ρ^λ on any permutation in the conjugacy class $\mathcal{C}_\mu$. (here, $|\mu| = |\lambda|$).

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If $|\lambda| = |\mu| + 1$, we have $$\chi^\lambda(\mu \cup (1)) = \sum_{\nu \colon \nu \nearrow \lambda} \chi^\nu(\mu).$$

Iterating the branching rule r times gives: if $|\lambda| = |\mu| + r$,

$$\chi^\lambda(\mu \cup (1^r)) = \sum_{\substack{\nu^{(0)},\dots,\nu^{(r-1)} \\ \nu^{(0)} \nearrow \nu^{(1)} \nearrow \dots \nearrow \lambda}} \chi^{\nu^{(0)}}(\mu)$$

Symmetric group characters

Facts from representation theory:

- Irreducible representation ρ^λ of the symmetric groups are indexed by Young diagrams λ;
- We are interested in computing the character $\chi^\lambda(\mu)$ of ρ^λ on any permutation in the conjugacy class $\mathcal{C}_\mu$. (here, $|\mu| = |\lambda|$).

Proposition (Branching rule)

If $|\lambda| = |\mu| + 1$, we have
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Iterating the branching rule r times gives: if $|\lambda| = |\mu| + r$,

$$\chi^\lambda(\mu \cup (1^r)) = \sum_{\substack{\nu^{(0)},\dots,\nu^{(r-1)} \\ \nu^{(0)} \nearrow \nu^{(1)} \nearrow \dots \nearrow \lambda}} \chi^{\nu^{(0)}}(\mu) = \sum_{\nu \colon |\nu| = |\mu|} f^{\lambda/\nu} \chi^\nu(\mu).$$

Normalized characters are shifted symmetric (OO '98)

Multiply previous equality by $\frac{H(\lambda)}{(|\lambda|-|\mu|)!} = \frac{(|\lambda| \downharpoonleft |\mu|)}{\dim(\rho^\lambda)}$, we get

$$(|\lambda| \downharpoonleft |\mu|) \frac{\chi^\lambda(\mu \cup (1^r))}{\dim(\rho^\lambda)} = \sum_{\nu:\, |\nu|=|\mu|} \left(\frac{H(\lambda)}{(|\lambda|-|\mu|)!} f^{\lambda/\nu} \right) \chi^\nu(\mu)$$

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$$= \sum_{\nu:\, |\nu|=|\mu|} s^\star_\nu(\lambda)\, \chi^\nu(\mu) = \mathrm{Ch}_\mu(\lambda),$$

where $\mathrm{Ch}_\mu = \sum_{\nu:\, |\nu|=|\mu|} \chi^\nu(\mu)\, s^\star_\nu$ is a shifted symmetric function.

Example (characters on transpositions):

$$\mathrm{Ch}_{(2)}(\lambda) = s^\star_{(2)} - s^\star_{(1,1)} = \sum_{i \geq 1} \left[(\lambda_i - i)^2 + \lambda_i - i^2 \right].$$

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We'll refer to Ch_μ as normalized characters: this will be our second favorite basis of $\Lambda^\star$.

Vanishing characterization of normalized characters

Reminder: $\mathrm{Ch}_\mu = \sum_{\nu:\, |\nu|=|\mu|} \chi^\nu(\mu)\, s^\star_\nu$.

Proposition (F., Śniady, 2015)

Ch_μ *is the unique shifted symmetric function* F *of degree at most* $|\mu|$ *such that*

1. $F(\lambda) = 0$ *if* $|\lambda| < |\mu|$;
2. *The top-degree component of* F *is* p_μ.

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Proof.

Easy to check that Ch_μ fulfills 1. and 2. from $\mathrm{Ch}_\mu = \sum_{\nu:\, |\nu|=|\mu|} \chi^\nu(\mu)\, s^\star_\nu$.

Uniqueness: if F_1 and F_2 are two such functions, then $F_1 - F_2$ has degree at most $|\mu| - 1$ and vanishes on all diagrams of size $|\mu| - 1$.
$\Rightarrow\ F_1 - F_2 = 0.$ □

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Examples

The two following formulas hold since their RHS fulfills 1. and 2.:

$$\mathrm{Ch}_{(2)}(\lambda) = \sum_{i\geq 1} \left[(\lambda_i - i)^2 + \lambda_i - i^2\right].$$

$$\mathrm{Ch}_{(3)}(\lambda) = \sum_{i\geq 1} \left[(\lambda_i - i)^3 - \lambda_i + i^3\right] - 3\sum_{i<j} (\lambda_i + 1)\lambda_j.$$

Transition

Multiplications tables

Multiplication tables

Question

Can we understand the multiplication tables of our favorite bases?

$$s^\star_\mu \, s^\star_\nu = \sum_{\rho : |\rho| \le |\mu| + |\nu|} c^\rho_{\mu,\nu} s^\star_\rho$$

$$\mathrm{Ch}_\mu \; \mathrm{Ch}_\nu = \sum_{\rho : |\rho| \le |\mu| + |\nu|} g^\rho_{\mu,\nu} \, \mathrm{Ch}_\rho$$

Are $c^\rho_{\mu,\nu}$ and $g^\rho_{\mu,\nu}$ integers? nonnegative? Do they have a combinatorial interpretation?

Note: when $|\rho| = |\mu| + |\nu|$, then $c^\rho_{\mu,\nu}$ is a Littlewood-Richardson coefficient (but $c^\rho_{\mu,\nu}$ is defined more generally when $|\rho| < |\mu| + |\nu|$).

Shifted Littlewood-Richardson coefficients

$$s^\star_\mu \, s^\star_\nu = \sum_{\rho : |\rho| \le |\mu| + |\nu|} c^\rho_{\mu,\nu} s^\star_\rho \qquad (2)$$

An easy proposition

1. $c^\rho_{\mu,\nu} = 0$ if $\rho \not\supseteq \mu$ or $\rho \not\supseteq \nu$;
2. $c^\nu_{\mu,\nu} = s^\star_\mu(\nu)$.

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Proof.

1. If $\lambda \not\supseteq \mu$ or $\lambda \not\supseteq \nu$, the LHS of (2) evaluated in λ vanishes (vanishing theorem). The same argument as in the uniqueness proof implies 1.

Shifted Littlewood-Richardson coefficients

$$s^\star_\mu s^\star_\nu = \sum_{\rho:|\rho|\leq|\mu|+|\nu|} c^\rho_{\mu,\nu} s^\star_\rho \tag{2}$$

An easy proposition

1. $c^\rho_{\mu,\nu} = 0$ if $\rho \not\supseteq \mu$ or $\rho \not\supseteq \nu$;
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Proof.

1. If $\lambda \not\supseteq \mu$ or $\lambda \not\supseteq \nu$, the LHS of (2) evaluated in λ vanishes (vanishing theorem). The same argument as in the uniqueness proof implies 1.
2. We evaluated (2) in $\lambda := \nu$. Only summands with $\rho \subseteq \nu$ survive. Combining with 1., only summand $\rho = \nu$ survives and the factor $s^\star_\nu(\nu)$ simplifies. □

Shifted Littlewood-Richardson coefficients

Manipulating further the vanishing theorem, one can prove

Proposition (Molev-Sagan '99)

$$c^{\rho}_{\mu,\nu} = \frac{1}{|\rho| - |\nu|} \left(\sum_{\nu^+ \nwarrow \nu} c^{\rho}_{\mu,\nu^+} - \sum_{\rho^- \nearrow \rho} c^{\rho^-}_{\mu,\nu} \right)$$

Allows to compute all $c^{\rho}_{\mu,\nu}$ by induction on $|\rho| - |\nu|$ (μ being fixed).

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Allows to compute all $c^{\rho}_{\mu,\nu}$ by induction on $|\rho| - |\nu|$ (μ being fixed).

Next slide: combinatorial formula for $c^{\rho}_{\mu,\nu}$.
Proof strategy: show that it satisfies the same induction relation.

Shifted Littlewood-Richardson coefficients

Theorem (Molev-Sagan, '99, Molev '09)

$$c^{\rho}_{\mu,\nu} = \sum_{T,R} \mathrm{wt}(T,R),$$

T: reverse semi-standard tableau with barred entries

$\bar{3}$	$\bar{3}$	1	$\bar{1}$
2	$\bar{1}$		
$\bar{1}$			

R: sequence

$$\nu \nearrow \nu^{(1)} \dots \nearrow \nu^{(r)} = \rho.$$

(The barred entries of T indicate in which row is the box $\nu^{(i+1)}/\nu^{(i)}$, so that R is in fact determined by T.)

$$\mathrm{wt}(T,R) := \prod_{\square \text{ unbarred}} \left[\nu^{(k)}_{T(\square)} - c(\square)\right].$$

We do not explain the rule to determine k in $\nu^{(k)}_{T(\square)}$.

Shifted Littlewood-Richardson coefficients

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- all barred entries → combinatorial rule for usual LR coefficients.
- no barred entries → combinatorial formula for $s^*_\mu(x_1,\dots,x_N)$.

Shifted Littlewood-Richardson coefficients

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$$\mathrm{wt}(T,R) := \prod_{\square \text{ unbarred}} \left[\nu^{(k)}_{T(\square)} - c(\square)\right].$$

If $\nu^{(k)}_{T(\square)} - c(\square) < 0$ for some unbarred box $\square$ in some tableau T, then it vanishes for another unbarred box in the same tableau. $\Rightarrow$ $c^{\rho}_{\mu,\nu} \in \mathbb{N}_{\geq 0}$.

Multiplication table of Ch_μ (Ivanov-Kerov, '99)

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Observation (on an example):

$$\mathrm{Ch}_{(2)}(\lambda) = n(n-1)\frac{\chi^\lambda(2,1^{n-2})}{\dim(\rho^\lambda)} = \frac{1}{\dim(\rho^\lambda)}\,\mathrm{tr}\left(\rho^\lambda\left(\sum_{1\le i\neq j\le n}(i,j)\right)\right).$$

But $\rho^\lambda\left(\sum_{1\le i\neq j\le n}(i,j)\right) = \text{const}\ \mathrm{id}_{V_\lambda}$ (Schur's lemma),
so $\mathrm{Ch}_{(2)}(\lambda)$ is simply the eigenvalue of $C\ell_{(2)} := \sum_{1\le i<j\le n}(i,j)$ on ρ^λ.

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Conclusion: in general, define $C\ell_\mu = \sum_{\substack{1\le a_1,\dots,a_{|\mu|}\le n \\ \textit{distinct}}} (a_1 \cdots a_{\mu_1})\cdots$

Then the multiplication table of Ch_μ is the same as $C\ell_\mu$.

$\rightarrow$ It has nonnegative integer coefficients and is related to the multiplication table of the center of the symmetric group algebra (computing the latter is a widely studied problem!)

Multiplication table of Ch_μ (an example)

$$C\ell_{(2)} \cdot C\ell_{(2)} = \sum_{\substack{1 \leq i,j \leq n \\ i \neq j}} (i\ j) \ \cdot \sum_{\substack{1 \leq k,l \leq n \\ k \neq l}} (k\ l).$$

This looks similar to $C\ell_{(2,2)}$, except that the indices (i, j, k, l) may not be disjoint.

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This looks similar to $C\ell_{(2,2)}$, except that the indices (i, j, k, l) may not be disjoint.

$\rightarrow$ We split the sum depending on which indices are equal. We get

$$\begin{aligned} C\ell_{(2)} \cdot C\ell_{(2)} &= \sum_{i,j,k,l \text{ distinct}} (i\ j)(k\ l) \\ &\quad + 4 \sum_{i,j,l \text{ distinct}} (i\ l\ j) + 2 \sum_{i \neq j} (i)(j) \\ &= C\ell_{(2,2)} + 4 C\ell_{(3)} + 2 C\ell_{(1,1)}. \end{aligned}$$

Thus $\mathrm{Ch}_{(2)}^2 = \mathrm{Ch}_{(2,2)} + 4\,\mathrm{Ch}_{(3)} + 2\,\mathrm{Ch}_{(1,1)}$.

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- Two nice bases of the shifted symmetric function ring: shifted Schur functions $s^\star_\mu$ and normalized characters Ch_μ;
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  - Littlewood-Richardson coefficients;
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We have seen

- Two nice bases of the shifted symmetric function ring: shifted Schur functions $s^\star_\mu$ and normalized characters Ch_μ;
- Vanishing characterization theorems for these two bases and several applications for shifted Schur functions;
- That the multiplication tables of these two bases contain nonnegative coefficients which provide information on
  - Littlewood-Richardson coefficients;
  - multiplication table of the center of the symmetric group algebra.

Tomorrow: combinatorial formulas for these bases.