Why the Genocchi Numbers are Integers?

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Tangent numbers, Genocchi numbers

The Genocchi numbers $\{G_{2n+2}\}_{n\geq 0}$ can be defined by the generating function:

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This implies

$$G_{2n+2}=\frac{(n+1)T_{2n+1}}{2^{2n}}, \quad \text{where } \tan x=\sum_{n\geq 0}T_{2n+1}\frac{x^{2n+1}}{(2n+1)!}.$$

$\{T_{2n+1}\}_{n\geq 0}$: the Tangent numbers.

Tangent numbers, Genocchi numbers

Here are the first values:

n	0	1	2	3	4	5	6
T_{2n+1}	1	2	16	272	7936	353792	22368256
G_{2n+2}	1	1	3	17	155	2073	38227

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- There are algebraic ways to prove the fact (Carlitz 1971, Riordan-Stein 1973).
- What about a combinatorial proof?

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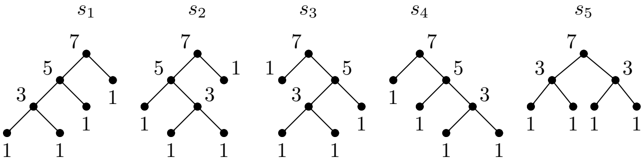

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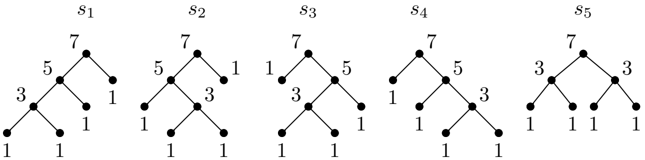

Hook length formula:

$$\#\mathcal{L}(t) = \frac{n!}{\prod_{v \in t} h_v}.$$

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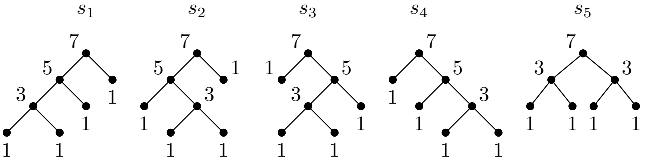
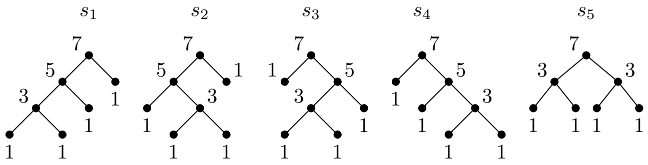
For example, let $2n+1=7$:



$$T_7 = \frac{7!}{7\times 5\times 3\times 1^4}\times 4 + \frac{7!}{7\times 3^2\times 1^4} = 272.$$

Combinatorial interpretation of the tangent numbers

We can partition the set of all complete binary trees with $2n+1$ vertices into equivalence classes:



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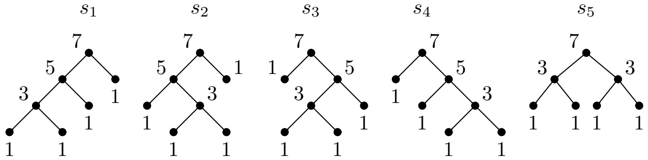

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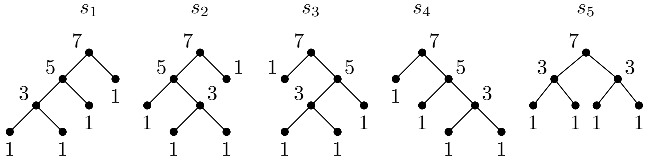

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- All trees in the same equivalence class share the same product of all hook lengths.

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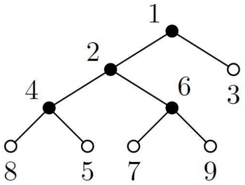

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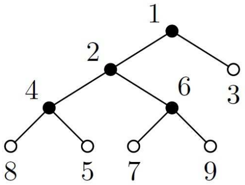

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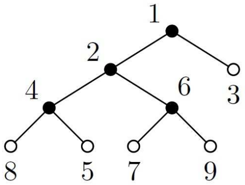

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While the shapes of these 2^n trees are all equivalent, this divisibility actually holds in each equivalence class $\bar{s}$.

$$2^n \mid T(\bar{s}).$$

where $T(\bar{s})$ stands for the number of increasing labeled trees whose shapes are in $\bar{s}$.

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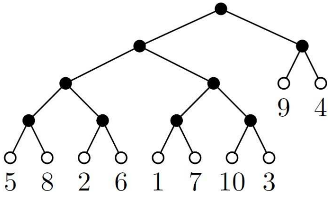

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\begin{aligned}
f(n) &= \sum_{\bar{s}} \frac{(n+1)T(\bar{s})H(\bar{s})}{2^{2n}} \\
&= \sum_{\bar{s}} \frac{(2n+2)! \times \#\bar{s}}{2^{2n+1}} \\
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While $\sum_{\bar{s}} \#\bar{s}$ equals to the Catalan number C_n, we can calculate that

$$
\begin{aligned}
f(n) &= \frac{(2n+2)!}{2^{2n+1}} \cdot C_n \\
&= \frac{(2n+2)!}{2^{2n+1}} \cdot \frac{1}{n+1}\binom{2n}{n} \\
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Let L_{kn+1} be the number of increasing labeled complete k-ary trees with $kn+1$ vertices, and

$$\phi(x) := \sum_{n \geq 0} L_{kn+1} \frac{x^{kn+1}}{(kn+1)!}$$

be the exponential generating function of L_{kn+1}.

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Let $M_{k^2n-kn+k}$ defined as

$$M_{k^2n-kn+k} := \frac{(k^2n-kn+k)!\, L_{kn+1}}{(k!)^{kn+1}(kn+1)!}$$

and

$$\psi(x) := \sum_{n\geq 0} M_{k^2n-kn+k} \frac{x^{k^2n-kn+k}}{(k^2n-kn+k)!}.$$

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Generalizations to k-ary trees

It's not difficult to calculate that

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Compare with our original problem:

the exponential generating function of T_{2n+1}, $y = \tan x$, satisfied the differential equation $y' = y^2 + 1$;

and the exponential generating function of G_{2n+2}, satisfied that

$$g(x) := \sum_{n \geq 0} G_{2n+2} \frac{x^{2n+2}}{(2n+2)!} = x \cdot \tan \frac{x}{2}.$$

Generalizations to k-ary trees

Theorem

(a) For each integer $k \geq 2$, the integer

$$\frac{(k^2n - kn + k)!\, L_{kn+1}}{(kn+1)!}$$

is divisible by $(k!)^{kn+1}$.
(b) Moreover, the quotient

$$M_{k^2n-kn+k} := \frac{(k^2n - kn + k)!\, L_{kn+1}}{(k!)^{kn+1}(kn+1)!} \equiv \begin{cases} 1 \pmod{k}, & k = p, \\ 1 \pmod{p^2}, & k = p^t,\ t \geq 2, \\ 0 \pmod{k}, & \textit{otherwise}, \end{cases}$$

where $n \geq 1$ and p is a prime number.

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- Partition the set of all complete k-ary trees with $kn+1$ vertices into equivalence classes, and restrict the divisibility in one equivalence class $\bar{r}$;

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- $k^{kn+1} \mid \dfrac{(k^2n-kn+k)!L(\bar{r})}{(kn+1)!(k-1)!^{kn+1}}$, while $H(\bar{r}) \equiv 1 \pmod{k}$, it's equivalent to

$$k^{kn+1} \mid (k^2n-kn+k)! \cdot \frac{L(\bar{r})H(\bar{r})}{(kn+1)!(k-1)!^{kn+1}}.$$

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Let r be a complete k-ary tree with $kn+1$ vertices. Add a new sheaf of leaves to r and put labels $\{1, 2, \cdots, k^2n - kn + k\}$ on these new leaves. Since such a leaf-labeled tree is totally determined by the order of labels as well as the shape of the original tree, we have the number of leaf-labeled trees is exactly

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While this leaf-labeled tree has $kn+1$ non-leaf vertices, and each time of subtree exchanging will make a difference on the order of labels, we can conclude that

$$(k!)^{kn+1} \mid (k^2n-kn+k)! \cdot \#\bar{r}.$$

Proof of (b)

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- Construct the weighted function

$$f(n) := \sum_{\bar{r}} \frac{(k^2n - kn + k)!\, L(\bar{r})H(\bar{r})}{(k!)^{kn+1}\,(kn+1)!} \equiv M_{k^2n-kn+k} \pmod{k}.$$

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- Calculate $f(n)$:

$$\begin{aligned} f(n) &= \sum_{\bar{r}} \frac{(k^2n - kn + k)!\, L(\bar{r})H(\bar{r})}{(k!)^{kn+1}\,(kn+1)!} \\ &= \sum_{\bar{r}} \frac{(k^2n - kn + k)! \times \#\bar{r}}{(k!)^{kn+1}} \\ &= \frac{(k^2n - kn + k)!}{(k!)^{kn+1}} C_k(n), \end{aligned}$$

Proof of (b)

- $C_k(n)$ is the number of all (unlabeled) complete k-ary trees, which is equal to the Fuss-Catalan number

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$$\begin{aligned} f(n) &= \frac{(k^2n-kn+k)!}{(k!)^{kn-n+1}(kn-n+1)!} \cdot \frac{(kn)!}{(k!)^n n!} \\ &= \prod_{i=0}^{kn-n} \binom{ik+k-1}{k-1} \times \prod_{j=0}^{n-1} \binom{jk+k-1}{k-1}. \end{aligned}$$

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$$\binom{ip+p-1}{p-1} = \frac{(ip+1)(ip+2)\cdots(ip+p-1)}{1\times 2\times\cdots\times(p-1)} \equiv 1 \pmod{p}.$$

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- If $k = p^t$, then

$$\binom{ip^t+p^t-1}{p^t-1} \equiv \binom{ip^{t-1}+p^{t-1}-1}{p^{t-1}-1} \equiv \cdots \equiv \binom{ip+p-1}{p-1}$$
$$\equiv 1 \pmod{p^2} \quad (p \geq 3),$$

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$$\begin{aligned}\binom{i2^t+2^t-1}{2^t-1} &\equiv \binom{i\cdot 2^{t-1}+2^{t-1}-1}{2^{t-1}-1} \equiv \cdots \equiv \binom{i\cdot 2+2-1}{2-1} \\ &= 2i+1 \pmod{4} \quad (p=2).\end{aligned}$$

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$$\binom{ip^t+p^t-1}{p^t-1} \equiv \binom{ip^{t-1}+p^{t-1}-1}{p^{t-1}-1} \equiv \cdots \equiv \binom{ip+p-1}{p-1} \\ \equiv 1 \pmod{p^2} \quad (p \geq 3),$$

$$\binom{i2^t+2^t-1}{2^t-1} \equiv \binom{i\cdot 2^{t-1}+2^{t-1}-1}{2^{t-1}-1} \equiv \cdots \equiv \binom{i\cdot 2+2-1}{2-1} \\ = 2i+1 \pmod{4} \quad (p = 2).$$

  In both cases we can conclude that $f(n) \equiv 1 \pmod{p^2}$.

Proof of (b)

- Suppose that k has more than one prime factors. Let q be one prime factor, write $k = bq^m$ with $b \geq 2$ and $(b, q) = 1$. Notice that $f(n) \mid f(n+1)$, we only need to show

$$f(1) = \frac{(k^2)!}{(k!)^{k+1}} \equiv 0 \pmod{q^m}.$$

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Lemma (Legendre's formula)

Suppose that p is a prime number. For each positive integer k, let $\alpha(k)$ be the highest power of p dividing $k!$ and $\beta(k)$ be the sum of all digits of k in base p. Then,

$$\alpha(k) = \sum_{i \geq 1} \left\lfloor \frac{k}{p^i} \right\rfloor = \frac{k - \beta(k)}{p - 1}.$$

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$$M_{k^2n-kn+k} \equiv 1, 1, 5, 5, 1, 1, 5, 5, \cdots \pmod{2^3}.$$

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  Remains to be solved.

Thank you all for your attention!