79th Sèminaire Lotharingien de Combinatoire and

XXI Incontro Italiano di Combinatoria Algebrica

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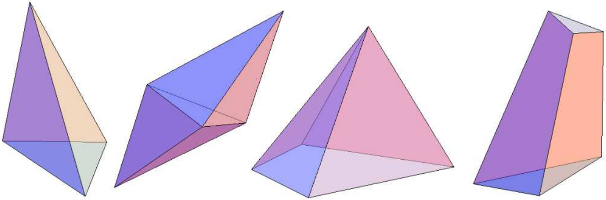


Slack ideals of Polytopes

Antonio Macchia

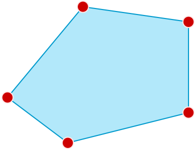
Università degli Studi di Bari

joint work with João Gouveia, Rekha Thomas, Amy Wiebe

Polytopes

A ***polytope*** P is a convex hull of a finite set of points in some $\mathbb{R}^d$:

$$P = \operatorname{conv}(p_1, \ldots, p_v)$$



$\mathcal{V}$-representation

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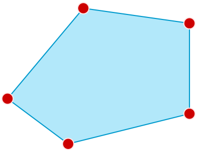

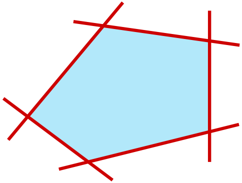

$\mathcal{V}$-representation ***$\mathcal{H}$-representation***

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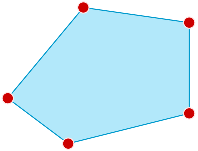
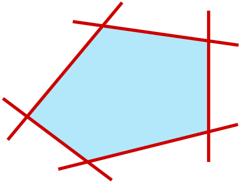


$\mathcal{V}$-representation



$\mathcal{H}$-representation

The ***dimension*** of a polytope is the dimension of its affine hull.

The ***face lattice*** of P is the set of all faces of P, ordered by inclusion.

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Find all polytopes of a fixed dimension:

- combinatorial types of polytopes $\leftrightarrow$ finite lattices corresponding to face lattices of polytopes,
- describe the set of all realizations of a given combinatorial type (realization space).

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Theorem (Steinitz, 1922) *A graph G is the edge graph of a 3-polytope $\Leftrightarrow$ G is simple, planar and 3-connected.*

Realization Space

Given: combinatorial type of a d-polytope

$\rightarrow$ face lattice or vertex-facet incidences

Want: all ways to realize this type in $\mathbb{R}^d$

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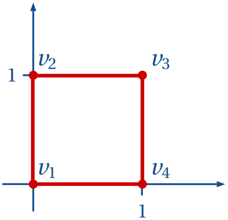
P is a quadrilateral
4 vertices $\{v_1, v_2, v_3, v_4\}$
4 facets $\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_1\}$



Combinatorial equivalence

$$Q \overset{c}{=} P \Leftrightarrow P \text{ and } Q \text{ have the same vertex-facet incidences}$$

All quadrilaterals are combinatorially equivalent to a square

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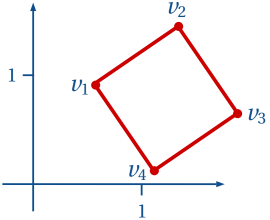

Affine equivalence

$$Q \overset{a}{=} P \Leftrightarrow Q = \psi(P), \;\; \psi(\boldsymbol{x}) = A\boldsymbol{x} + \boldsymbol{b}$$

preserves parallel lines, e.g. scaling, rotation, reflection, translation

Parallelograms are affinely equivalent to a square

Realization Space

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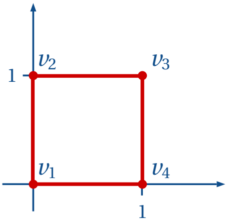


Projective equivalence

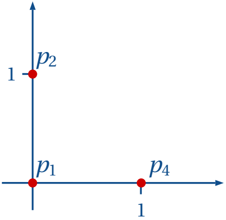
$$Q \overset{p}{=} P \Leftrightarrow Q = \phi(P), \;\; \phi(\boldsymbol{x}) = \frac{A\boldsymbol{x} + \boldsymbol{b}}{\boldsymbol{c}^\intercal + d}, \;\; \det \begin{bmatrix} A & \boldsymbol{b} \\ \boldsymbol{c}^\intercal & d \end{bmatrix} \neq 0$$

All quadrilaterals are projectively equivalent to a square.
A square is ***projectively unique***.

Realization Space

Set of all realizations of polytopes combinatorially equivalent to P

Mod out affine transformation:
fix an affine basis B of $d+1$ labelled vertices



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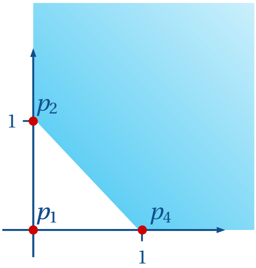

Realization space

$$\mathcal{R}(P, B) = \left\{ Q = \operatorname{conv}\{q_1, \ldots, q_n\} \subset \mathbb{R}^d : q_i = p_i \; \forall p_i \in B, \, Q \overset{c}{=} P \right\}$$

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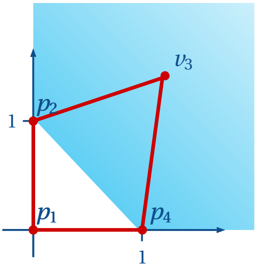

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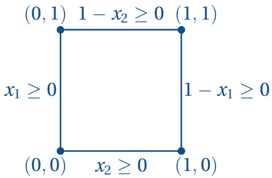
Slack Matrices

P: d-polytope

vertices: $\{p_1, \ldots, p_v\}$

facet inequalities:

$$\begin{gathered}\beta_1 - \boldsymbol{a}_1^\intercal \boldsymbol{x} \geq 0 \\ \vdots \\ \beta_f - \boldsymbol{a}_f^\intercal \boldsymbol{x} \geq 0\end{gathered}$$



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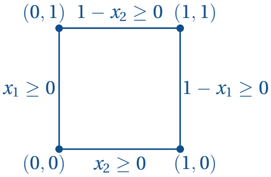

Slack matrix

$$S_P = \begin{pmatrix} & \vdots & \\ \cdots & \beta_j - a_j^\intercal p_i & \cdots \\ & \vdots & \end{pmatrix}$$

$$S_P = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

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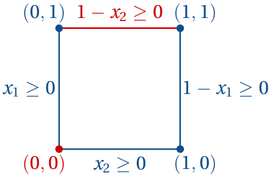

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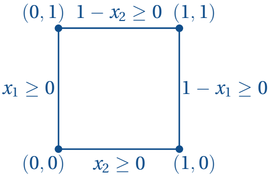

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zero pattern $\leftrightarrow$ combinatorics

$\mathrm{rank}(S_P) = d + 1$

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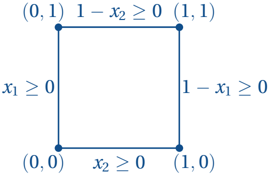

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`zero pattern ↔ combinatorics`

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$\mathtt{rank}(\mathtt{S_P}) = \mathtt{d} + \mathtt{1}$

Theorem (GGKPRT, 2013) *A nonnegative matrix S is the slack matrix of some realization of P if and only if*

1. $\operatorname{supp}(S) = \operatorname{supp}(S_P)$;
2. $\operatorname{rank}(S) = \operatorname{rank}(S_P) = d + 1$;
3. *the all ones vector lies in the column span of S.*

Slack Ideal

Symbolic slack matrix

Replace nonzero entries of S_P by distinct variables.

$$S_P = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \rightarrow S_P(\boldsymbol{x}) = \begin{pmatrix} 0 & x_1 & x_2 & 0 \\ 0 & 0 & x_3 & x_4 \\ x_5 & 0 & 0 & x_6 \\ x_7 & x_8 & 0 & 0 \end{pmatrix}$$

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Slack ideal

$$I_P = \langle (d+2)\text{-minors of } S_P(\boldsymbol{x}) \rangle : \left(\prod x_i \right)^\infty$$

$$I_P = \langle x_2 x_4 x_5 x_8 - x_1 x_3 x_6 x_7 \rangle$$

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Theorem (Gouveia, M, Thomas, Wiebe, 2017)

$\mathcal{V}_+(I_P)/(\mathbb{R}^v_{>0} \times \mathbb{R}^f_{>0}) \xleftrightarrow{1:1}$ *classes of projectively equivalent polytopes of the same combinatorial type as P.*

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A new computational tool for testing polytopal properties.

Application 1: Realizability

Steinitz problem *Check whether an abstract polytopal complex is the boundary of an actual polytope.*

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[Altshuler, Steinberg, 1985]: 4-polytopes and 3-spheres with 8 vertices.

The smallest non-polytopal 3-sphere has vertex-facet non-incidence matrix

$$S_P(\boldsymbol{x}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & * \\ 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 \\ 0 & * & 0 & * & 0 & 0 & 0 & * & * & * \\ * & 0 & * & 0 & 0 & * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & * & * & * & 0 \end{pmatrix}.$$

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Proposition *P is realizable $\iff$ $\mathcal{V}_+(I_P) \neq \varnothing$.*

In this case, $I_P = \langle 1 \rangle \Rightarrow$ no rank 5 matrix with this support $\Rightarrow$ no polytope with the given facial structure.

Application 2: Prescribability of faces

Lemma *F face of $P \Rightarrow S_F$ submatrix of S_P and $I_F \subset I_P$.*
F prescribable in $P \Leftrightarrow \mathcal{V}_+(I_F) = \mathcal{V}_+(I_P \cap \mathbb{C}[\boldsymbol{x}_F])$.

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[Barnette, 1987]: 4-dimensional prism over a square pyramid with a non-prescribable cubical facet F

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$S_F(\boldsymbol{x})$

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$$\dim(I_P \cap \mathbb{C}[\boldsymbol{x}_F]) = 15, \dim(I_F) = 16 \Rightarrow I_F \neq I_P \cap \mathbb{C}[\boldsymbol{x}_F]$$

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$$S_P(\boldsymbol{x}) = \begin{pmatrix} x_1 & 0 & 0 & 0 & x_2 & x_3 & 0 \\ x_4 & 0 & 0 & 0 & 0 & x_5 & x_6 \\ x_7 & 0 & 0 & x_8 & 0 & 0 & x_9 \\ x_{10} & 0 & 0 & x_{11} & x_{12} & 0 & 0 \\ x_{13} & 0 & x_{14} & 0 & 0 & 0 & 0 \\ 0 & x_{15} & 0 & 0 & x_{16} & x_{17} & 0 \\ 0 & x_{18} & 0 & 0 & 0 & x_{19} & x_{20} \\ 0 & x_{21} & 0 & x_{22} & 0 & 0 & x_{23} \\ 0 & x_{24} & 0 & x_{25} & x_{26} & 0 & 0 \\ 0 & x_{27} & x_{28} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$S_F(\boldsymbol{x})$

$$\dim(I_P \cap \mathbb{C}[\boldsymbol{x}_F]) = 15, \dim(I_F) = 16 \Rightarrow I_F \neq I_P \cap \mathbb{C}[\boldsymbol{x}_F]$$

Computations show that the cube admits further realizations than are possible as a face of $P \Rightarrow \mathcal{V}_+(I_F) \neq \mathcal{V}_+(I_P \cap \mathbb{R}[\boldsymbol{x}_F])$

Application 3: Rationality

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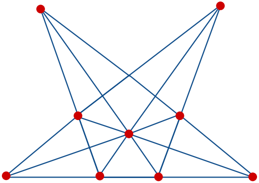

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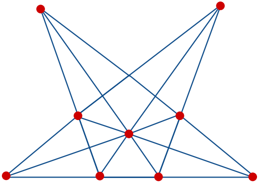

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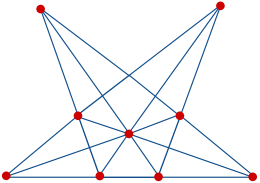

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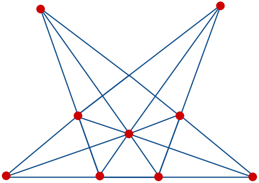

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What else can we determine from I_P?

How nice can I_P be?

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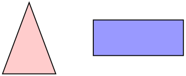
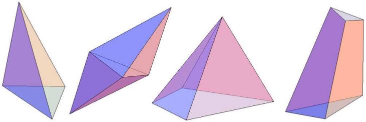
- $d = 2$: 
- $d = 3$: 
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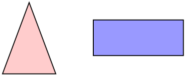
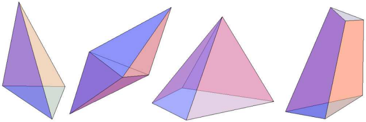
- $d = 2$: 
- $d = 3$: 
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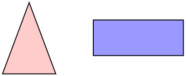
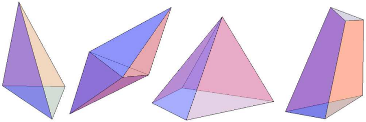
- $d = 2$: 
- $d = 3$: 
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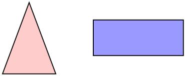
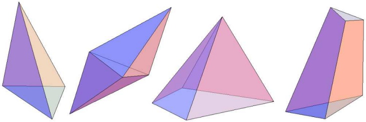
- $d = 2$: 
- $d = 3$: 
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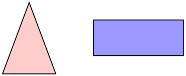
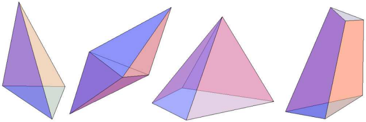
Always true?

- $d = 2$: 
- $d = 3$: 
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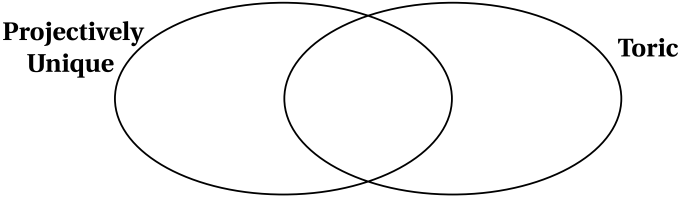

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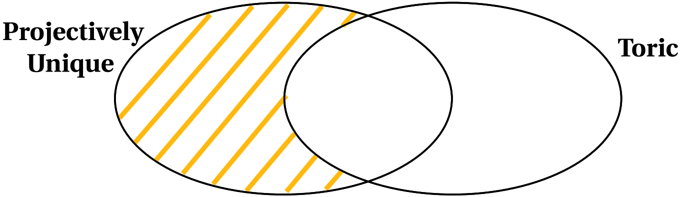

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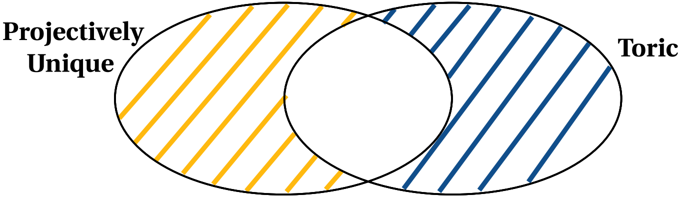

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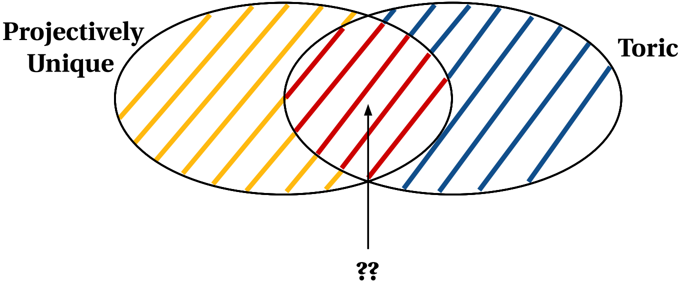

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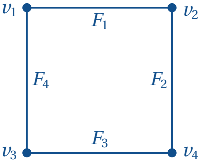
- ***Which polytopes are toric and PU?***

Non-incidence graph of a polytope

Polytope P $\longleftrightarrow$ Slack Matrix S_P



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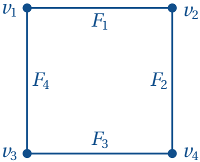

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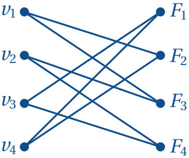

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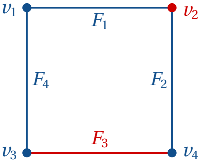

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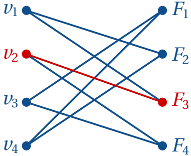

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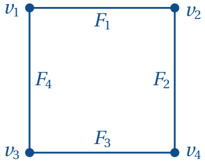

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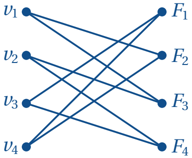

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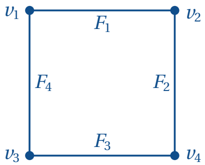

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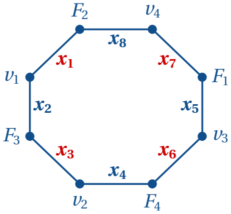

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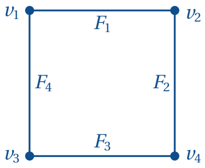

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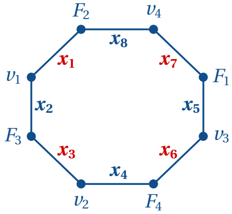

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What is the relation between I_P and T_P?

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Geometric meaning of polytopes for which $I_P \subseteq T_P$.

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Theorem (Gouveia, M, Thomas, Wiebe, 2017)

1. *A polytope P is morally 2-level* $\Leftrightarrow I_P \subseteq T_P$.
2. I_P *is graphic* $\Leftrightarrow I_P$ *toric and P projectively unique.*

Conclusions

- Slack matrix encodes the combinatorics of polytopes
- Positive part of the slack variety as a model of the realization space for modding out projective equivalence
- Slack ideals gives new computational framework for classic polytopal questions
- New characterization of class of projectively unique polytopes via slack ideal: graphic polytopes are PU

What next?

- Continue to improve this new dictionary between algebra and combinatorics of polytopes

Thank you for listening!