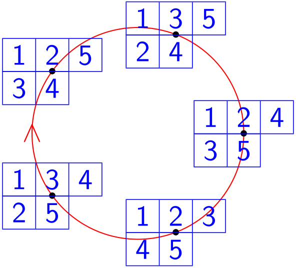
On Cyclic Descents of SYT

Yuval Roichman

Bar-Ilan University

Based on joint works with

Ron Adin, Sergi Elizalde and Vic Reiner



SLC 79, Bertinoro, Sep '17

Cyclic descents

Descents and cyclic descents of permutations

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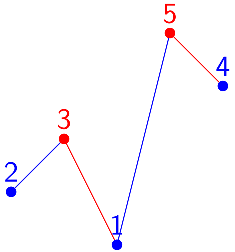

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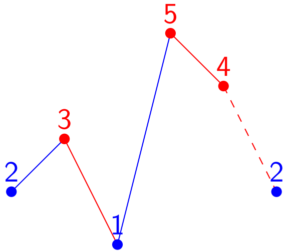

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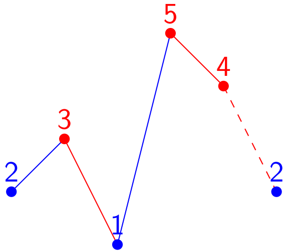

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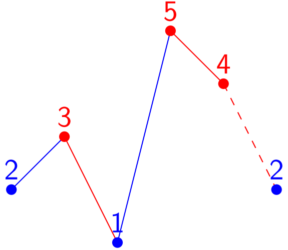

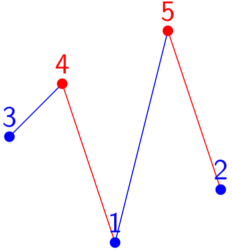

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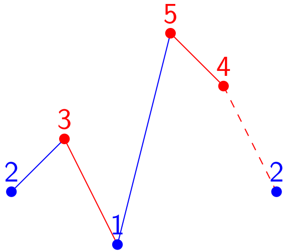

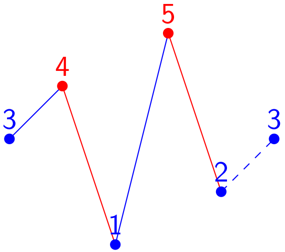

Consider $\mathbb{Z}$-actions on $\mathfrak{S}_n$ and on $2^{[n]}$, where the generator p of $\mathbb{Z}$ acts by

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A non-Escher property

"Ascending and Descending", M. C. Escher

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The paradox of $\mathrm{cDes}(\pi) = \varnothing$ and $\mathrm{cDes}(\pi) = [n]$.

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$$T = \begin{array}{ccccc} & 1 & 2 & 4 \\ & 3 & 6 & \\ 5 & & & \end{array} \in \mathsf{SYT}((4,3,1)/(1,1))$$

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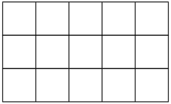
Motivating Problem:

Define a cyclic descent set for SYT of any shape λ/μ.

SYT of rectangular shapes



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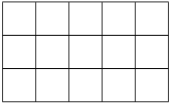

Theorem (Rhoades '10)

There exists a *cyclic descent map* $\mathrm{cDes} : \mathrm{SYT}(r^{n/r}) \to 2^{[n]}$ *s.t.*
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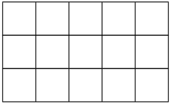

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$$\begin{aligned} \mathrm{cDes}(T) \cap [n-1] &= \mathrm{Des}(T) \\ \mathrm{cDes}(p(T)) &= p(\mathrm{cDes}(T)), \end{aligned}$$

where p acts on $\mathrm{cDes}(T)$ *by adding* $1 \pmod{n}$ *to each element, and acts on SYT by Schützenberger's promotion operator.*

SYT of rectangular shapes

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A $\mathbb{Z}$-orbit:

$$\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&6\\\hline\end{array} \to \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&6\\\hline\end{array} \to \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5&6\\\hline\end{array} \to \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&6\\\hline\end{array}$$

$$\{1,4\} \qquad \{2,5\} \qquad \{3,{\color{red}6}\} \qquad \{1,4\}$$

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\text{(extension)} \quad & \mathrm{cDes}(T) \cap [n-1] = \mathrm{Des}(T), \\
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Examples

- $\mathcal{T} = \mathfrak{S}_n$, with Cellini's cyclic descent set and $\mathbb{Z}$-action by cyclic rotation.
- $\mathcal{T} = \mathrm{SYT}(r^{n/r})$, with Rhoades' cyclic descent set and $\mathbb{Z}$-action by promotion.

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Recall the axioms: for all $T \in \mathrm{SYT}(\lambda/\mu)$,

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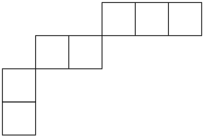

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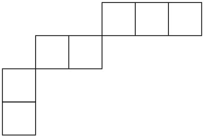

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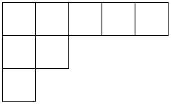

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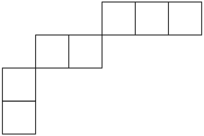

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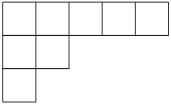

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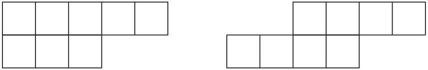

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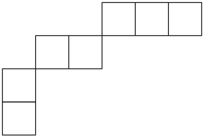

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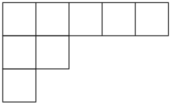

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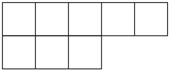

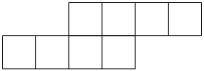

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The proofs are explicit and combinatorial.

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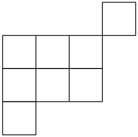
Example

$$(3,3,1)^{\square} =$$



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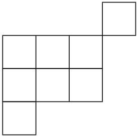
$(3,3,1)^\square =$ 

Theorem (Elizalde-Roichman '15)

For every partition $\lambda \vdash n-1$ there exists a cyclic descent extension on $\mathrm{SYT}(\lambda^\square)$.

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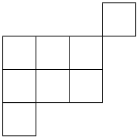

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So far - so good!

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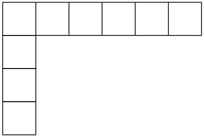

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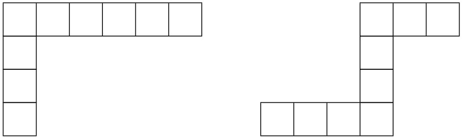

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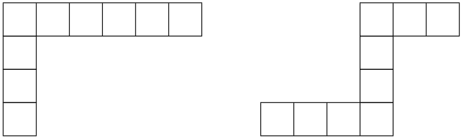

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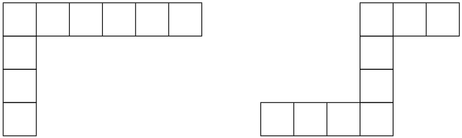

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Oops !!!

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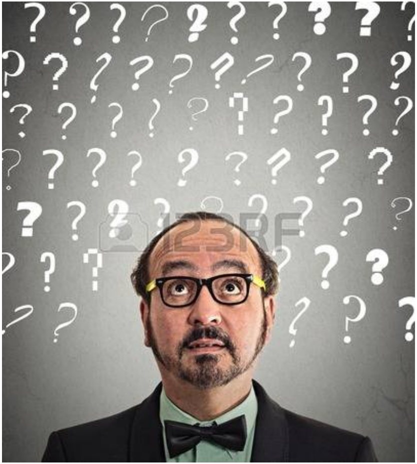

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For every non-hook partition $\lambda \vdash n$*, the set* $\mathrm{SYT}(\lambda)$ *has a cyclic descent extension; namely,*
$\exists\ \mathrm{cDes} : \mathrm{SYT}(\lambda) \to 2^{[n]}$ *and a bijection* $p : \mathrm{SYT}(\lambda) \longrightarrow \mathrm{SYT}(\lambda)$,
s.t. $\forall\ T \in \mathrm{SYT}(\lambda)$

$$\begin{aligned}
\text{(extension)} \quad & \mathrm{cDes}(T) \cap [n-1] = \mathrm{Des}(T), \\
\text{(equivariance)} \quad & \mathrm{cDes}(p(T)) = p(\mathrm{cDes}(T)), \\
\text{(non-Escher)} \quad & \varnothing \subsetneq \mathrm{cDes}(T) \subsetneq [n].
\end{aligned}$$

Affine ribbon Schur functions

Ribbon Schur functions

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For a subset $J = \{j_1 < j_2 < \ldots < j_t\} \subseteq [n-1]$ define the associated composition

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Theorem (Gessel '83)

For any skew shape λ/μ and $J \subseteq [n]$,

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For a subset $\varnothing \neq J = \{j_1 < j_2 < \ldots < j_t\} \subseteq [n]$ define the associated cyclic composition

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and the corresponding affine ribbon Schur function

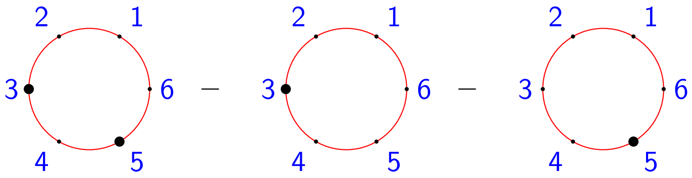
$$\tilde{s}_{\mathrm{cc}(J)} := \sum_{\varnothing \neq I \subseteq J} (-1)^{|J \setminus I|} h_{\mathrm{cc}(I)}.$$

Affine ribbon Schur functions

Example

Let $n = 6$ and $J = \{3, 5\}$. The affine ribbon Schur function is

$$\begin{aligned}\tilde{s}_{\mathrm{cc}(\{3,5\})} &= h_{\mathrm{cc}(\{3,5\})} - h_{\mathrm{cc}(\{3\})} - h_{\mathrm{cc}(\{5\})}\\ &= h_{(2,4)} - h_{(6)} - h_{(6)}.\end{aligned}$$



Theorem (Adin-Reiner-Roichman '16)

A skew shape λ/μ has a cyclic descent extension if and only if

$$\langle s_{\lambda/\mu}, \tilde{s}_{\mathrm{cc}(J)} \rangle \geq 0 \qquad (\forall \varnothing \subsetneq J \subsetneq [n]),$$

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Example For $n = 6$ and $J = \{3, 5\}$,

$$\tilde{s}_{cc(\{3,5\})} = s_{4,2} + s_{5,1} - s_6.$$

Main results:

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existence and uniqueness

Existence

Recall that the cyclic ribbon Schur functions $\tilde{s}_{\mathrm{cc}(J)}$ are not always Schur positive. Can this be made more precise?

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Theorem (ARR, Postnikov '05, McNamara '06)

For all $\varnothing \neq J \subseteq [n]$ of size $k > 0$

$$\tilde{s}_{cc(J)} + \sum_{i=0}^{k-1} (-1)^{k-i} s_{(n-i,1^i)}$$

is Schur positive (and hook-free).

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$$\tilde{s}_{cc(J)} = s_{\lambda/1/\lambda} + (-1)^{|J|-1} p_n,$$

where $s_{\lambda/1/\lambda}$ is a special case of Postnikov's toric Schur functions and

$$p_n = x_1^n + x_2^n + \ldots = \sum_{i=0}^{n-1} (-1)^i s_{(n-i,1^i)}$$

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Postnikov proved that, letting $x_{k+1} = x_{k+2} = \ldots = 0$,

$$s_{\lambda/d/\mu}(x_1, \ldots, x_k) = \sum_{\nu \subseteq k \times (n-k)} C_{\mu,\nu}^{\lambda,d} s_\nu(x_1, \ldots, x_k),$$

where $C_{\mu,\nu}^{\lambda,d} \geq 0$ are the Gromov-Witten invariants.

Main theorem: existence

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For every skew shape λ/μ of size n, which is not a connected ribbon, there exists a cyclic descent extension;

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Theorem (Adin-Reiner-Roichman '16)

For every skew shape λ/μ of size n, which is not a connected ribbon, there exists a cyclic descent extension; namely, $\exists\, \mathrm{cDes} : \mathrm{SYT}(\lambda/\mu) \longrightarrow 2^{[n]}$ *and* $p : \mathrm{SYT}(\lambda/\mu) \longrightarrow \mathrm{SYT}(\lambda/\mu)$, *s.t.* $\forall\ T \in in\,\mathrm{SYT}(\lambda)$

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($\Longrightarrow$ Equidistribution results)

Corollary

$$\sum_{\pi\in\mathfrak{S}_n}\mathbf{x}^{\mathrm{cDes}(\pi)} = \sum_{\substack{\text{non-hook}\\ \lambda\vdash n}} f^\lambda \sum_{T\in\mathrm{SYT}(\lambda)}\mathbf{x}^{\mathrm{cDes}(T)} + \sum_{k=1}^{n-1}\binom{n-2}{k-1}\sum_{T\in\mathrm{SYT}((n-k+1,1^k)/(1))}\mathbf{x}^{\mathrm{cDes}(T)},$$

where $f^\lambda = |\,\mathrm{SYT}(\lambda)|$.

Summary and open problems

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- For almost all skew shapes λ/μ, the fiber distribution of this cyclic extension is unique.
- The proof (of existence) involves toric Schur functions and the nonnegativity of the Gromov-Witten invariants.

Epilogue

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- Applications to Schur-positivisty (with Elizalde)

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Problem
Find bijective proofs to resulting equidistribution identities.

THANK YOU ! and ...

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a tribute to Bertinoro


BERTINORO


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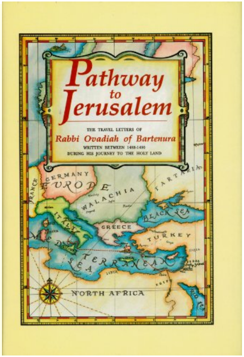
פירוש המשניות
פה ויניציא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WRITTEN BETWEEN 1488-1490
DURING HIS JOURNEY TO THE HOLY LAND
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ITALY
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