On zeros of irreducible characters of the symmetric group S_n

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Universität Wien

79th Séminaire Lotharingien de Combinatoire
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Irreducible characters of S_n

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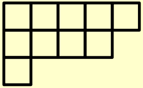
$$\chi^\lambda : S_n \to \mathbf{Z}$$

Murnaghan–Nakayama rule for computing $\chi^{\lambda}(g)$

λ



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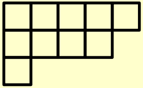

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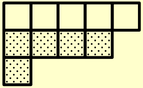

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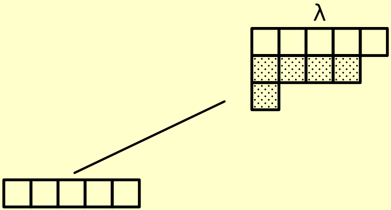

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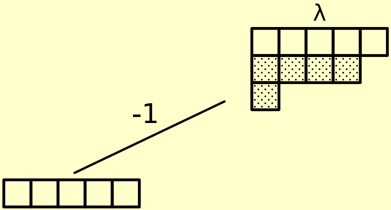

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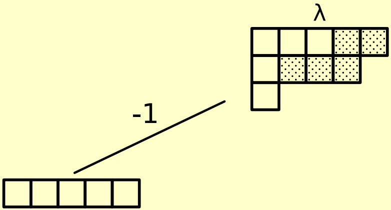

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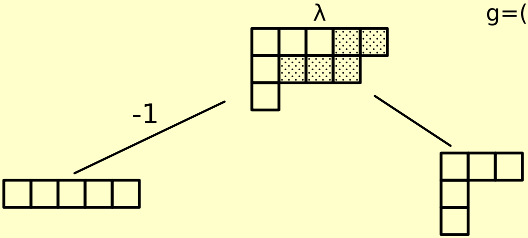

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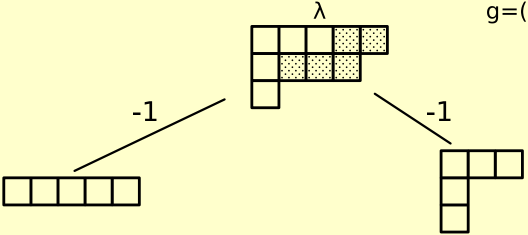

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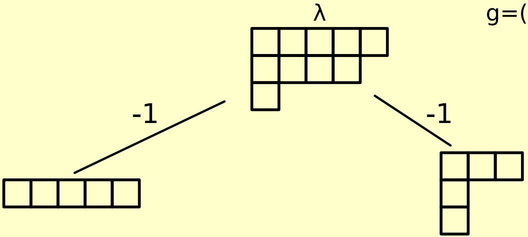

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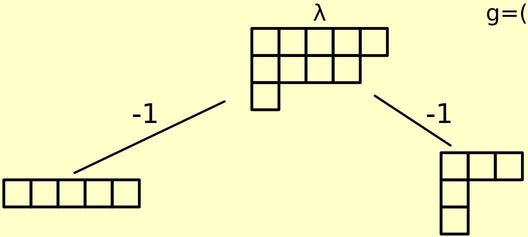

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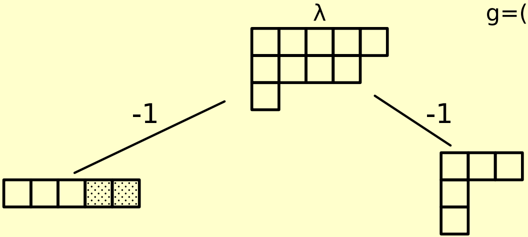

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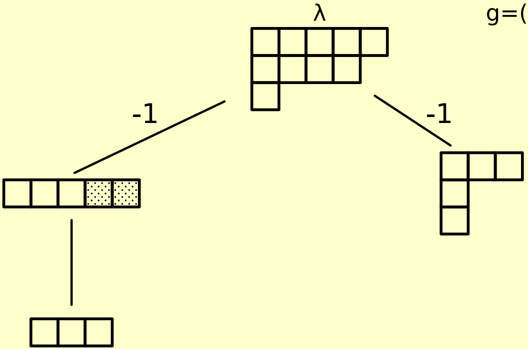

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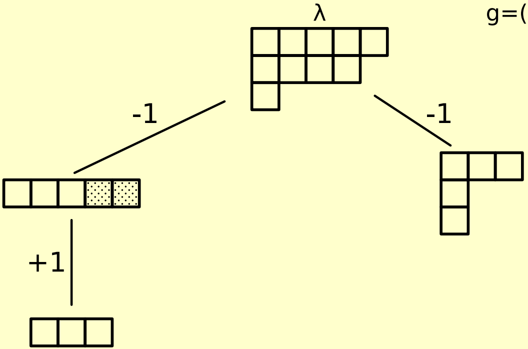

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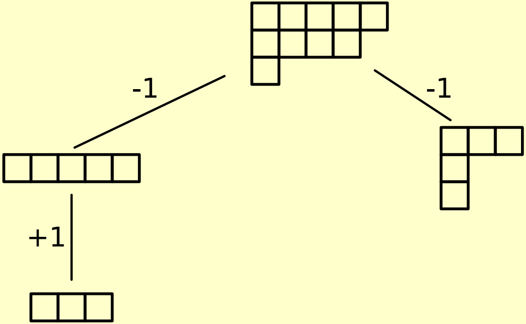

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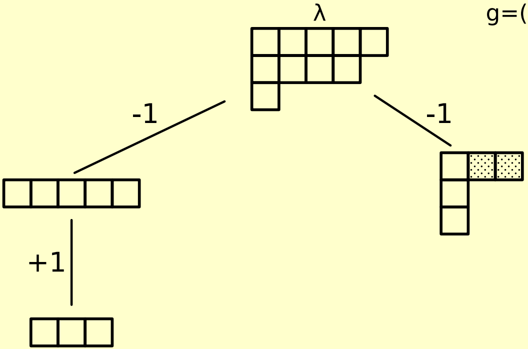

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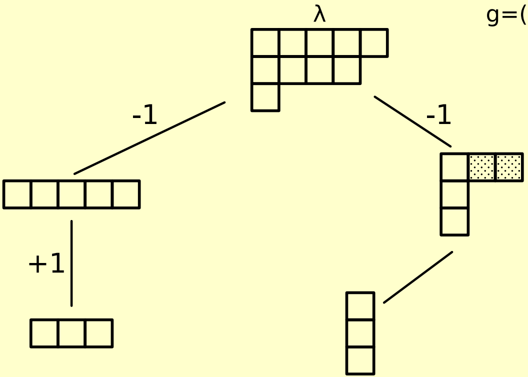

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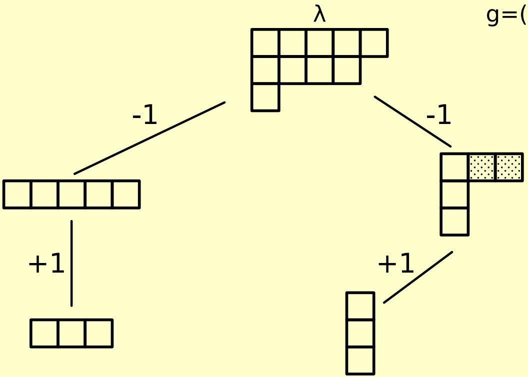

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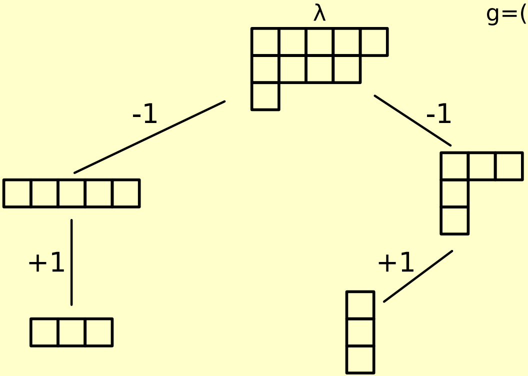

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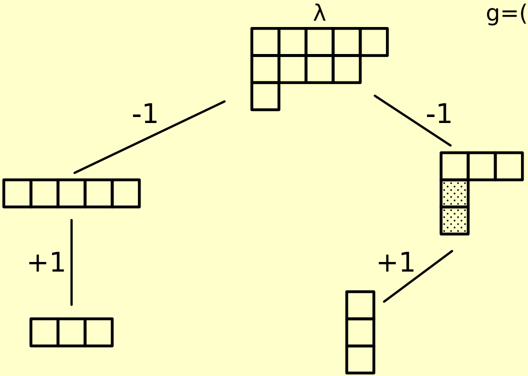

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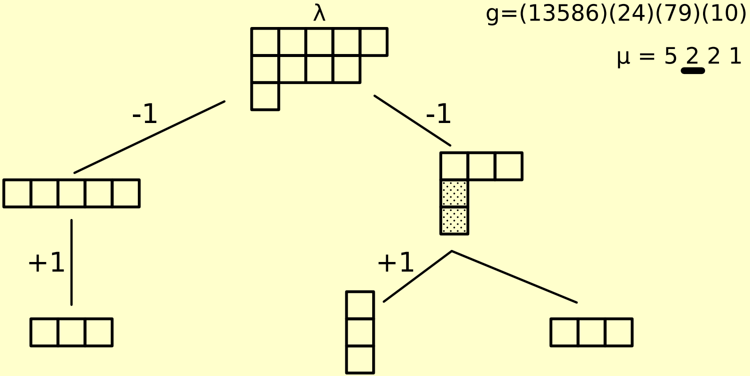

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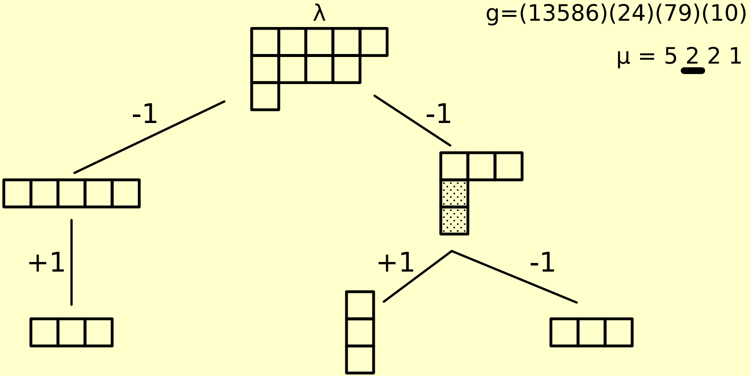

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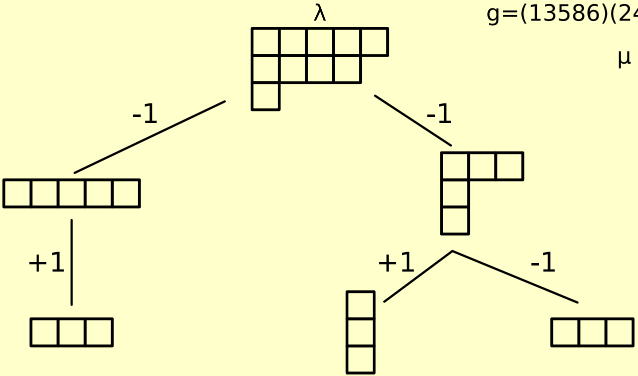

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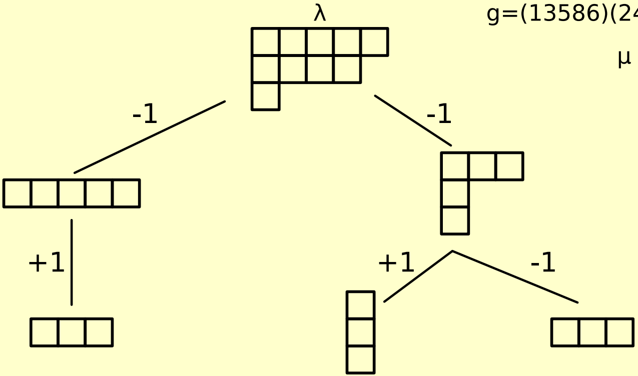

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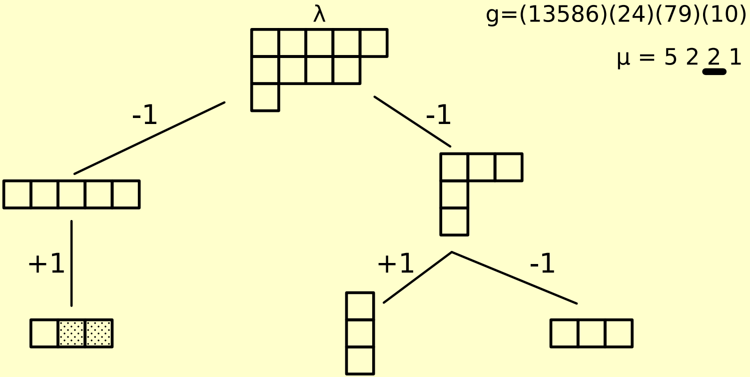

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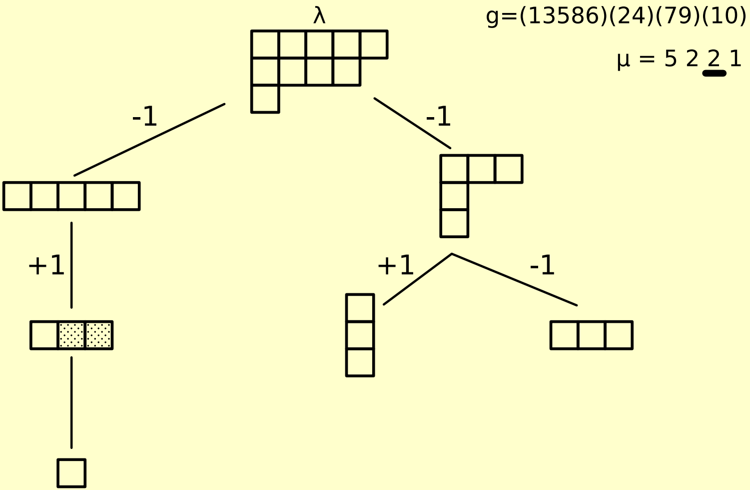

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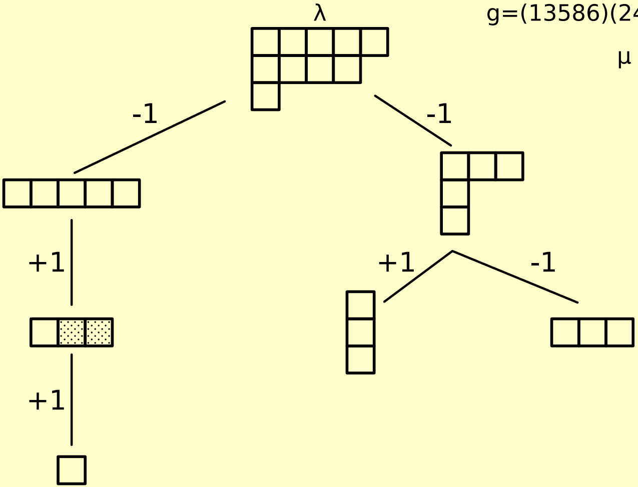

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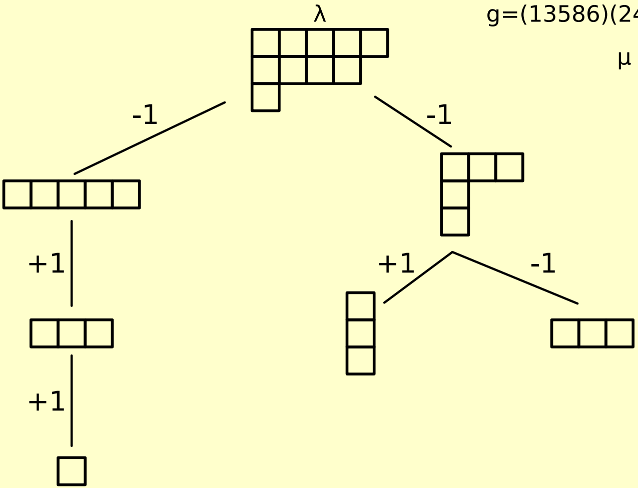

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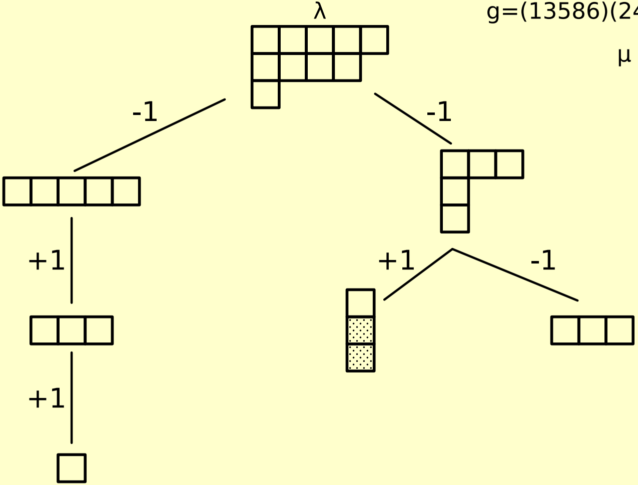

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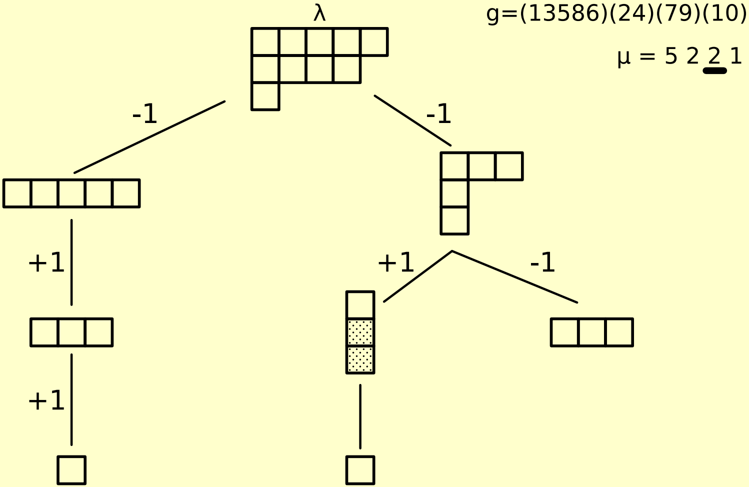

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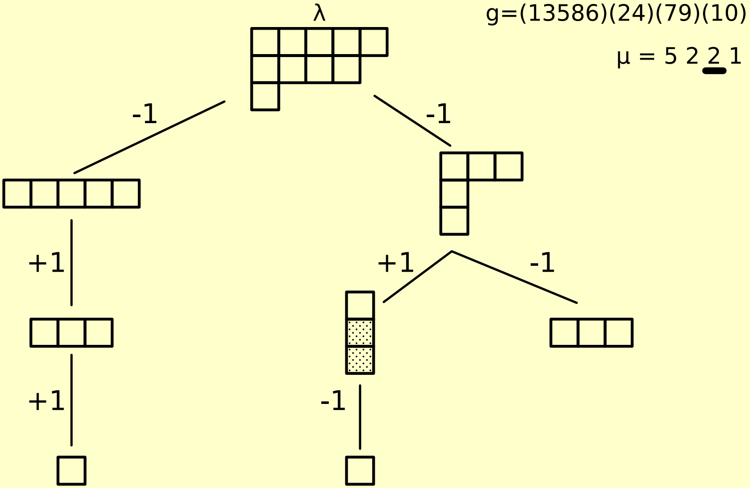

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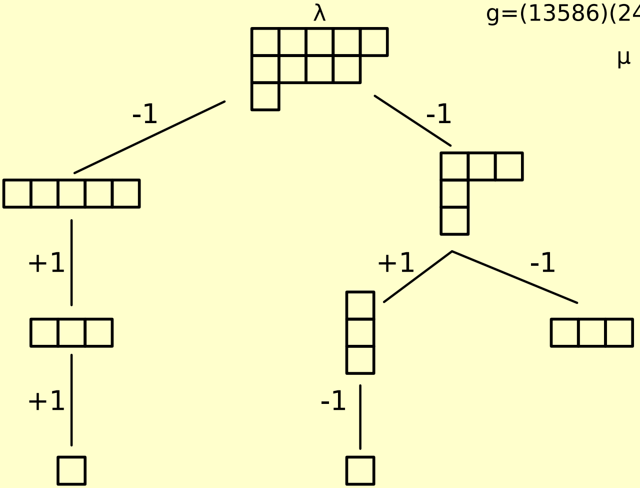

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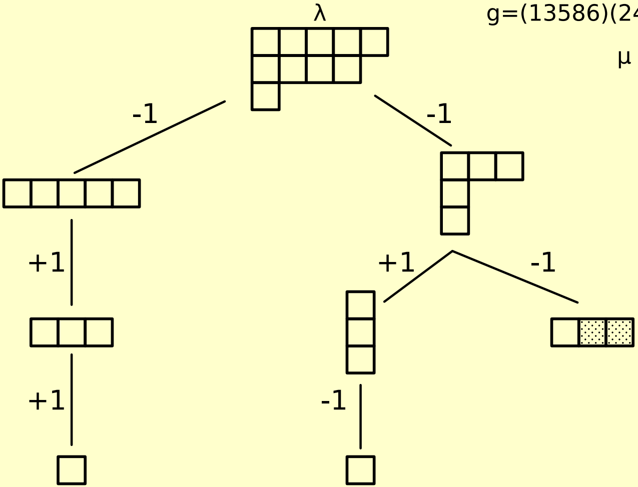

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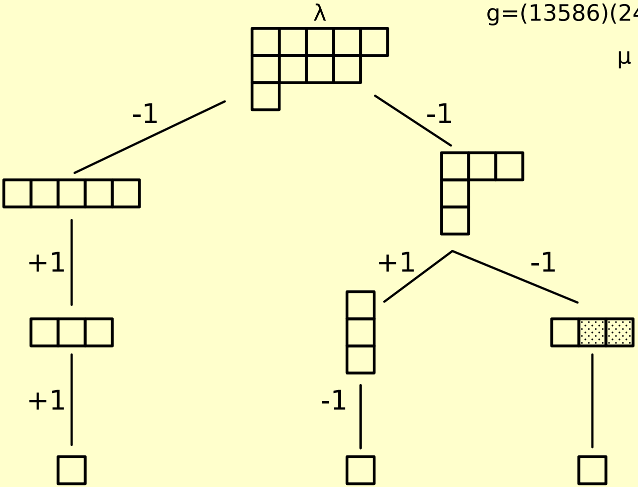

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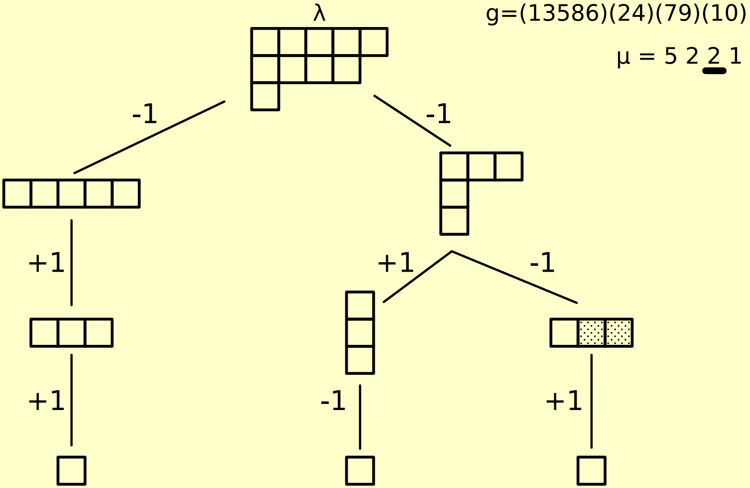

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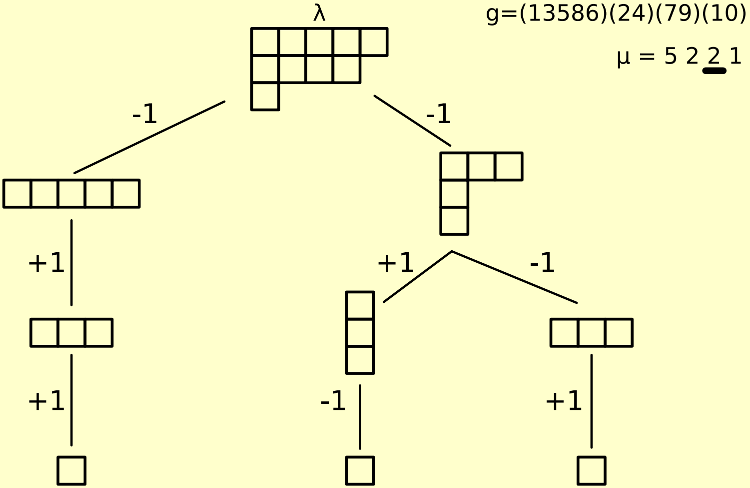

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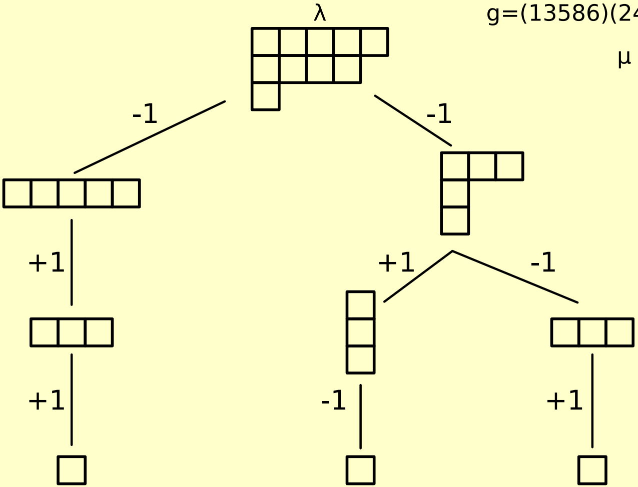

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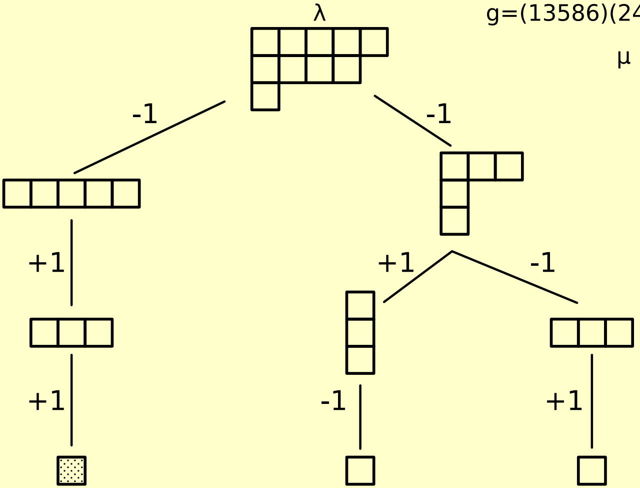

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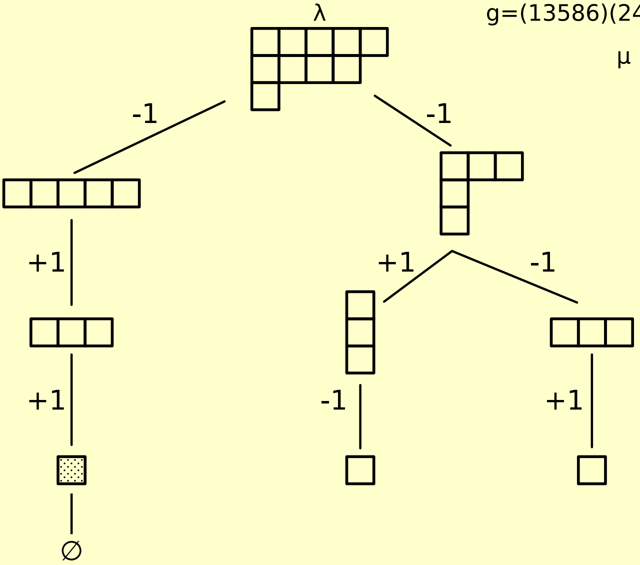

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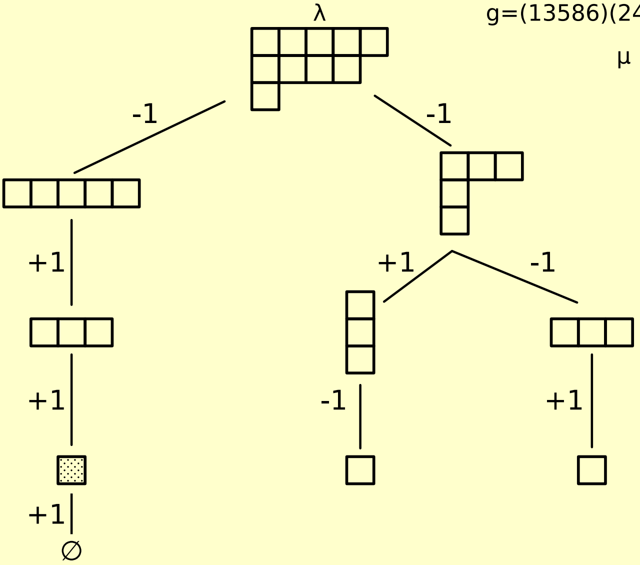

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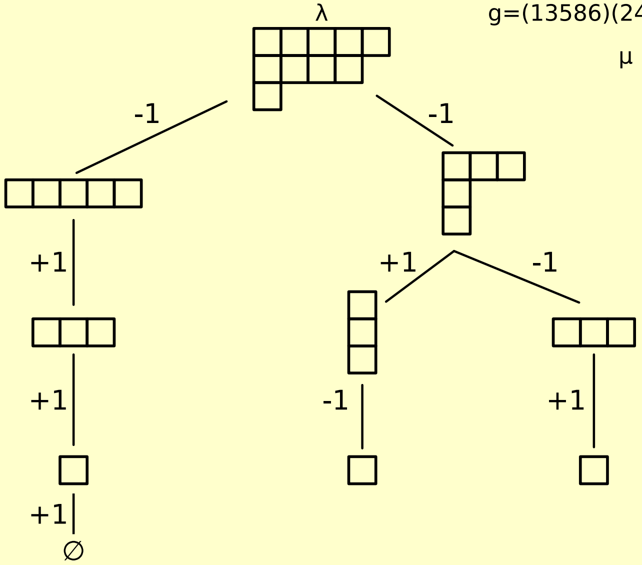

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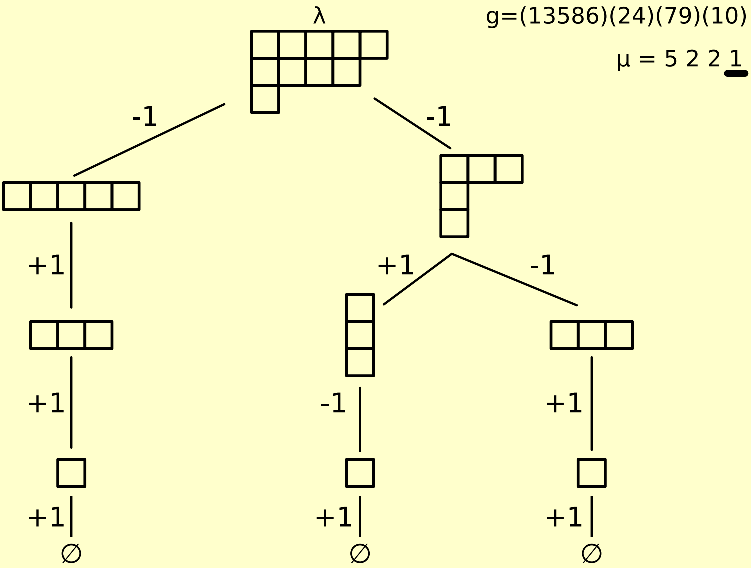

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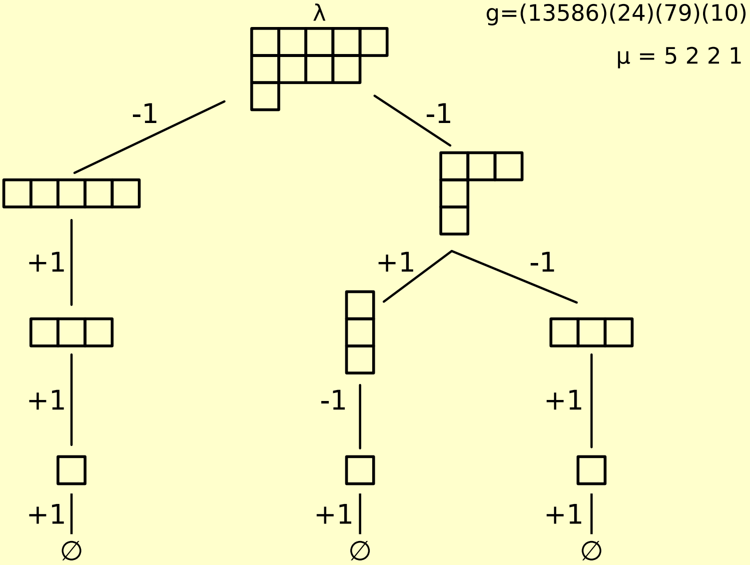
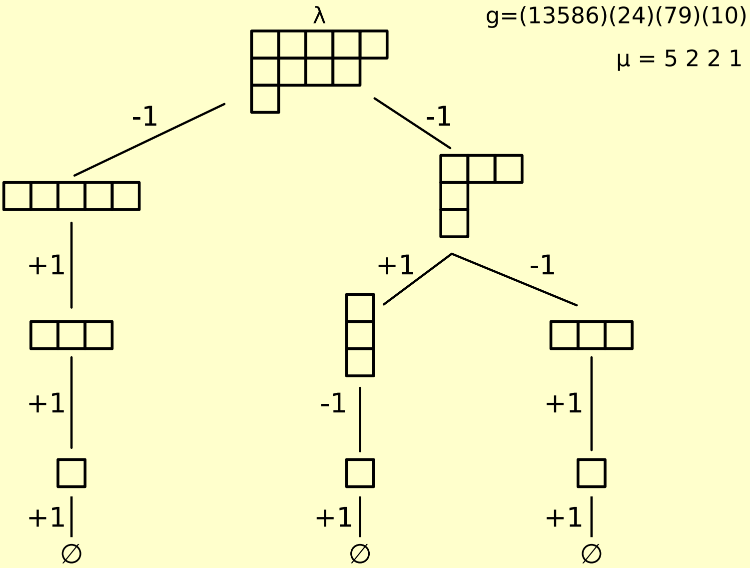


Murnaghan–Nakayama rule for computing $\chi^{\lambda}(g) =$



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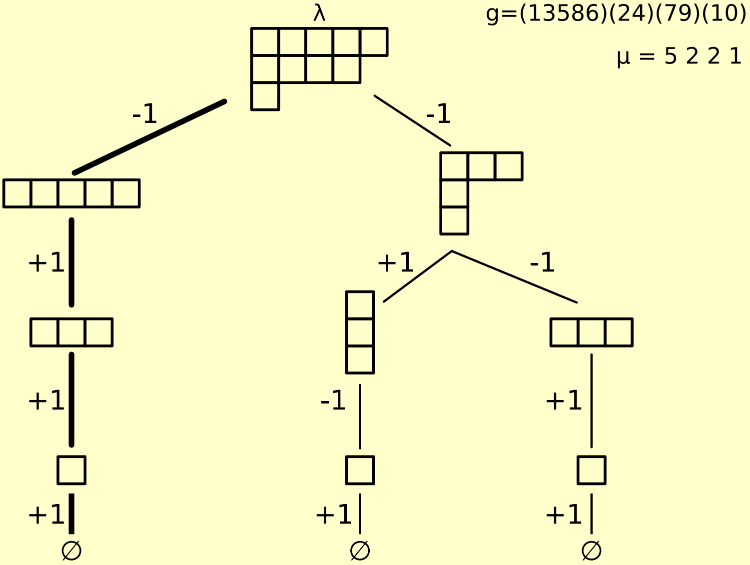

Murnaghan–Nakayama rule for computing $\chi^{\lambda}(g) = -1$

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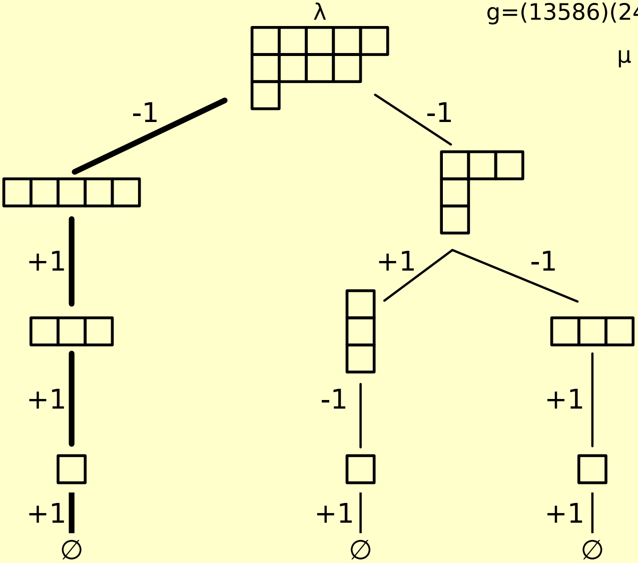

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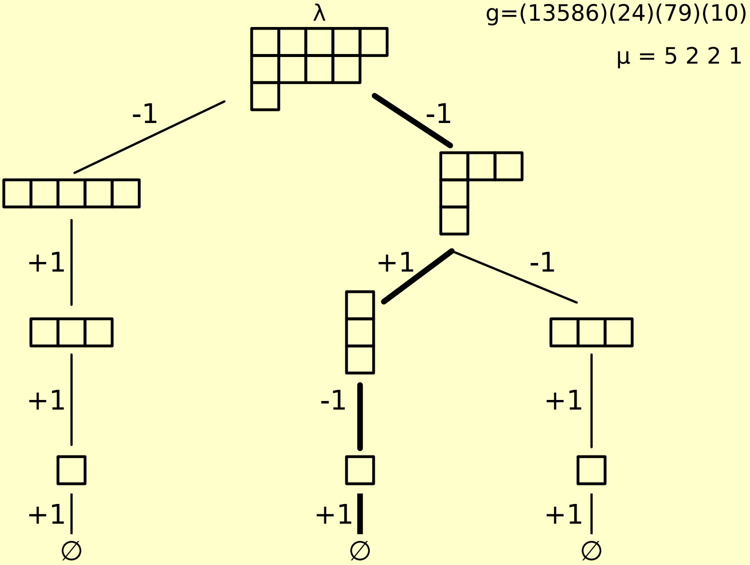

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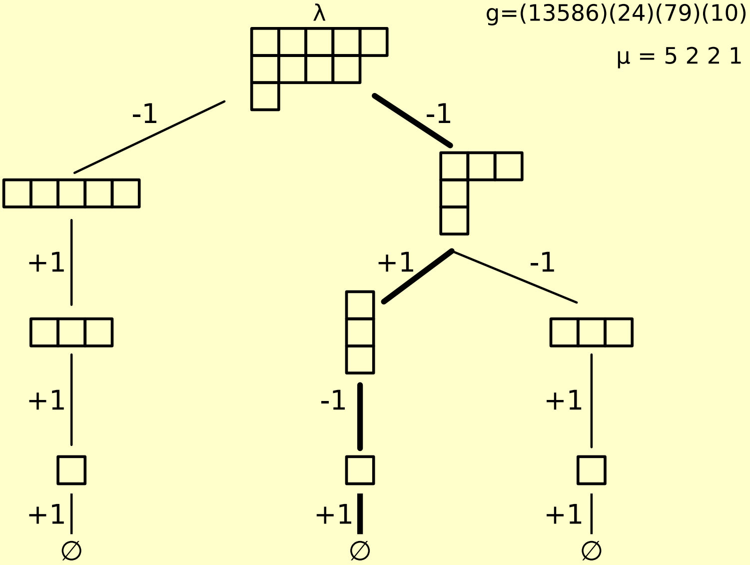

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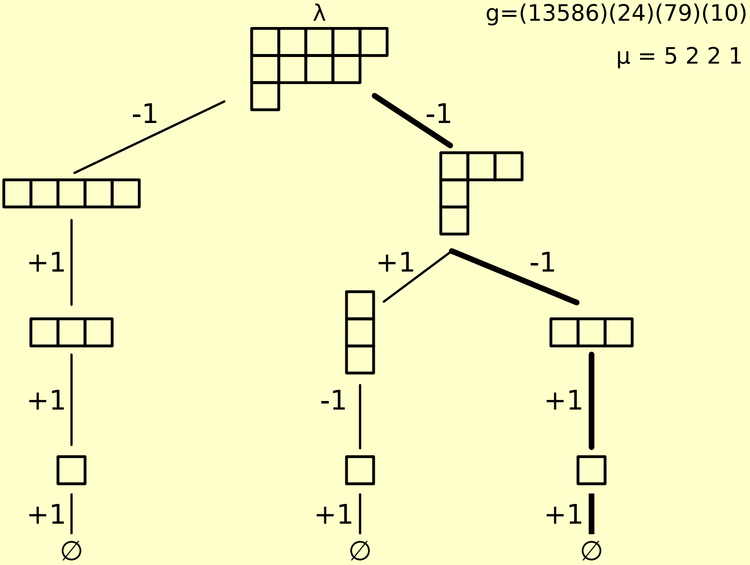

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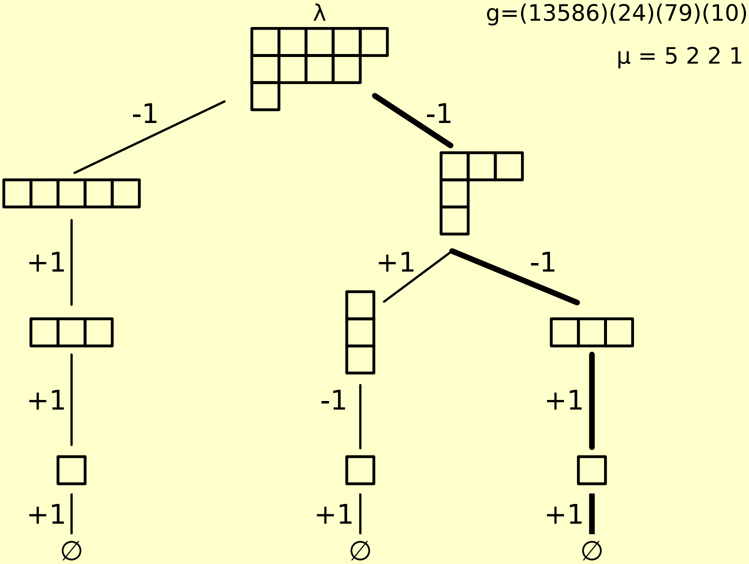
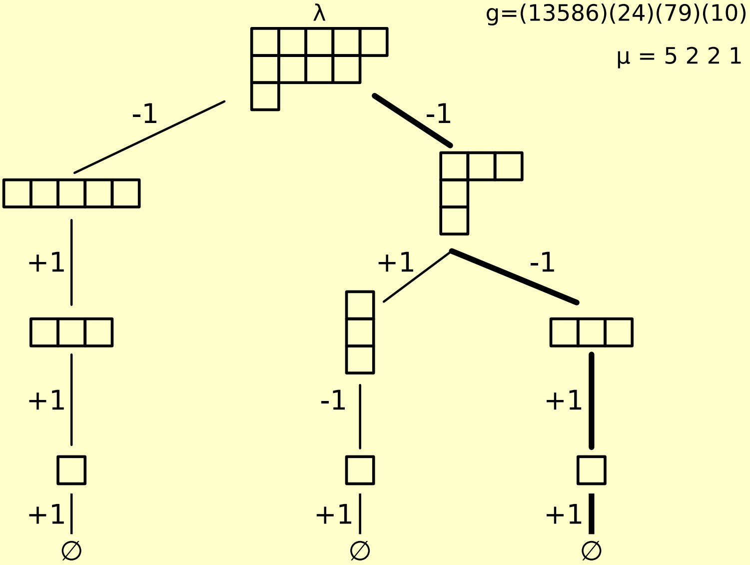


Murnaghan–Nakayama rule for computing $\chi^{\lambda}(g) = -1 + 1 + 1 = 1$



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Theorem (Miller)
For the symmetric group S_n we have that Prob($\chi(g) = 0$) $\to 1$ as $n \to \infty$.

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Example Character table of S_5:

$\lambda \backslash \mu$	$\ydiagram{1,1,1,1,1}$	$\ydiagram{2,1,1,1}$	$\ydiagram{2,2,1}$	$\ydiagram{3,1,1}$	$\ydiagram{3,2}$	$\ydiagram{4,1}$	$\ydiagram{5}$
$\ydiagram{5}$	1	1	1	1	1	1	1
$\ydiagram{4,1}$	4	2	0	1	−1	0	−1
$\ydiagram{3,2}$	5	1	1	−1	1	−1	0
$\ydiagram{3,1,1}$	6	0	−2	0	0	0	1
$\ydiagram{2,2,1}$	5	−1	1	−1	−1	1	0
$\ydiagram{2,1,1,1}$	4	−2	0	1	1	0	−1
$\ydiagram{1,1,1,1,1}$	1	−1	1	1	−1	−1	1

So for example $\chi^{\ydiagram{3,2}}((1\,2)(3\,5)(4)) = \chi^{\ydiagram{3,2}}(\ydiagram{2,2,1}) = 1$.

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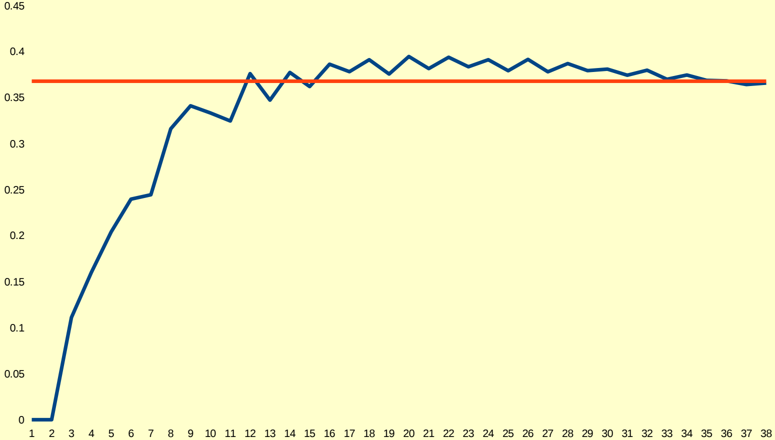

Part 3. $\mathrm{Prob}(\chi^\lambda(\mu) > 0 \mid \chi^\lambda(\mu) \neq 0)$

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$\lambda \backslash \mu$	$\ydiagram{1,1,1,1,1}$	$\ydiagram{2,1,1,1}$	$\ydiagram{2,2,1}$	$\ydiagram{3,1,1}$	$\ydiagram{3,2}$	$\ydiagram{4,1}$	$\ydiagram{5}$
$\ydiagram{5}$	1	1	1	1	1	1	1
$\ydiagram{4,1}$	4	2	0	1	-1	0	-1
$\ydiagram{3,2}$	5	1	1	-1	1	-1	0
$\ydiagram{3,1,1}$	6	0	-2	0	0	0	1
$\ydiagram{2,2,1}$	5	-1	1	-1	-1	1	0
$\ydiagram{2,1,1,1}$	4	-2	0	1	1	0	-1
$\ydiagram{1,1,1,1,1}$	1	-1	1	1	-1	-1	1

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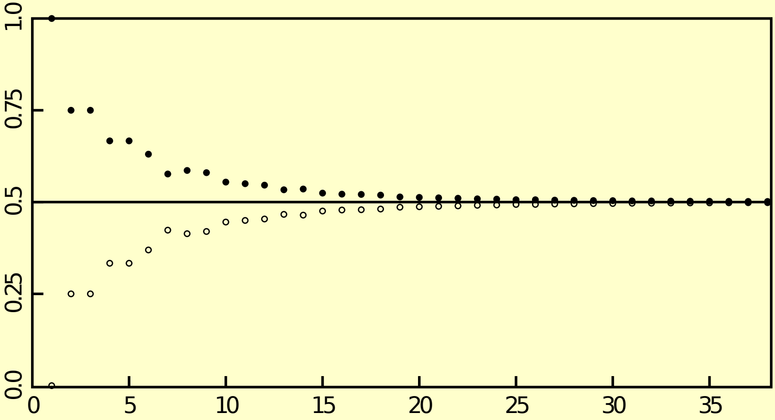
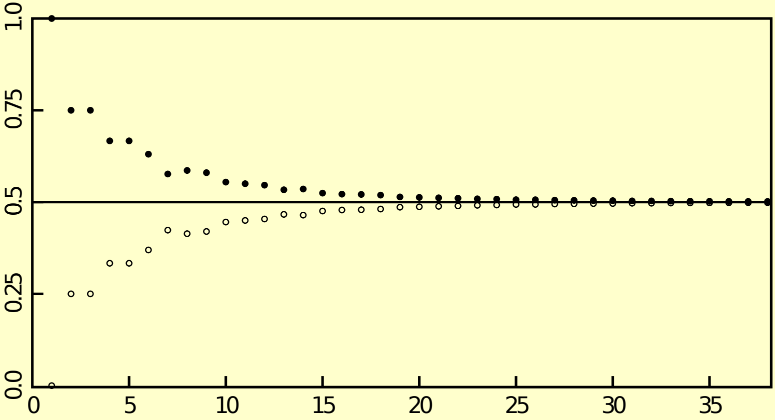


Part 3. $\mathrm{Prob}(\chi^\lambda(\mu) > 0 \mid \chi^\lambda(\mu) \neq 0)$

Conjecture (Miller) $\mathrm{Prob}(\chi^\lambda(\mu) > 0 \mid \chi^\lambda(\mu) \neq 0) \to 1/2$ as $n \to \infty$.



Part 4. $\mathrm{Prob}(\chi^\lambda(\mu) \equiv 0 \pmod 2)$

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$\lambda\backslash\mu$	$\ydiagram{1,1,1,1,1}$	$\ydiagram{2,1,1,1}$	$\ydiagram{2,2,1}$	$\ydiagram{3,1,1}$	$\ydiagram{3,2}$	$\ydiagram{4,1}$	$\ydiagram{5}$
$\ydiagram{5}$	1	1	1	1	1	1	1
$\ydiagram{4,1}$	4	2	0	1	-1	0	-1
$\ydiagram{3,2}$	5	1	1	-1	1	-1	0
$\ydiagram{3,1,1}$	6	0	-2	0	0	0	1
$\ydiagram{2,2,1}$	5	-1	1	-1	-1	1	0
$\ydiagram{2,1,1,1}$	4	-2	0	1	1	0	-1
$\ydiagram{1,1,1,1,1}$	1	-1	1	1	-1	-1	1

Part 4. $\mathrm{Prob}(\chi^\lambda(\mu) \equiv 0 \pmod 2)$

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(Proportion of the character table of S_n covered by even integers)

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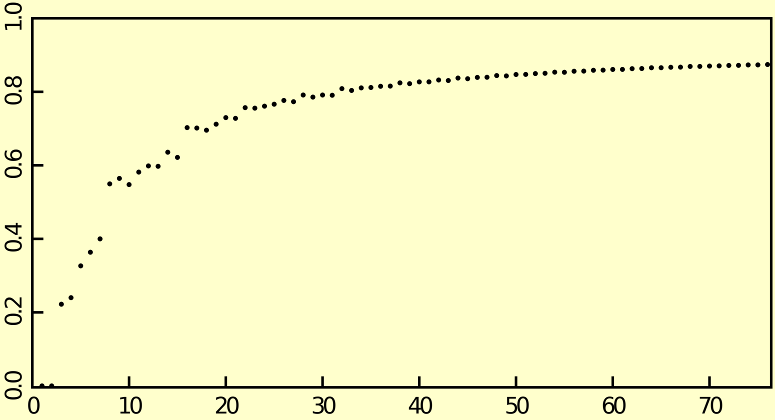
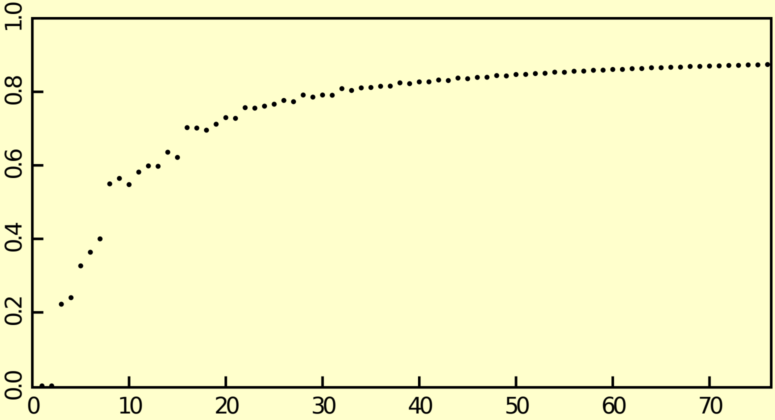


Part 4. $\mathrm{Prob}(\chi^\lambda(\mu) \equiv 0 \pmod 2)$

(Proportion of the character table of S_n covered by even integers)

Conjecture (Miller) $\mathrm{Prob}(\chi^\lambda(\mu) \equiv 0 \pmod 2) \to 1$ as $n \to \infty$.



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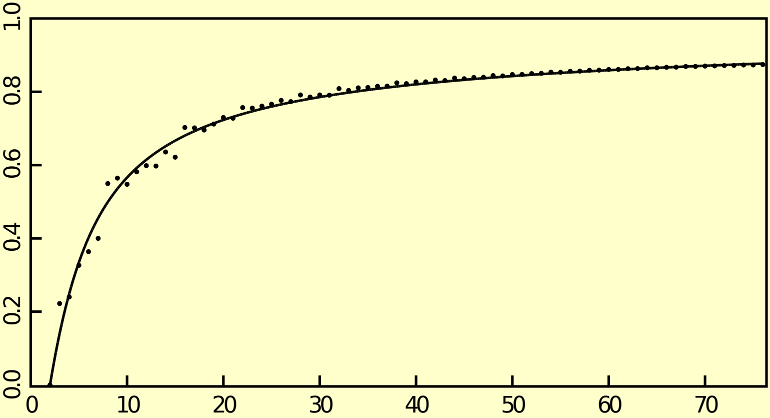

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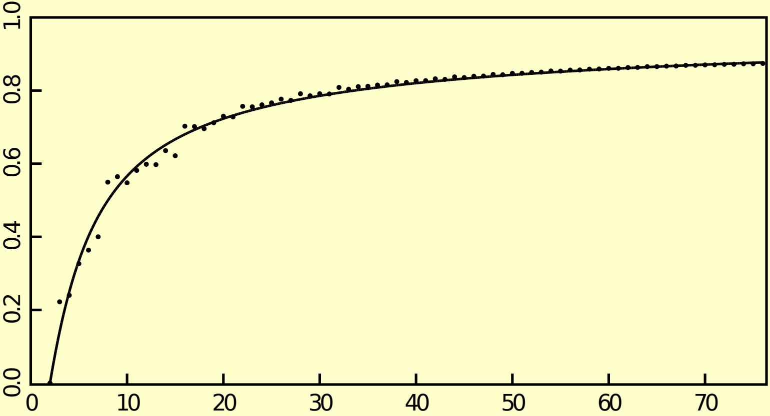

$\mathrm{Prob}(\chi^\lambda(\mu) \equiv 0 \pmod 2)$ and $2\pi^{-1}\,\mathrm{arctan}(\sqrt{n/2} - 1)$ for $2 \le n \le 76$