Flip Graphs, Yoke Graphs and Diameter

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Flip graphs

Flip graphs are graphs on sets of objects in which the adjacency relation reflects a minor change in adjacent objects.

Typical problems

- Metric properties: distance, diameter, finding antipodes and counting geodesics between them.
- Algebraic properties: presentations as Cayley/Schreier graphs, automorphism groups, eigenvalues.

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2. We extend known results, especially the diameter, to this new family.

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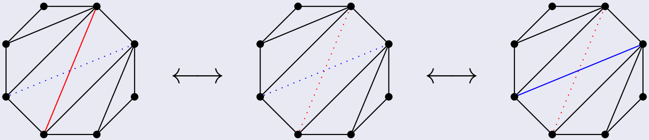
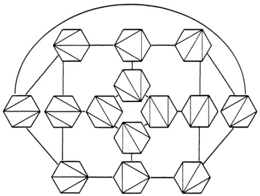




Triangulations of a Hexagon

Many variations of the triangulations flip graph have been studied. One such example is **the colored flip graph of triangle-free triangulations** studied by Adin, Firer and Roichman.

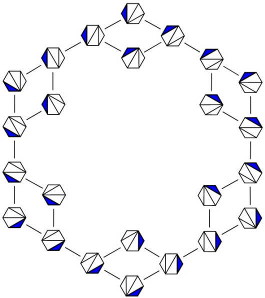
Coloured Triangle Free Triangulations (CTFT)

Triangle Free Triangulation

A triangulation is called **triangle-free**, if it contains no triangle with 3 internal edges (diagonals).

Fact

Each triangle free triangulation induces two opposite linear orders on its chords. A **coloring** *of a triangulation is a labeling of the chords by one of these orders.*



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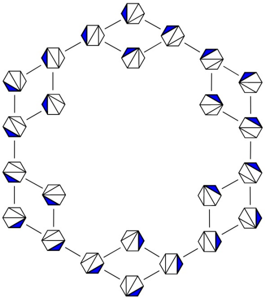


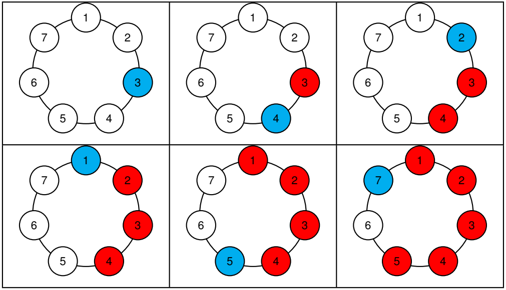
Arc Permutations

Arc Permutation

A permutation $\pi \in S_n$ is called an **arc permutation** if every prefix (and suffix) in π forms an interval in $\mathbb{Z}_n$.

Example

$\pi = 3421576$ is an arc permutation in S_7.



Arc Permutations

Cayley Flip Graphs

A (right) Cayley graph $X(G, S)$, where S is a symmetric generating set of G, is an algebraic flip graph in which right multiplication by one of the generators is the flip operation.

Arc Permutations Graph

A graph on the arc permutations of S_n. Two arc permutations π and σ are connected by an edge if $\pi = \sigma \circ (i, i+1)$ for some $1 \leq i \leq n-1$.

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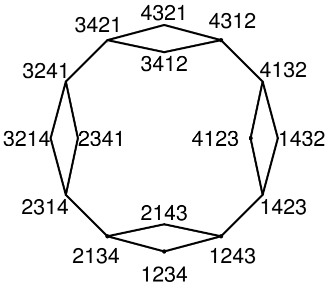
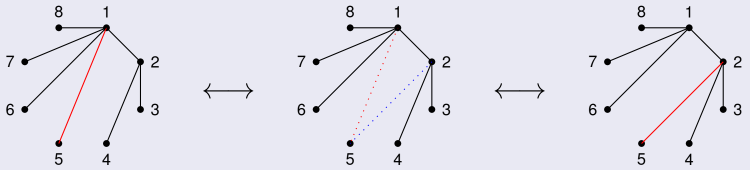


Caterpillars

Caterpillar

A (geometric) caterpillar is a non-crossing geometric tree, whose vertices are drawn on a circle, such that the non-leaves form an interval.

Flip Action



Known Results

Diameter

	Aux.	Card.	Diameter	Proof
Arc permutations	S_n	$n2^{n-2}$	$\frac{n(n-1)}{2}$	Similarity with the dominance order on $\mathbb{Z}^{n-1}$.
Caterpillars	n-trees	$n2^{n-3}$	$\lfloor\frac{n(n-2)}{2}\rfloor$	The Hurwitz graph $\mathcal{H}(S_n)$ on maximal chains in the non crossing partition lattice of S_n.
CTFT	n-gon	$n2^{n-4}$	$\frac{n(n-3)}{2}$	The weak order on $\widetilde{C}_{n-4}$.

They are all Schreier Graphs

In all of the cases an affine Weyl group acts transitively on the vertices of the graph.

- Arc permutations: $\widetilde{C}_{n-2}$.
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Yoke Graph $\mathscr{Y}_{n,m}$

Vertices

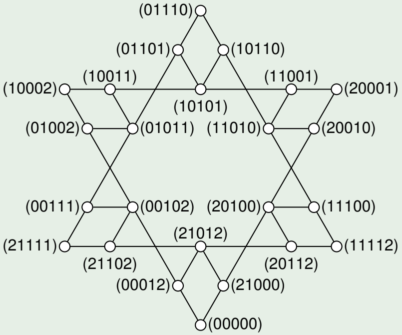
$u \in \mathscr{Y}_{n,m} \subseteq \mathbb{Z}_n \times \{0,1\}^m \times \mathbb{Z}_n$
$\sum_{i=0}^{m+1} u_i \equiv 0 (\bmod n)$.

Adjacency (Flip)

$u \sim v$ if for some $0 \leq i \leq m$

- $u_j = v_j$ $(\forall j \notin \{i, i+1\})$ and either
- $u_i = v_i + 1$ and $u_{i+1} = v_{i+1} - 1$ (**left shift**) or
- $u_i = v_i - 1$ and $u_{i+1} = v_{i+1} + 1$ (**right shift**).

Example. $\mathscr{Y}_{3,3}$



Proposition.

CTFT, Arc permutation and Caterpillar graphs are Yoke graphs.

Bijection for Arc permutations

The Arc permutations graph A_n is isomorphic to $\mathscr{Y}_{n,n-2}$.

$$\psi : A_n \to \mathbb{Z}_n \times \{0,1\}^{n-2} \times \mathbb{Z}_n$$

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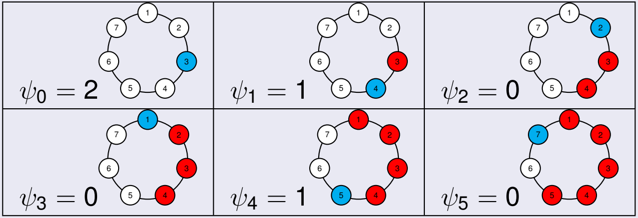

ψ respects the adjacency relation

Right multiplication by a simple reflection $(i, i+1)$ corresponds to a unit shift between ψ_i and ψ_{i+1}.
For example:

- $\psi(3421576) = (2, 1, 0, 0, 1, 0, 3)$.
- $\psi((3421576)(1, 2)) = (3, 0, 0, 0, 1, 0, 3)$.

For all of the examples $m < n$

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$\mathscr{Y}_{n,m}$ is a Schreir graph of the affine Weyl group $\widetilde{C}_m$.

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The diameter of Yoke graphs

Theorem

If $n \geq m$ then $\mathrm{diam}(\mathscr{Y}_{n,m}) = \lfloor \frac{n(m+1)}{2} \rfloor$.

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If $n \leq m$ and $2 | m - n$ or $n \leq \lceil \frac{m+1}{2} \rceil$, then

$$\mathrm{diam}(\mathscr{Y}_{n,m}) = d = \binom{\lceil \frac{m-n}{2} \rceil + 1}{2} + \binom{\lfloor \frac{m+n}{2} \rfloor + 1}{2}.$$

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Sketch of proof

The eccentricity of 0 in $\mathscr{Y}_{n,m}$

It can be shown that $\mathrm{ecc}_{\mathscr{Y}_{n,m}}(0)$ is equal to the value of the diameter in the theorem. Alas, we couldn't prove that 0 is an antipode in $\mathscr{Y}_{n,m}$.

Definition (*dYoke* graphs $\mathscr{Z}_{n,m}$)

- **Vertices**
  $u \in \mathscr{Z}_{n,m} \subseteq \mathbb{Z}_n \times \{0, \pm 1\}^m \times \mathbb{Z}_n$ such that $\sum_{i=0}^{m+1} u_i \equiv 0 \pmod{n}$.
- **Adjacency (Flip)** is the same as in Yoke graphs.

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Observation

$$\begin{aligned}\varphi_u : \mathscr{Y}_{n,m} &\to \mathscr{Z}_{n,m}\\ v &\mapsto v - u\end{aligned}$$

is a faithful, injective homomorphism for every $u \in \mathscr{Y}_{n,m}$.

And the images of φ_u cover $\mathscr{Z}_{n,m}$.

Lemma

For every $v, u \in \mathscr{Y}_{n,m}$, $d_{\mathscr{Y}_{n,m}}(v, u) = d_{\mathscr{Z}_{n,m}}(v - u, 0)$.

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