Maule

Tilings, Young and Tamari lattices under the same roof (part I)

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augmented set of slides with comments and references added 3 October 2017

X **cloud** is a finite subset of the square lattice $\mathbb{Z} \times \mathbb{Z}$

<u>Definition</u>

Γ - move

X cloud. Let $\alpha, \beta, \gamma \in X$ in Γ- position, that is

$$\begin{array}{ccc} \alpha & \cdots & \beta \\ \vdots & & \\ \gamma & & \end{array}$$

Suppose that all the vertices of the rectangle

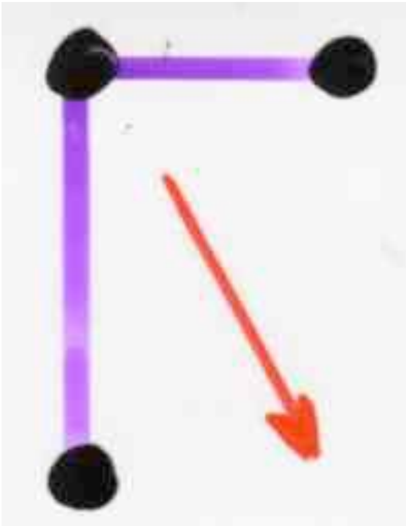
$$\begin{array}{ccc} \alpha & \cdots & \beta \\ \vdots & & \vdots \\ \gamma & \cdots & \delta \end{array}$$

, except α, β, γ, are empty (denoted $\times$)

Definition Γ- move

X, Y clouds $\quad Y = \Gamma(X)$

Γ- move



Definition Γ- move

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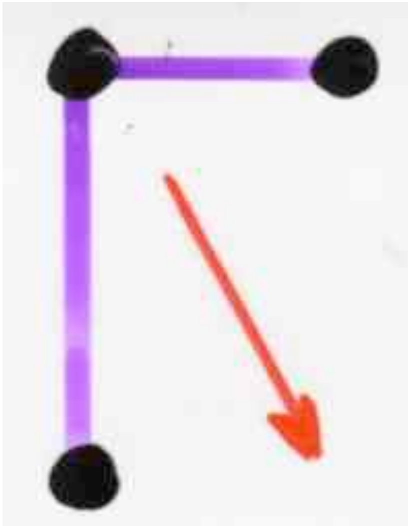
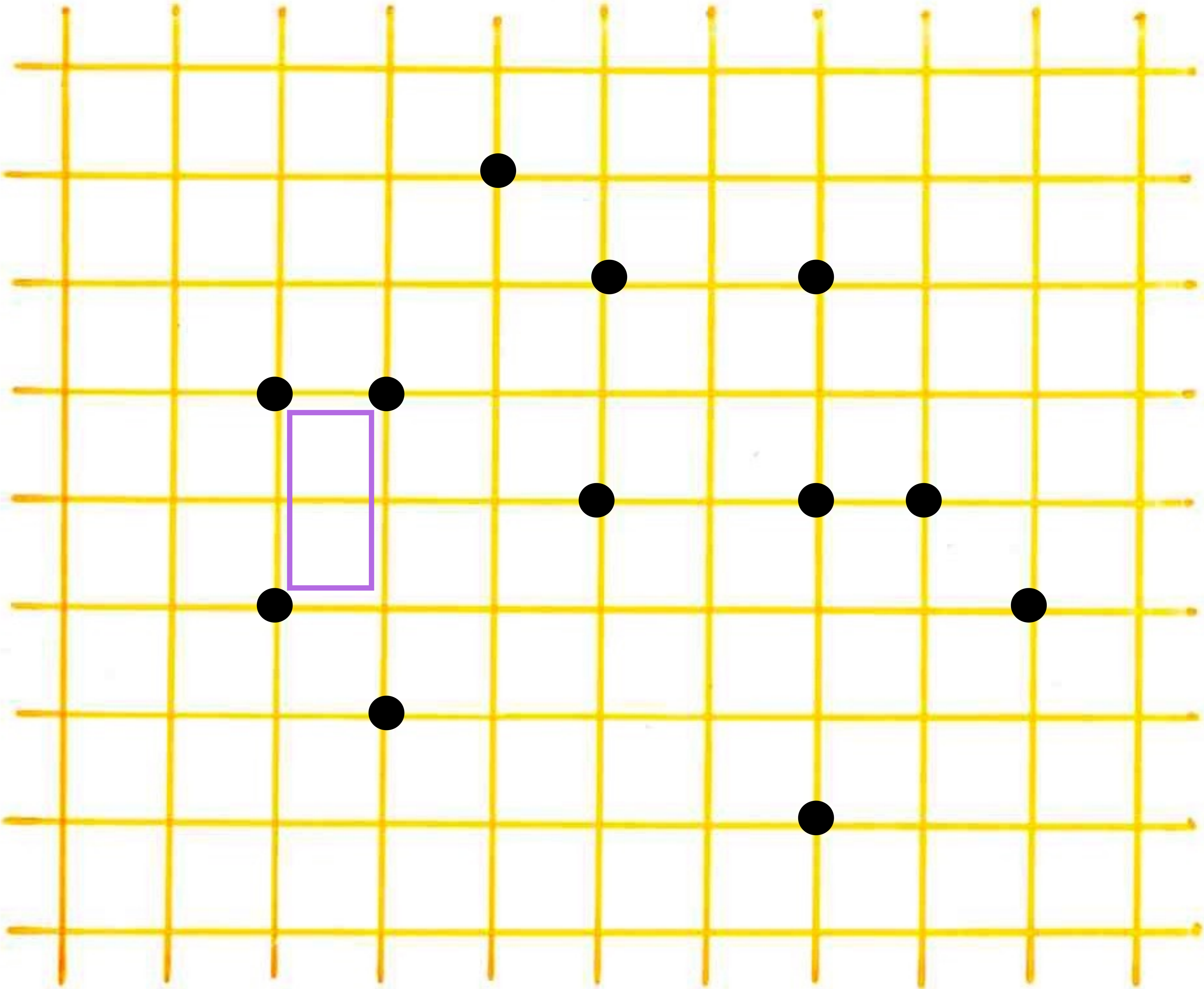
Γ - move

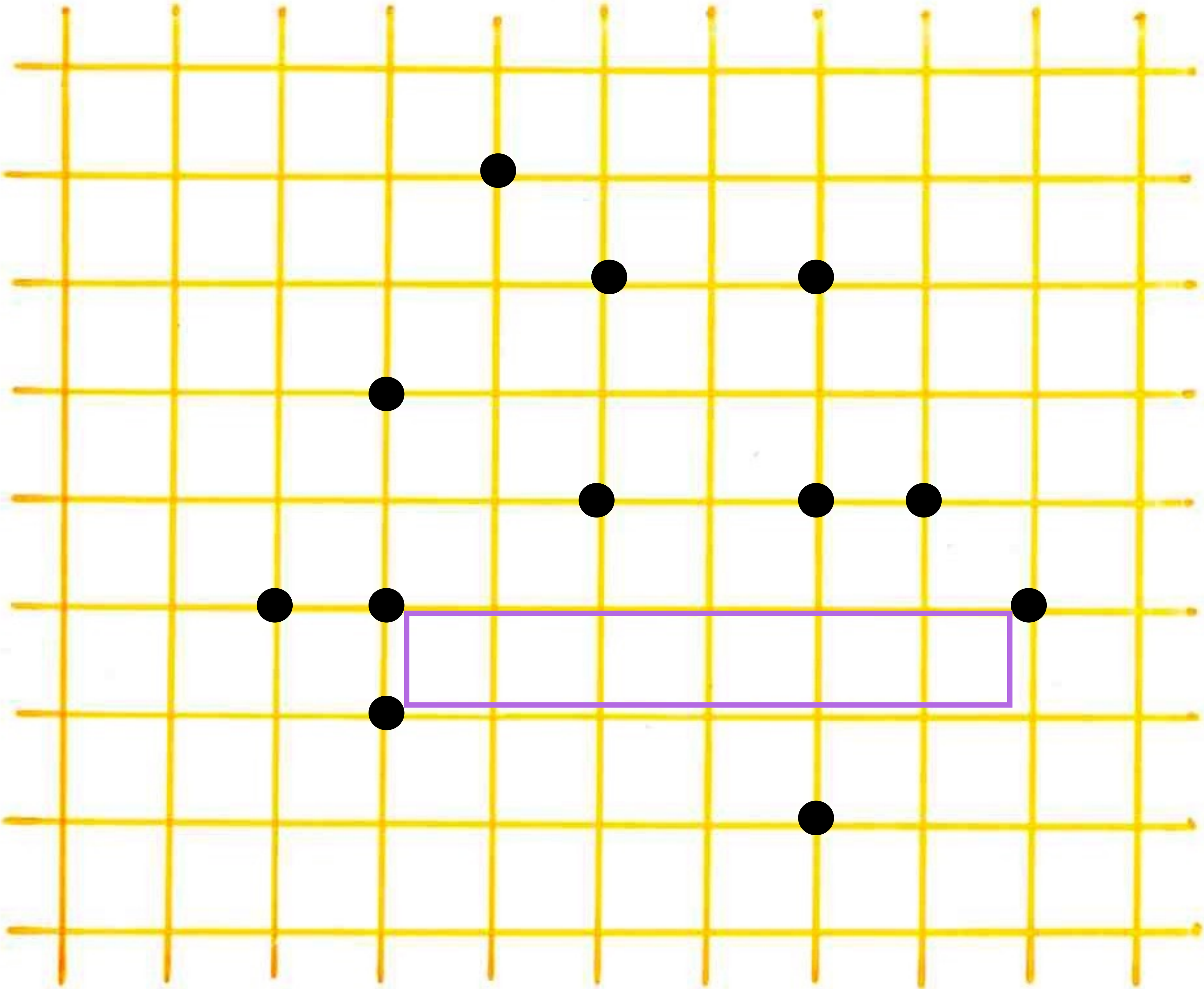


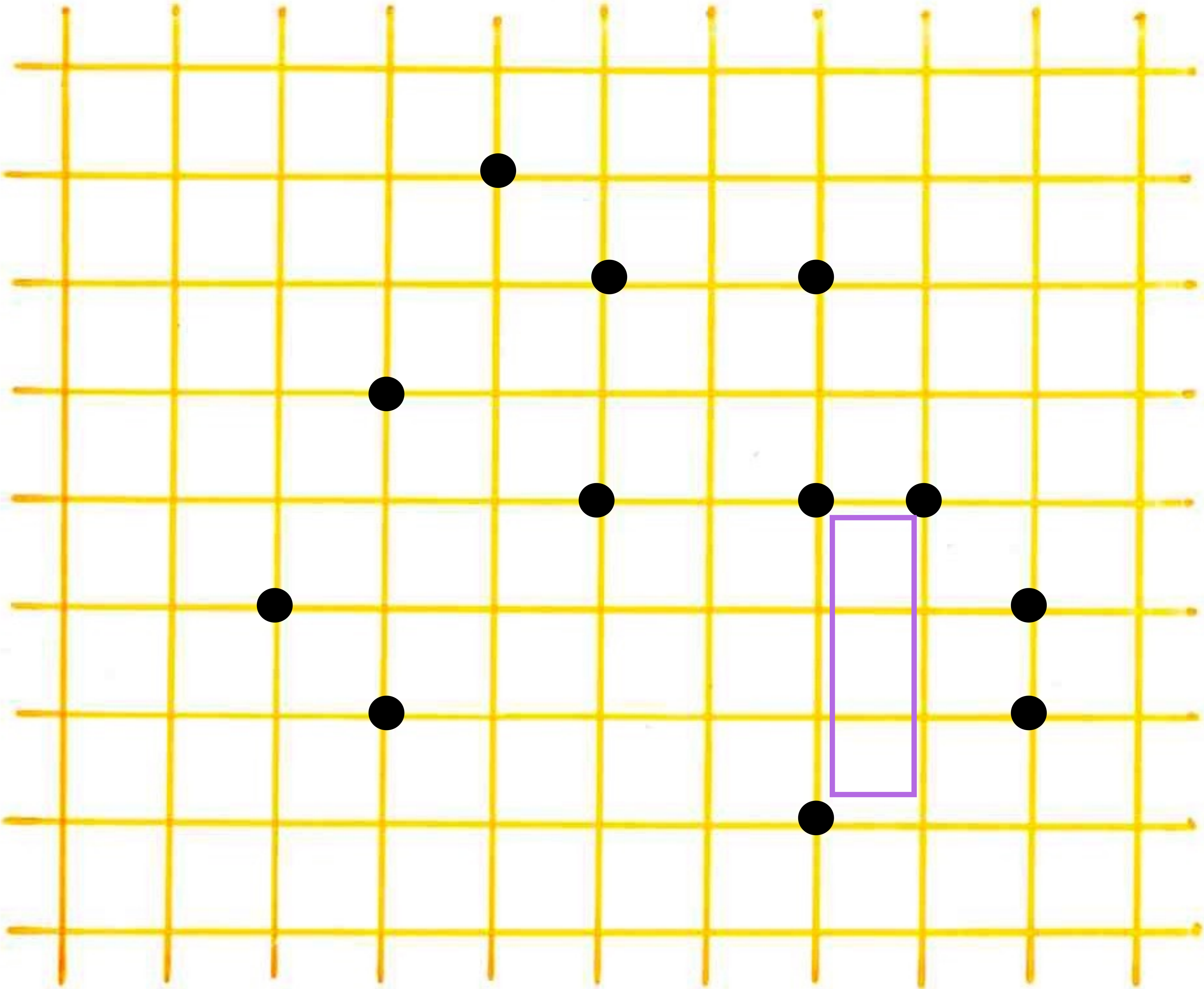
notation $X \xrightarrow{\Gamma} Y$

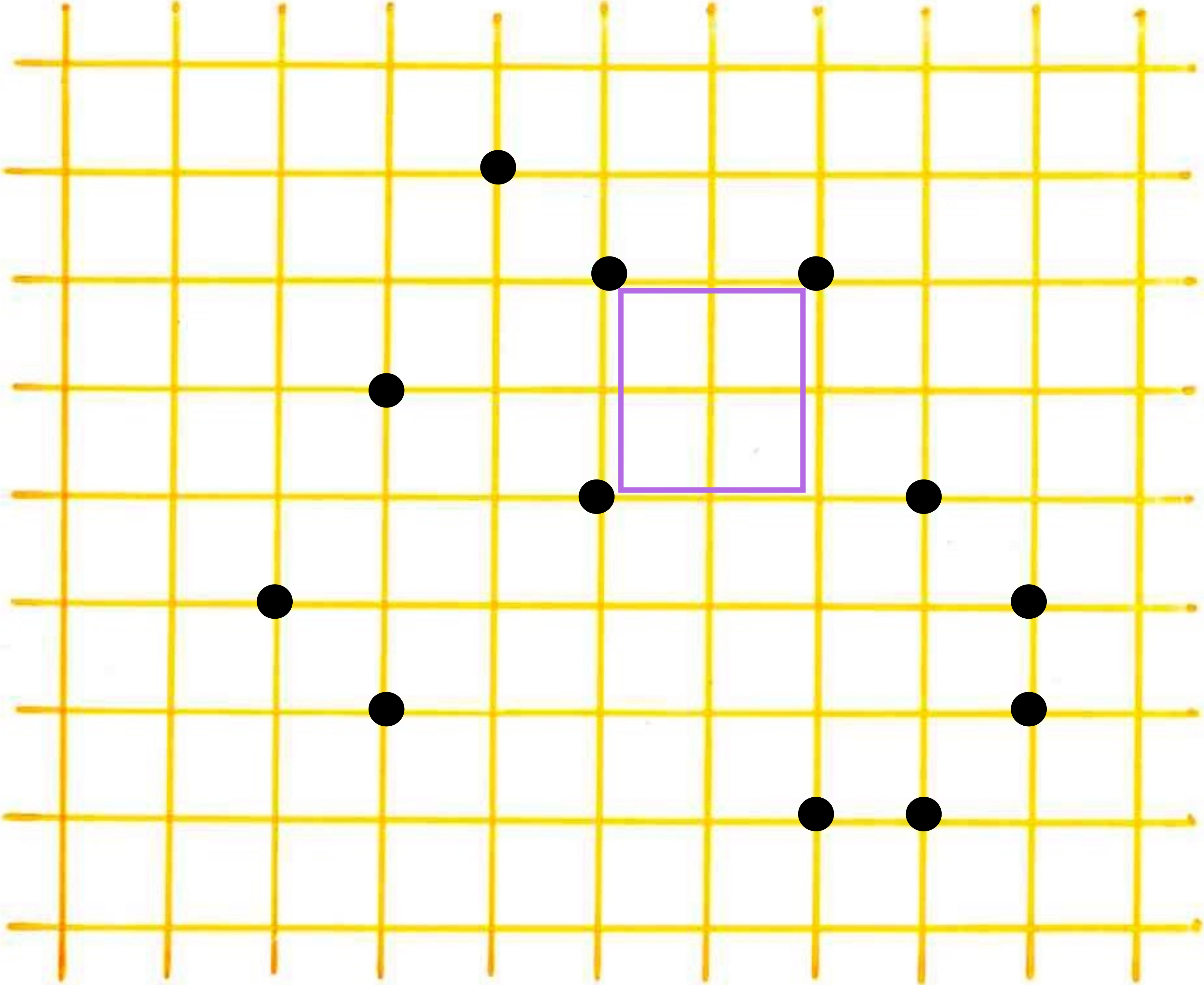
Definition The relation $X \xrightarrow{\Gamma^*} Y$ is the transitive closure of the relation $X \xrightarrow{\Gamma} Y$

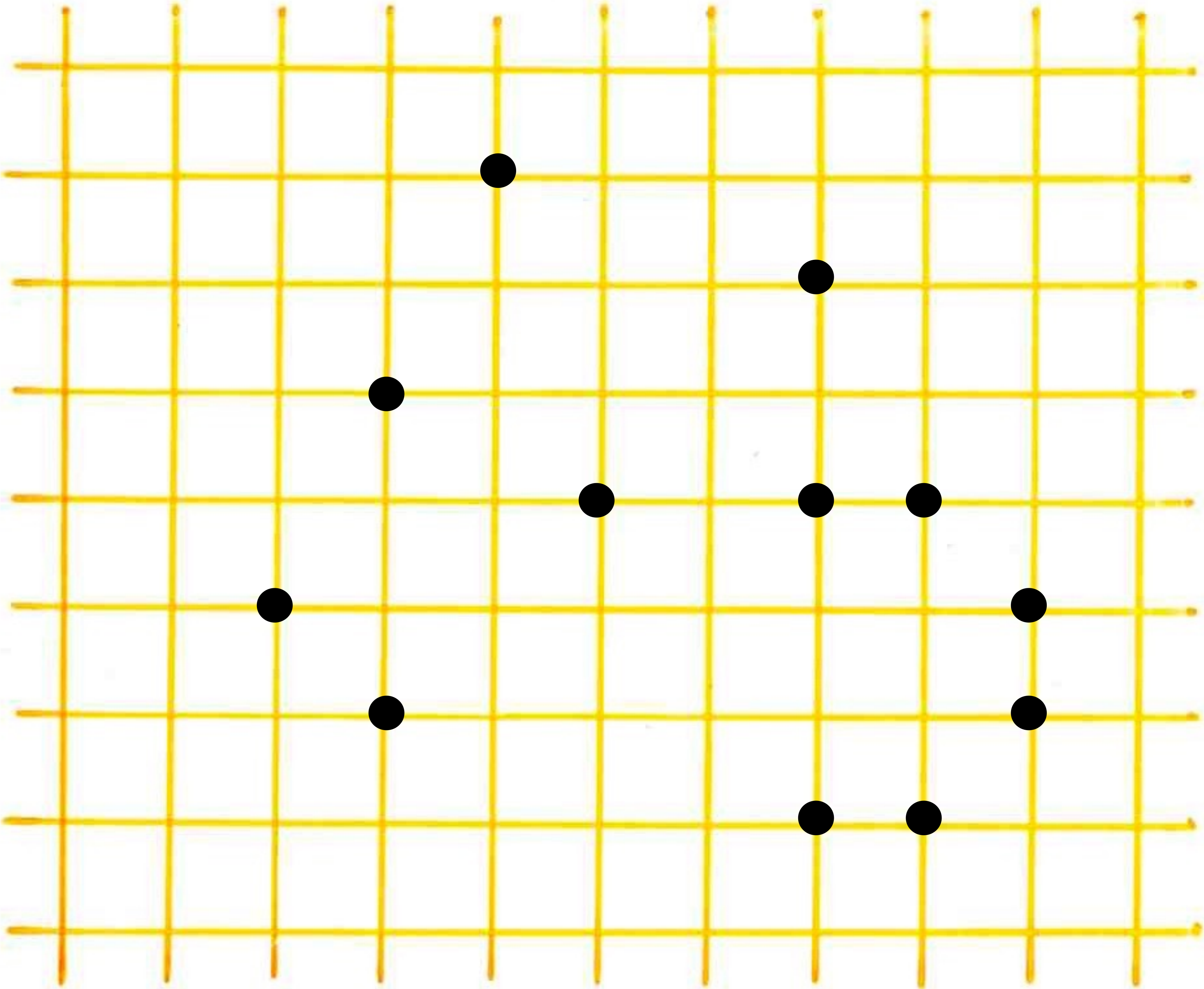
- $X \xrightarrow{\Gamma^*} Y$ is an order relation



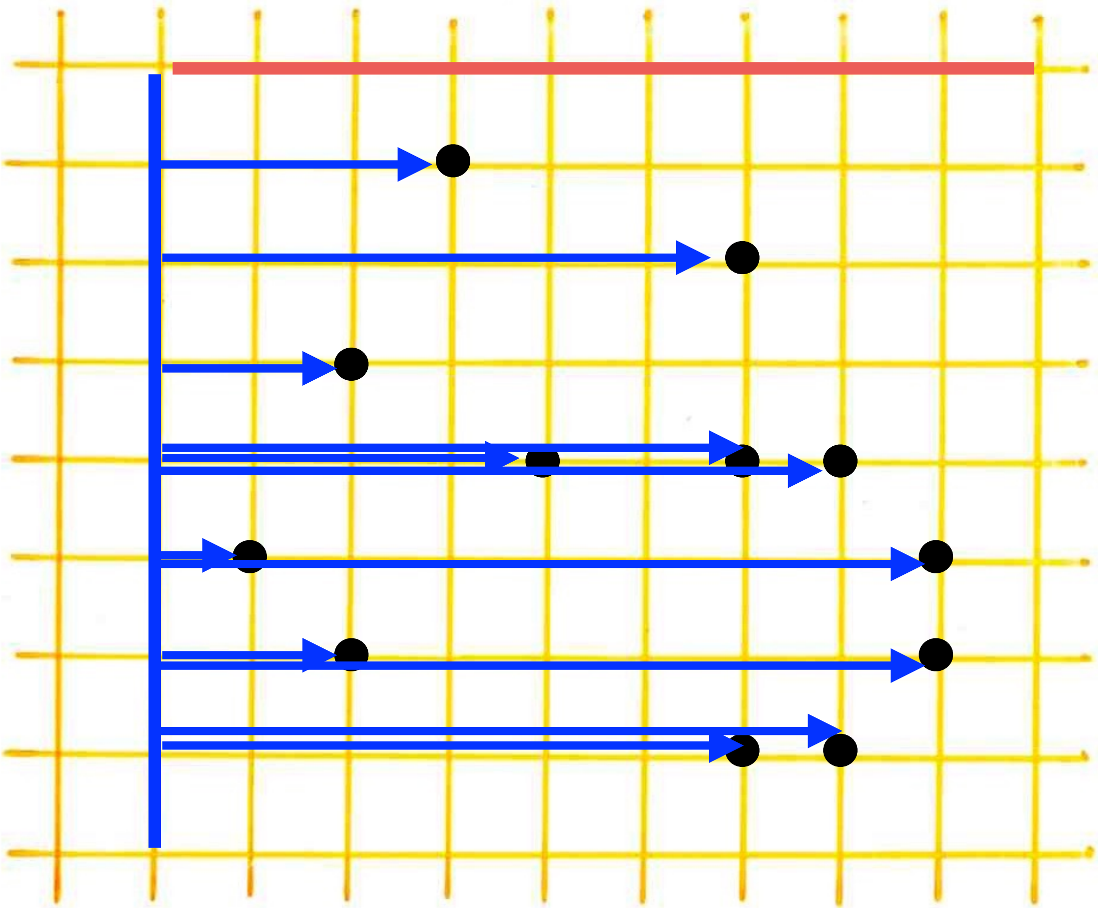








• $X \xrightarrow{\Gamma^*} Y$ is an order relation

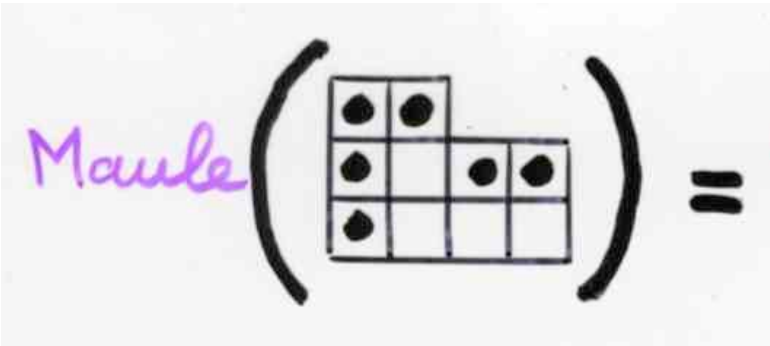


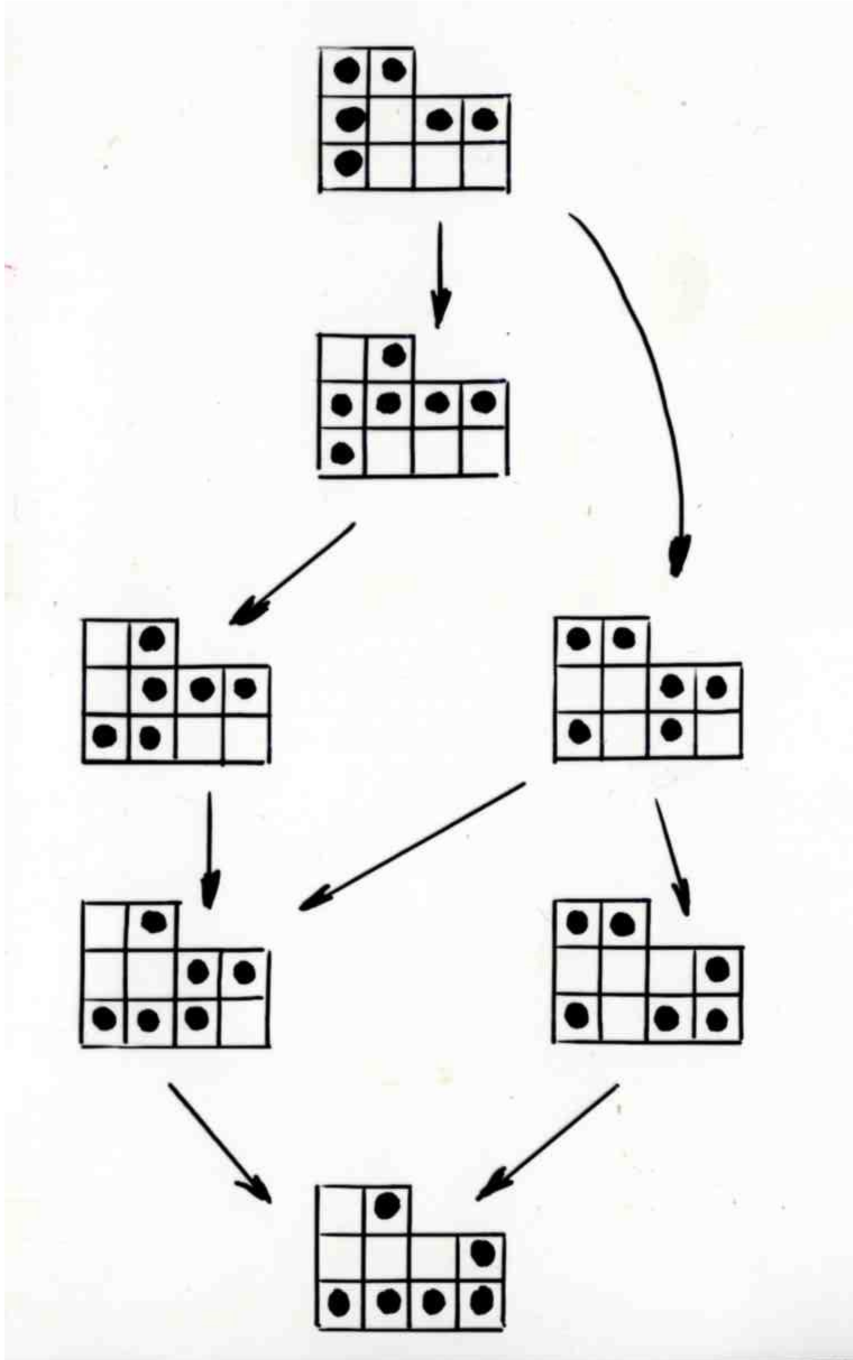
After a Γ-move, the sum of the distances of the points of the cloud to the blue vertical line will increase at least by one, and thus no cycles are possible.

Main definition The poset Maule (X) is the set of all clouds obtained from X by a succession of Γ-moves, (i.e. $X \xrightarrow{\Gamma^*} Y$) equipped with the order relation $Y \xrightarrow{\Gamma^*} Z$ for $Y, Z \in$ Maule(X).

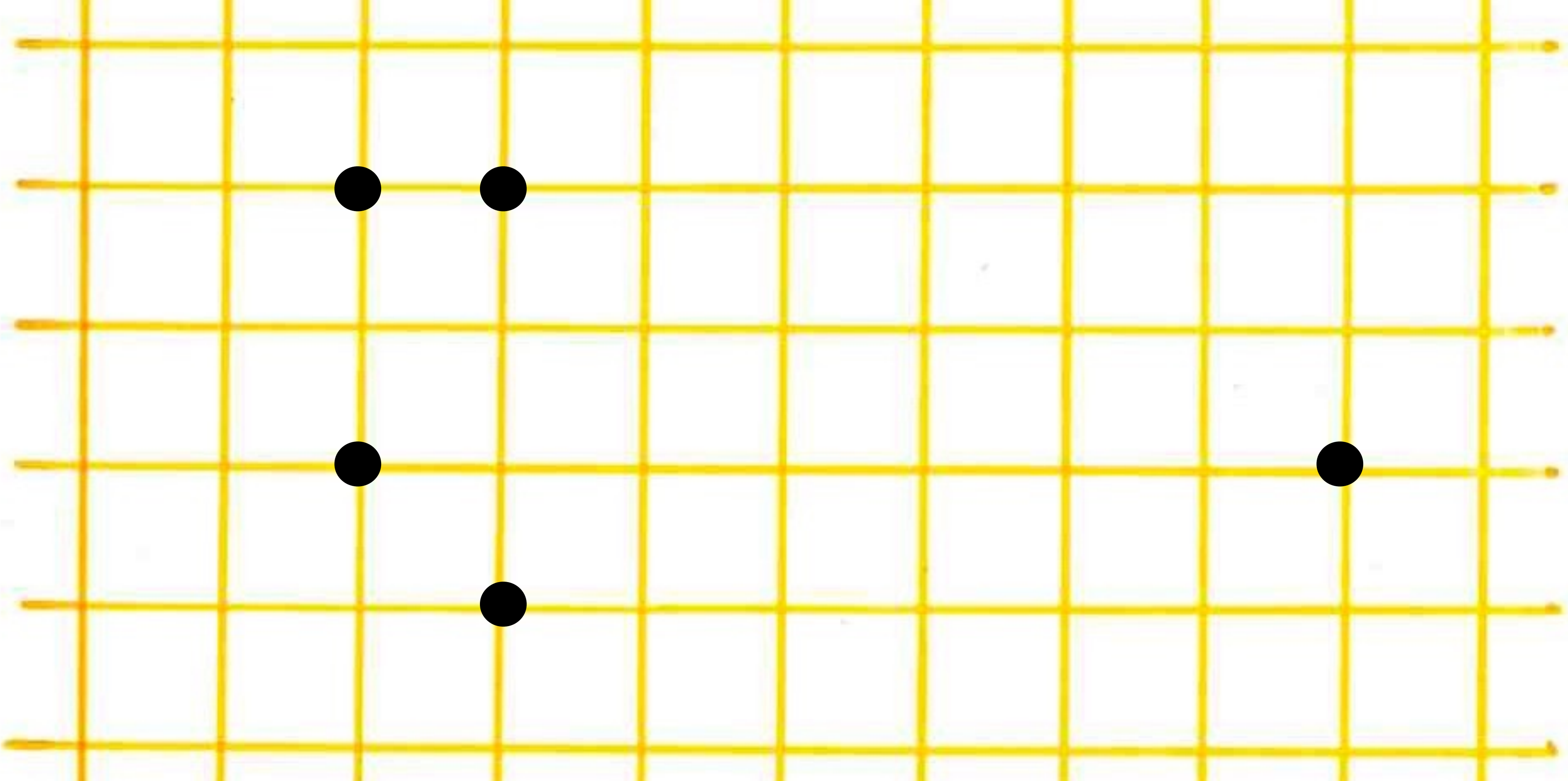
- The relation $Y \xrightarrow{\Gamma} Z$ (for $Y, Z \in$ Maule(X)) is the covering relation of the poset Maule (X).

an example





- The relation $Y \xrightarrow{\Gamma} Z$ (for $Y, Z \in \mathrm{Maule}(X)$) is the covering relation of the poset $\mathrm{Maule}(X)$.



two (or more) Γ-moves cannot be reduced to a single Γ-move

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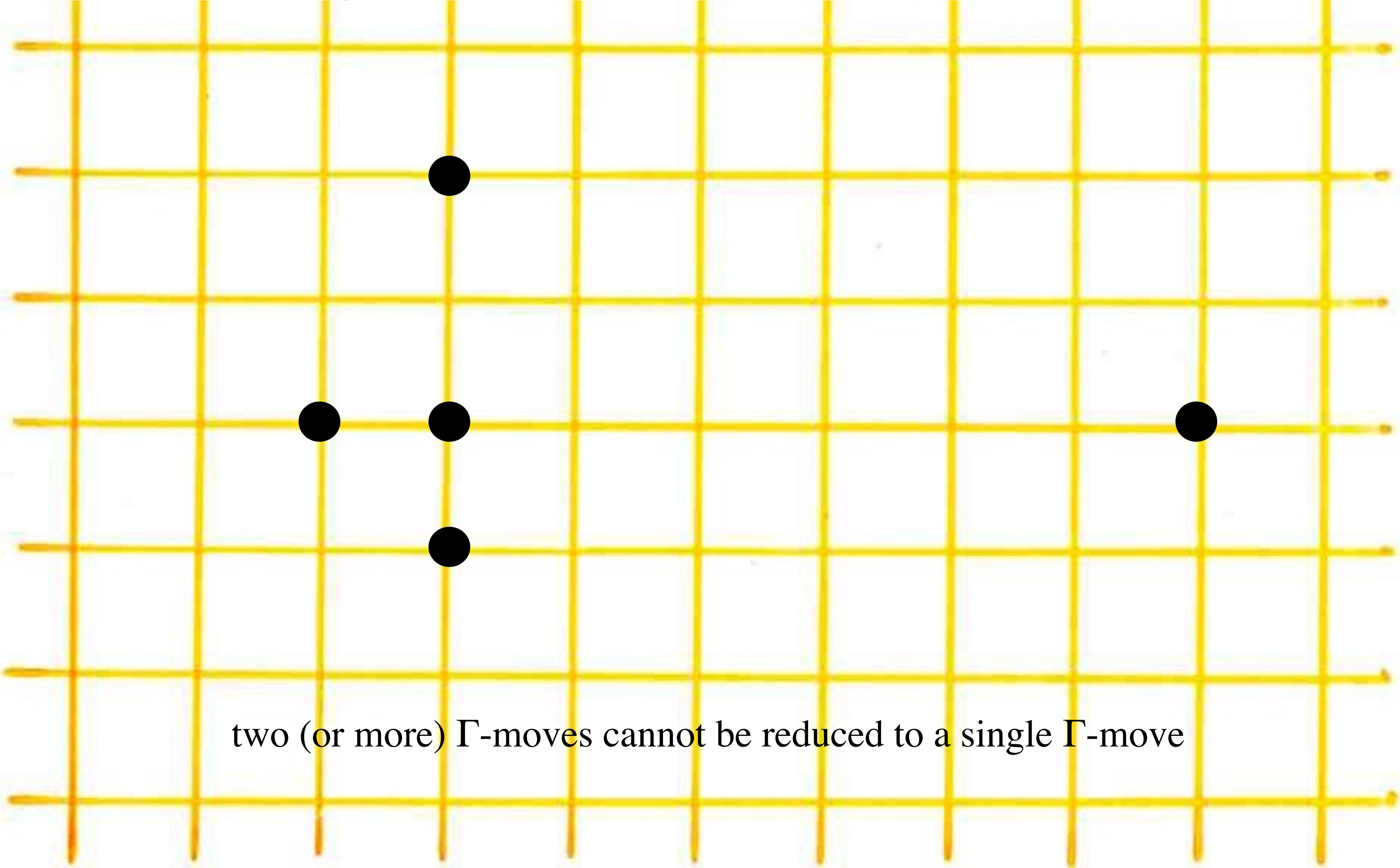

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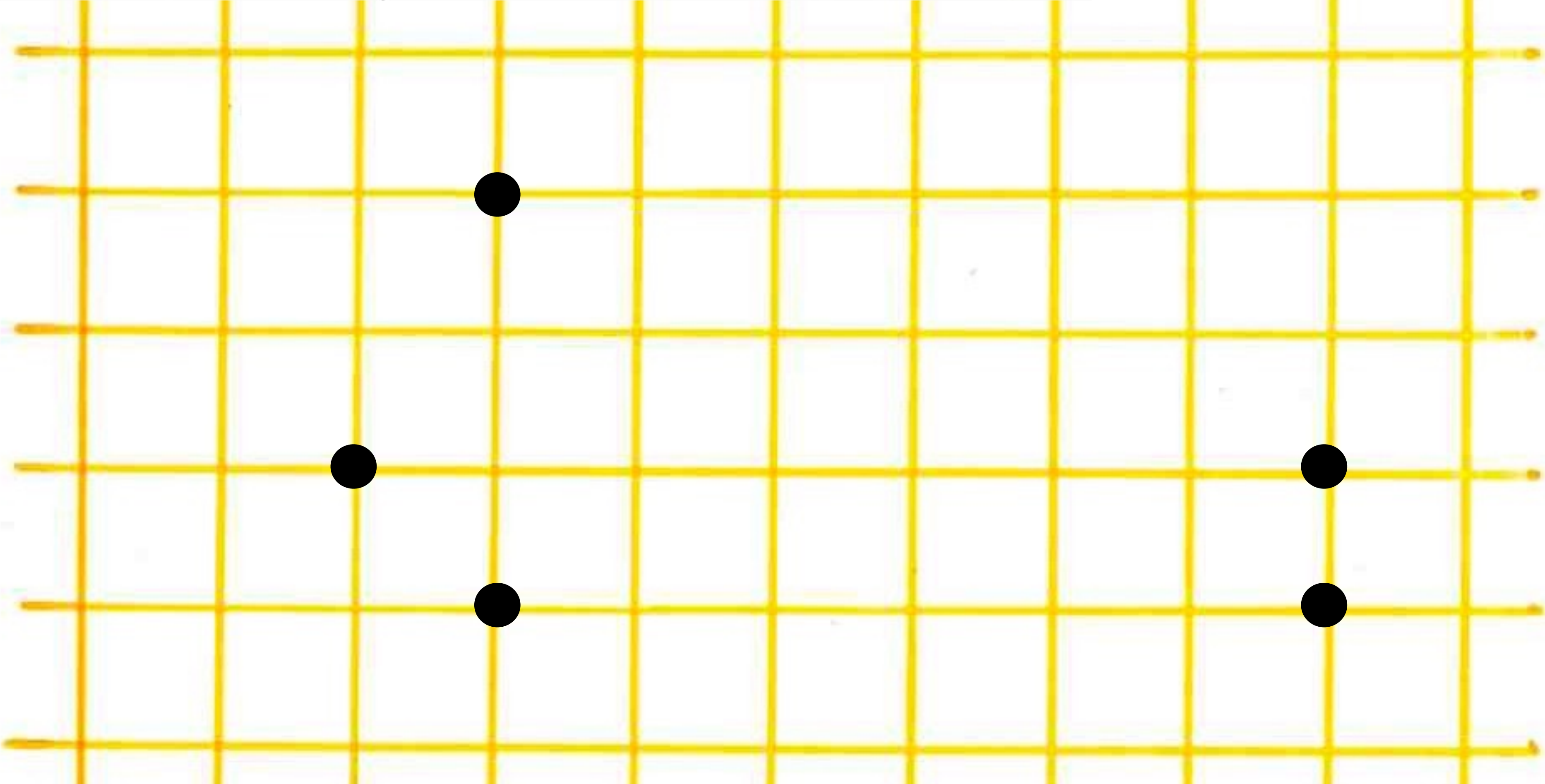


two (or more) Γ-moves cannot be reduced to a single Γ-move

Remark Maule

- name of an area in Chile where this research was started, thanks to an invitation of Luc Lapointe (Talca Univ.)
- also the name of the river crossing this area

Mapuche name : pronounce Ma-ou-lé
signification: rainy

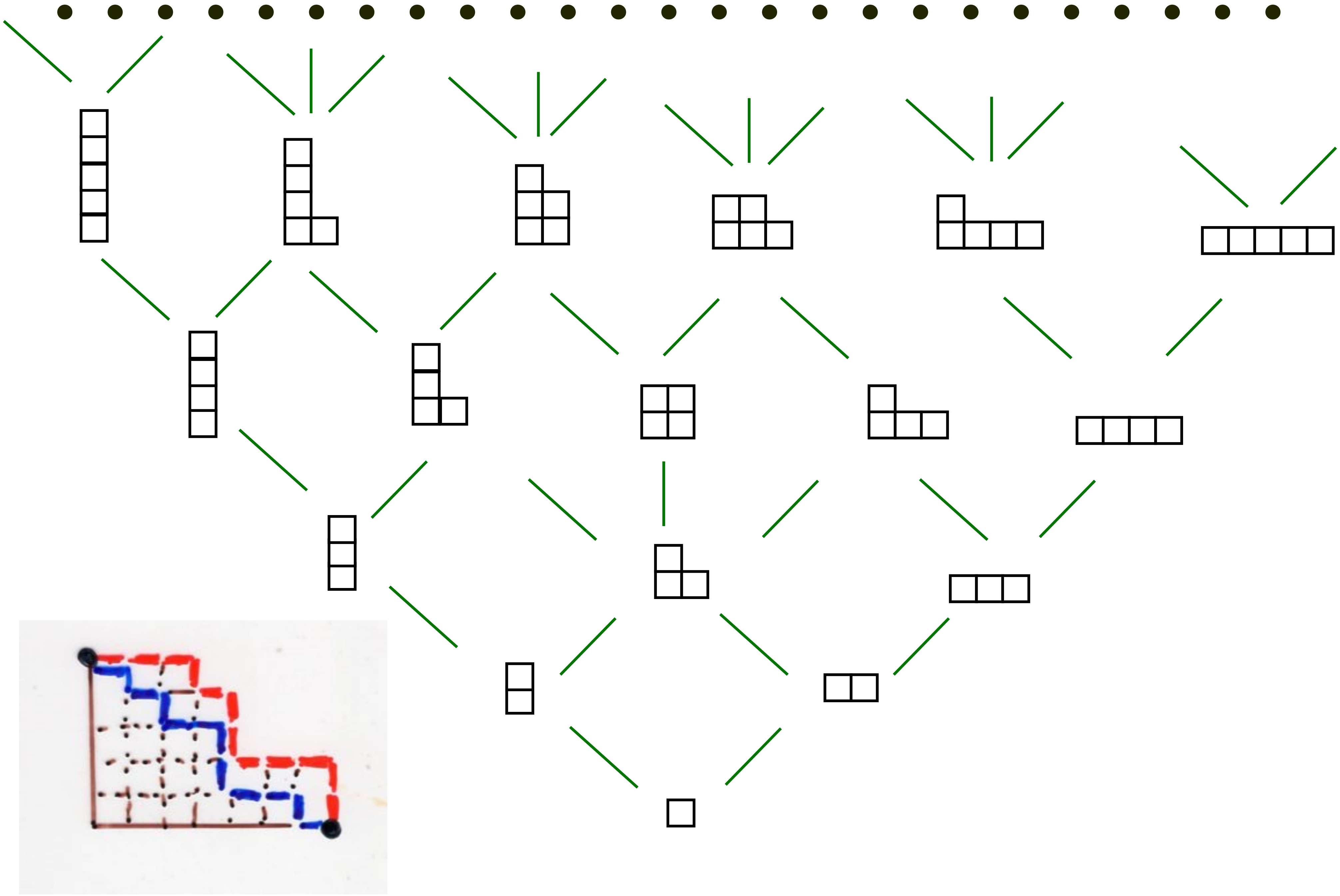

OCEANO
PACIFICO
CORDILLERA DE LOS ANDES
Lago Rapel
Lago Vichuquén
Lago Colbún
Río

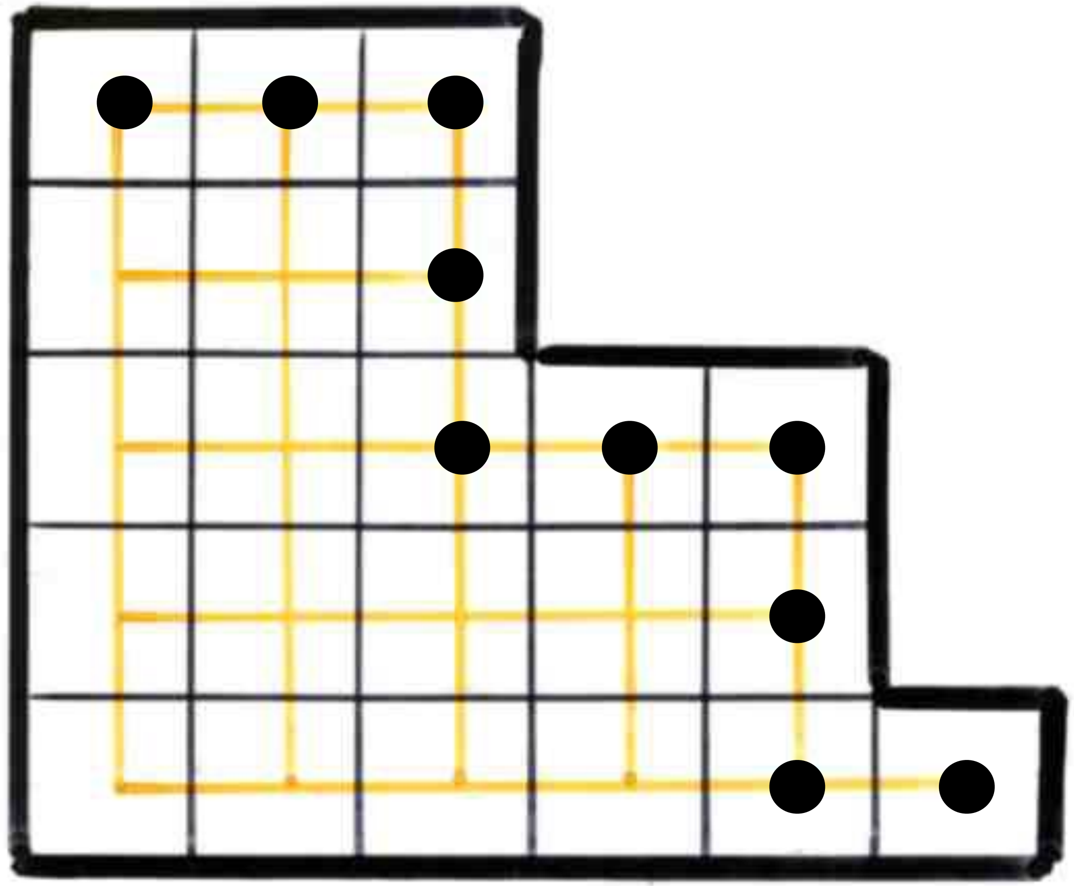
Maule valley

From Talca to Constitution

Young lattice

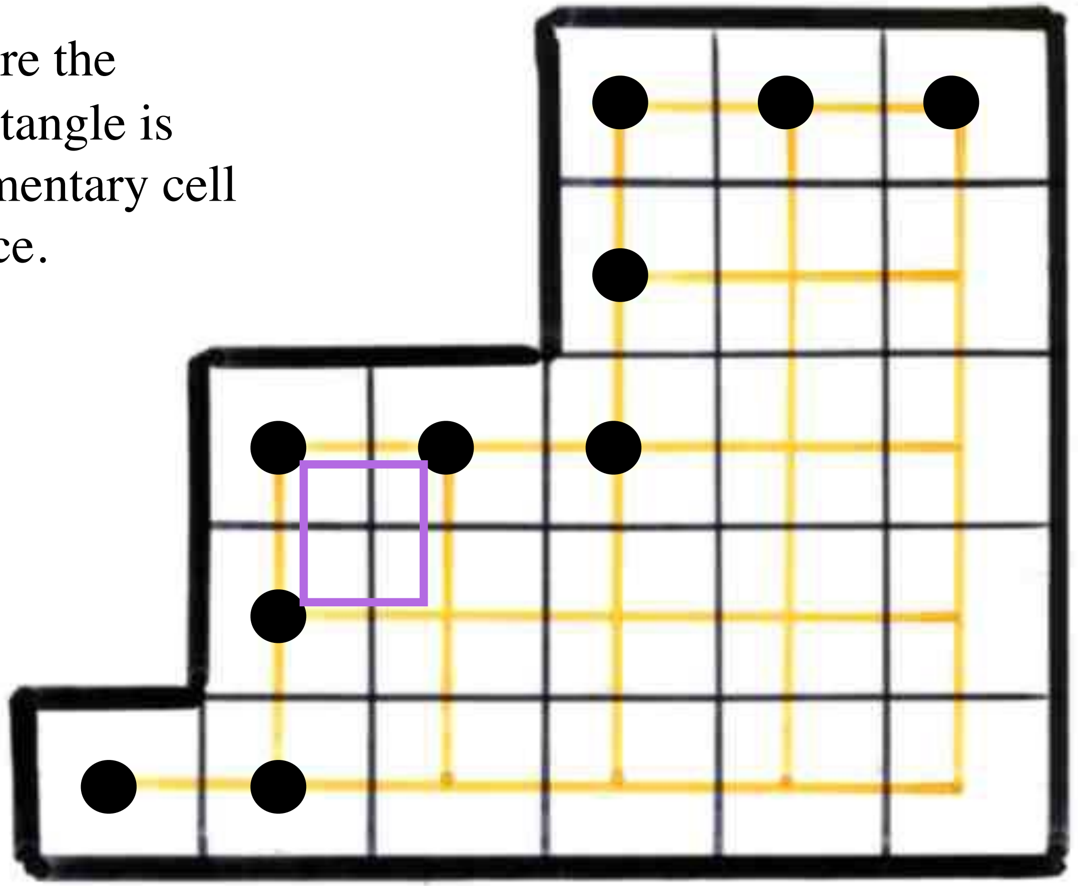
Young lattice



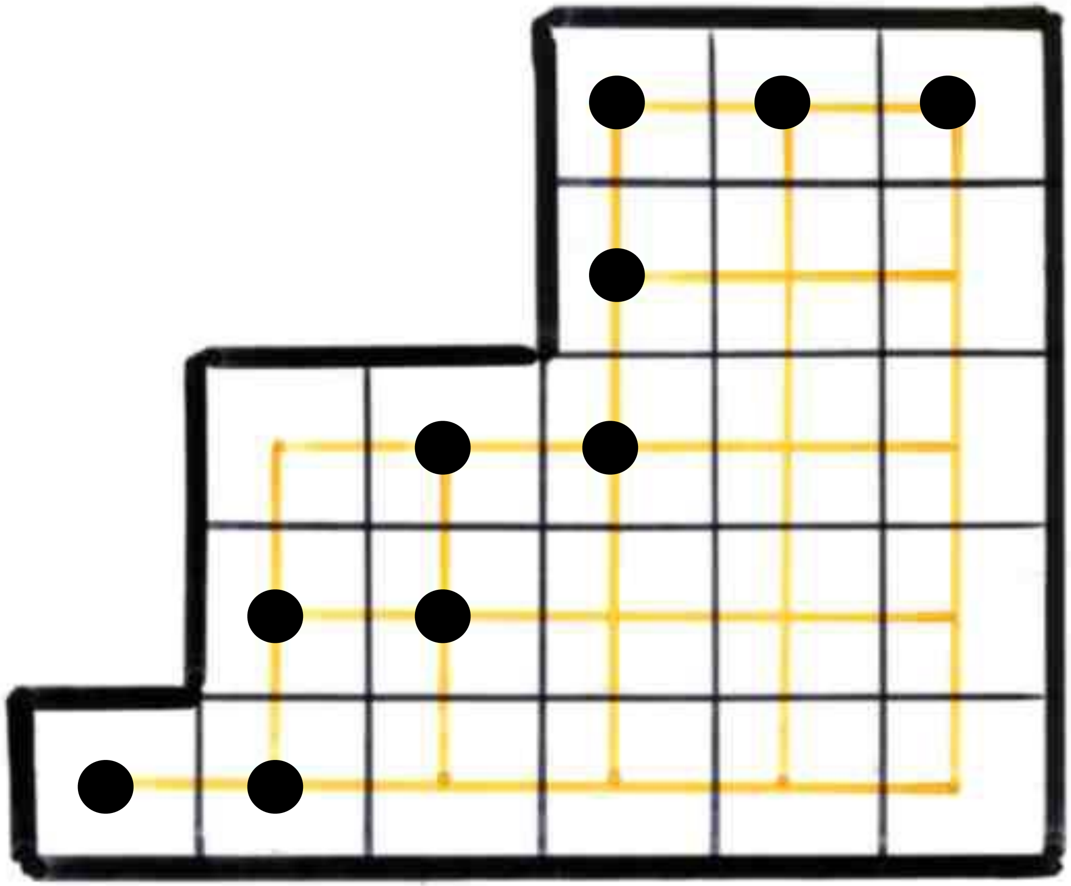


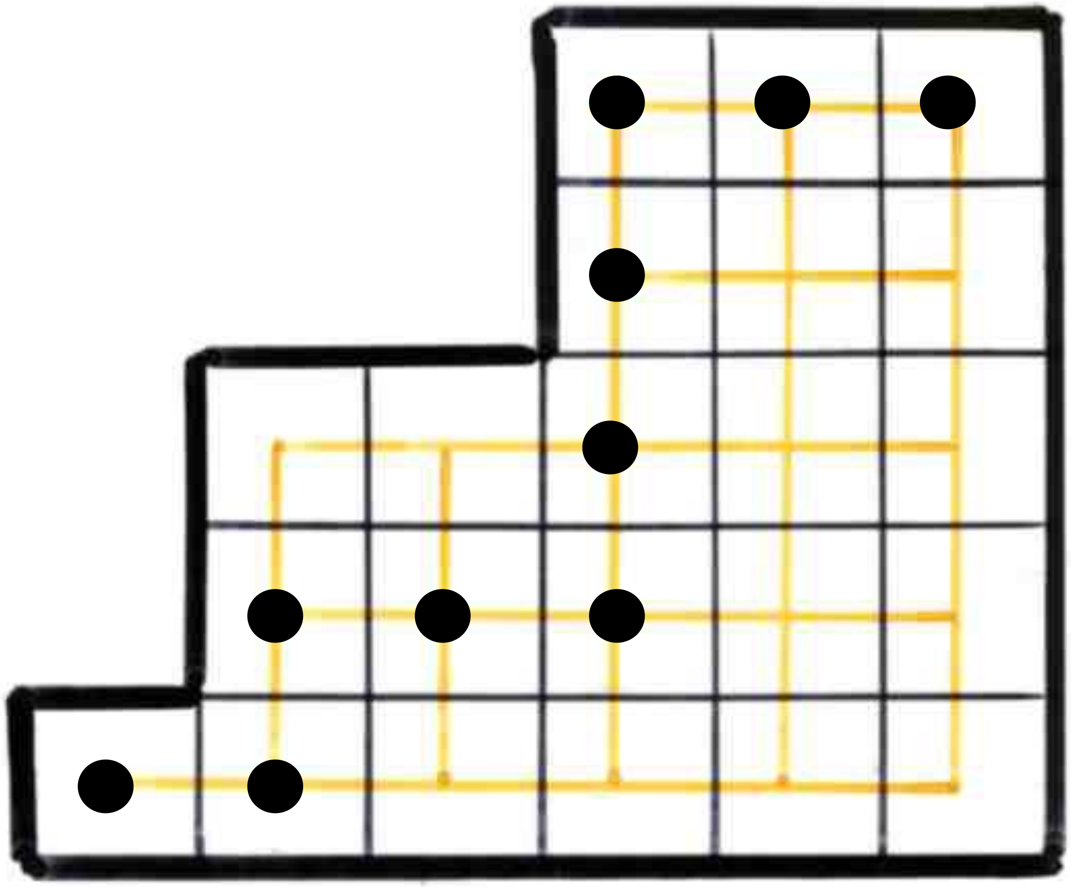
In the case of the maule generated by this cloud of points
Γ-moves are only elementary Γ-moves,

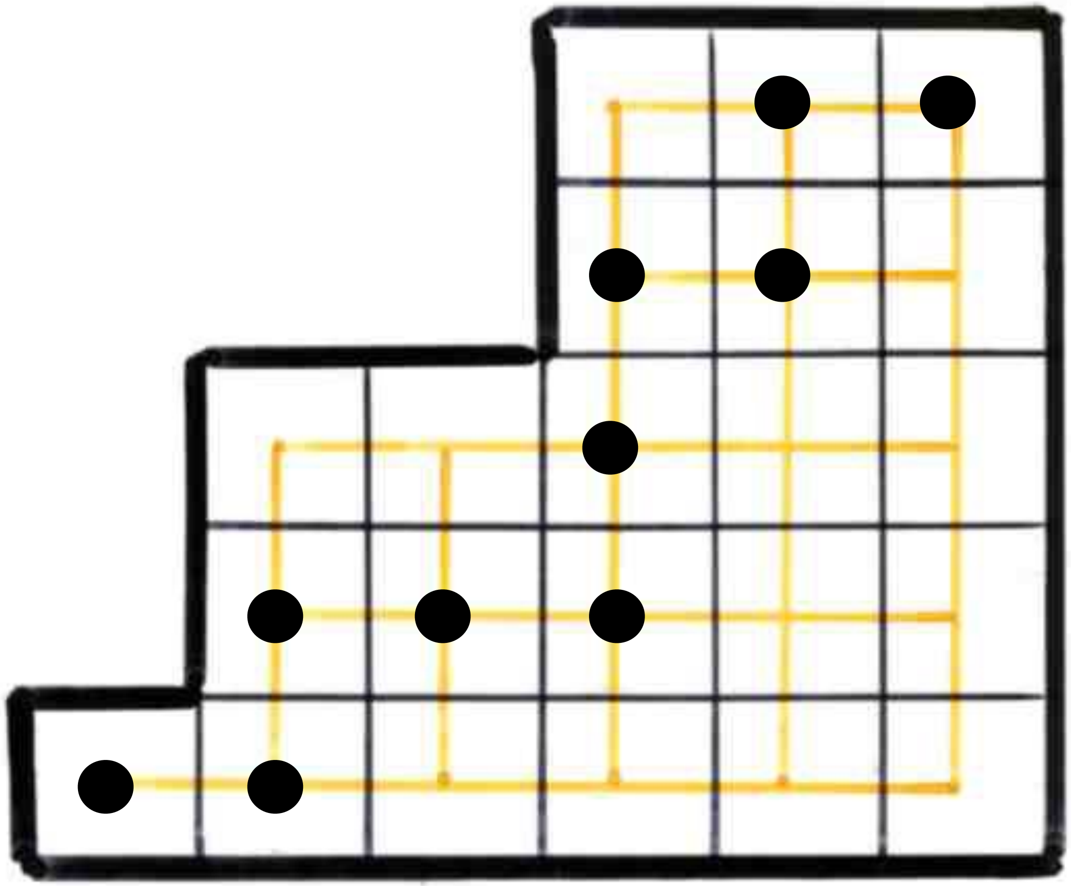
that is moves where the corresponding rectangle is reduced to an elementary cell of the square lattice.

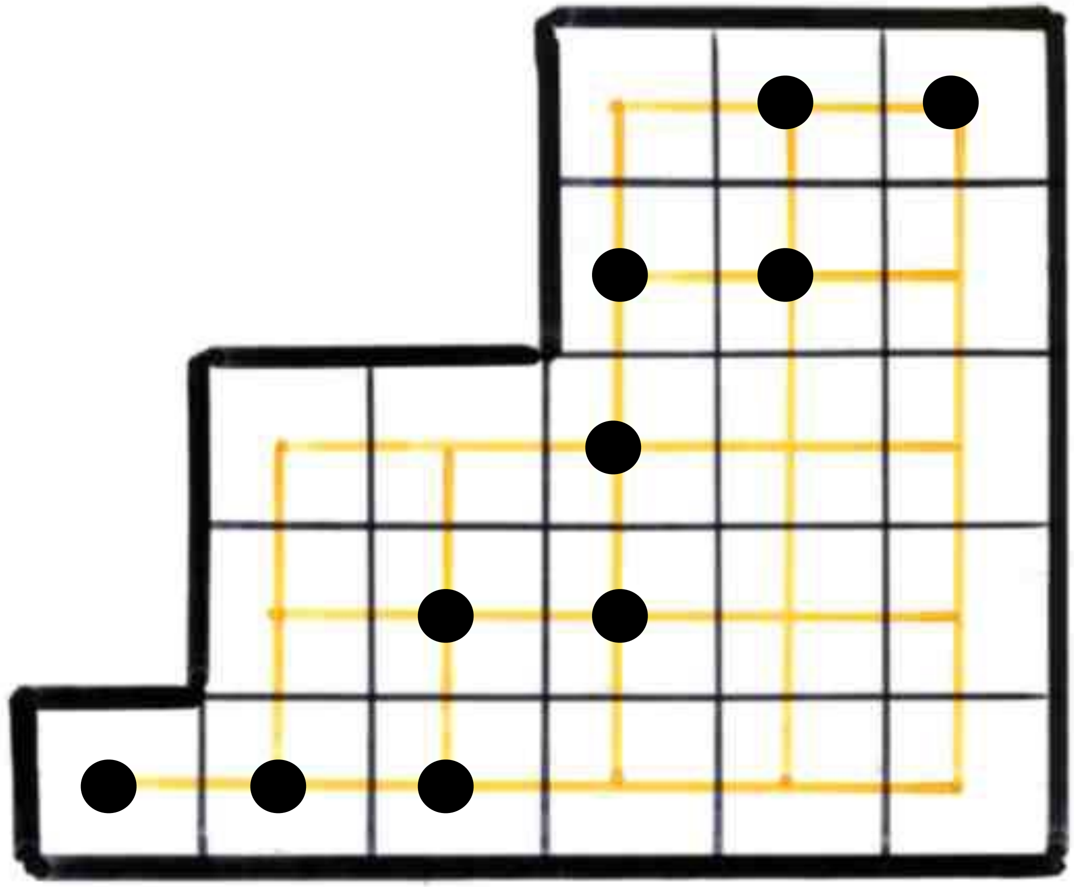


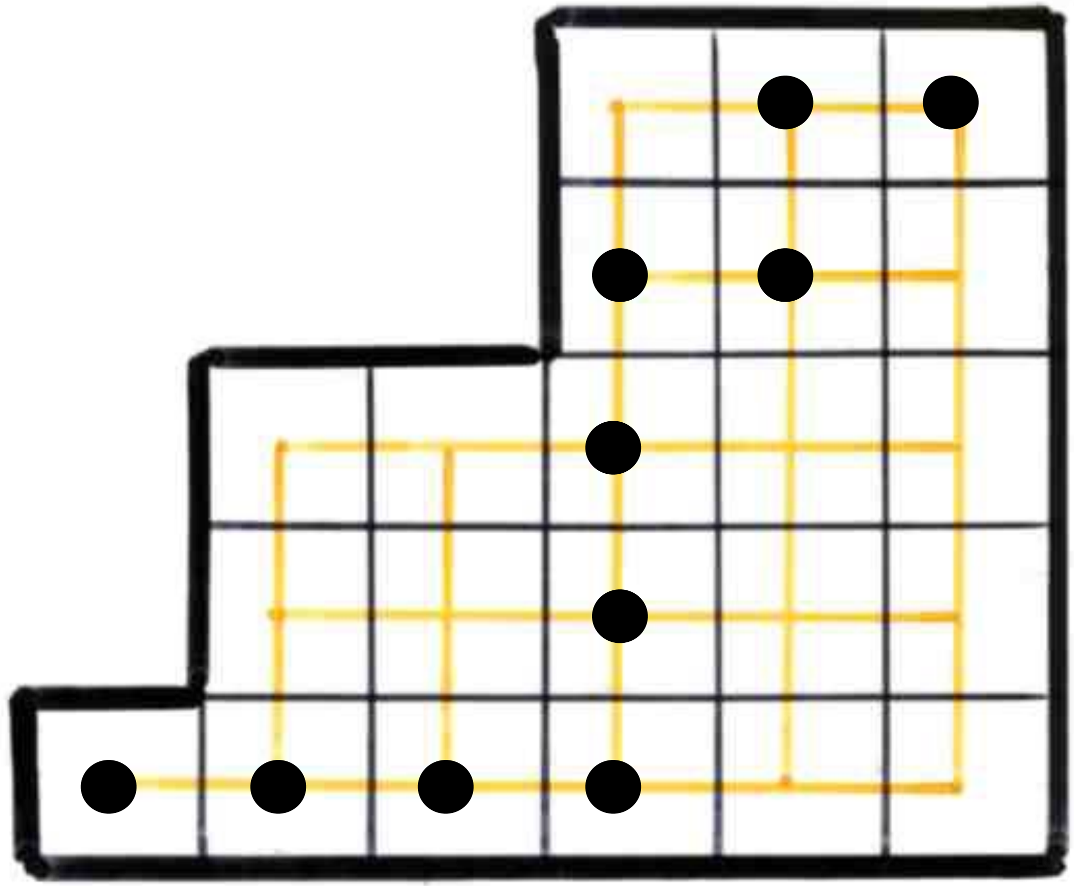
Such maules will be called simple maule.

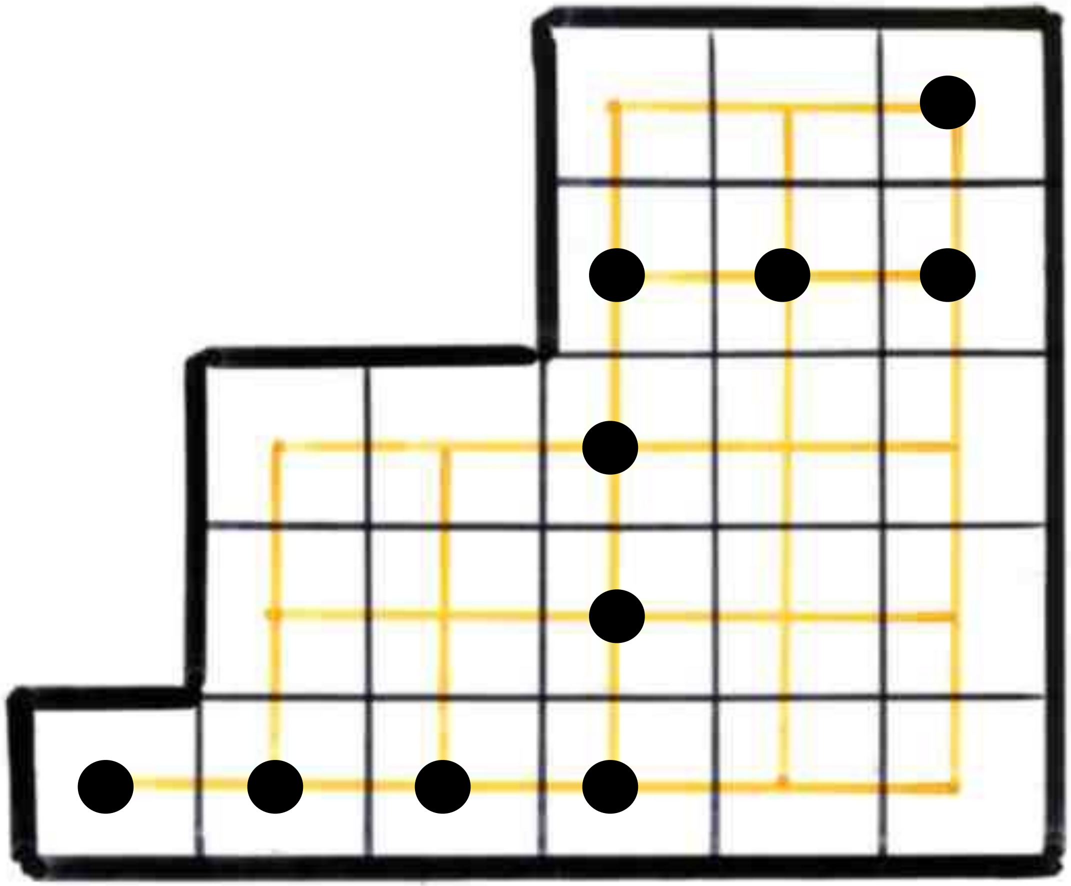


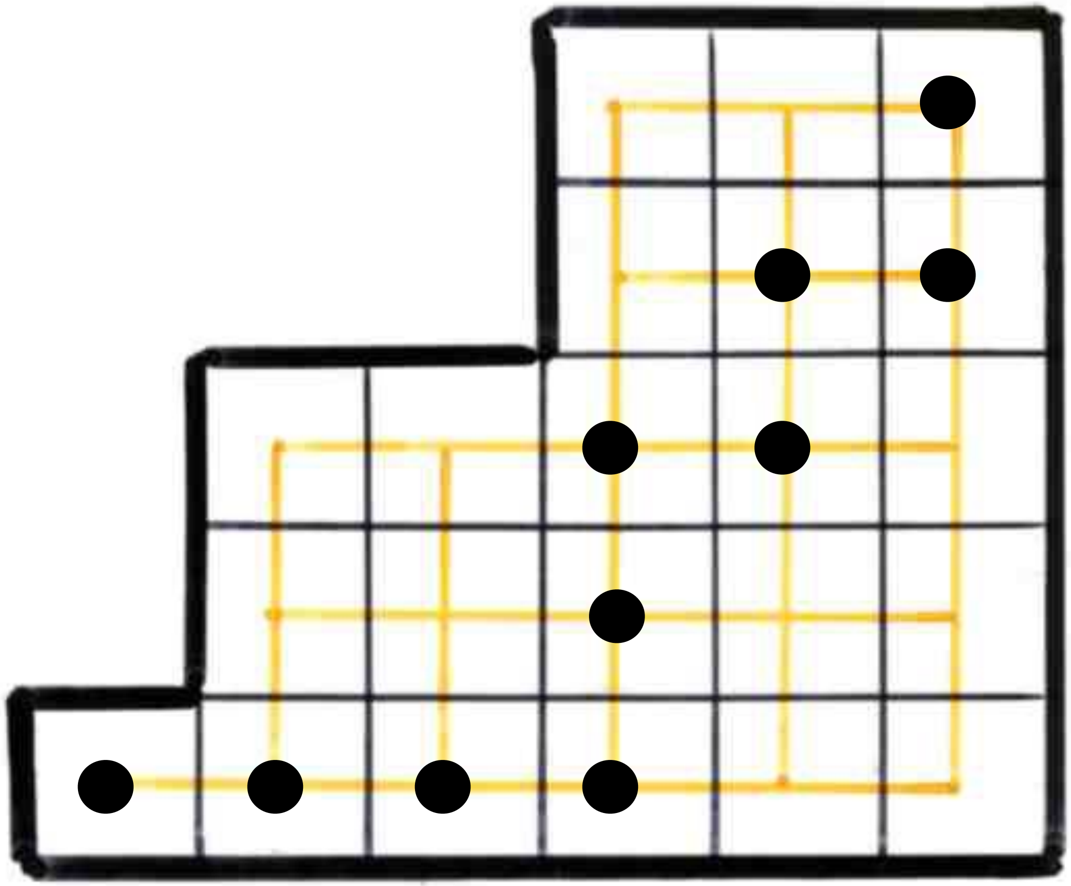


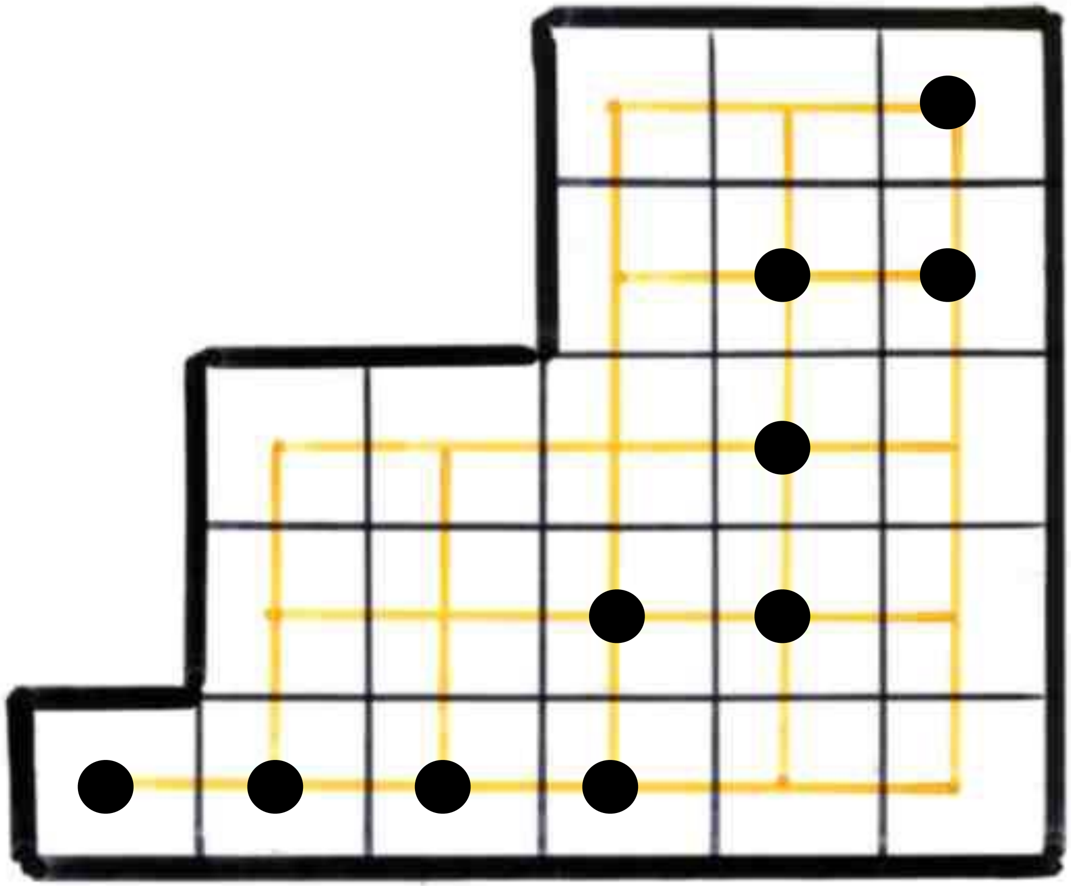


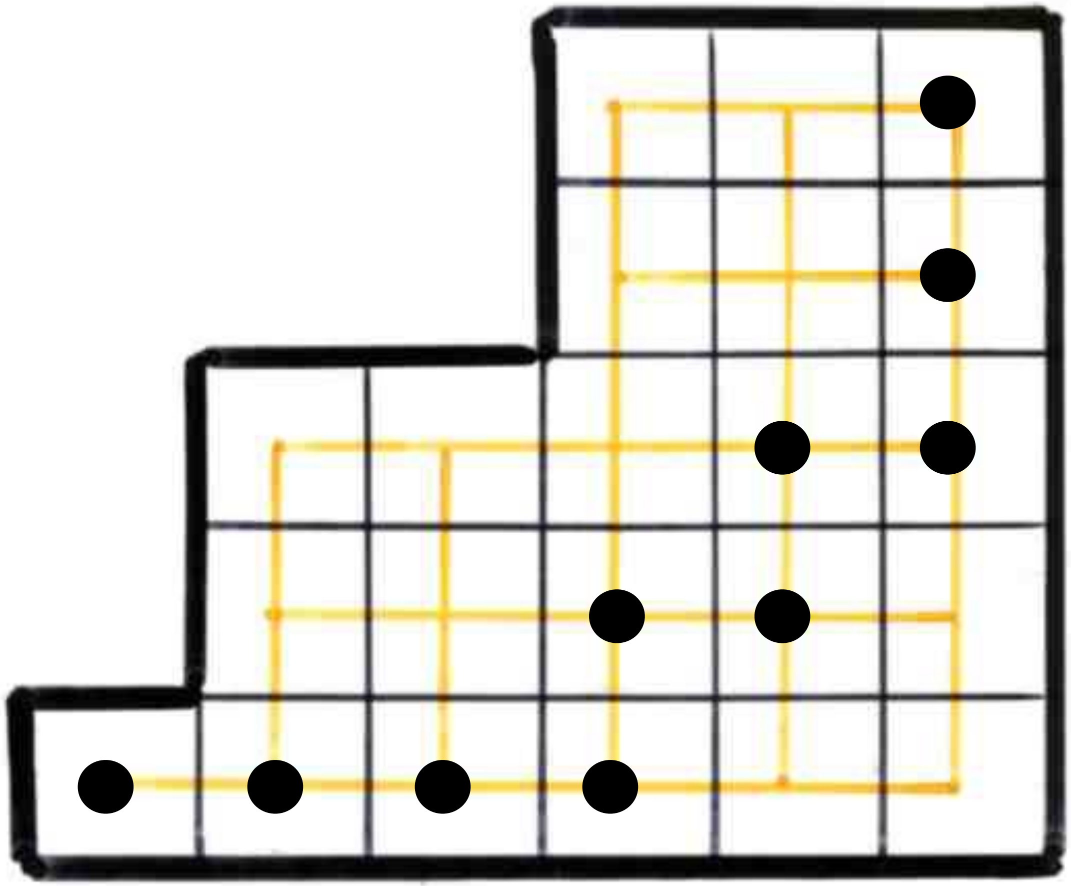


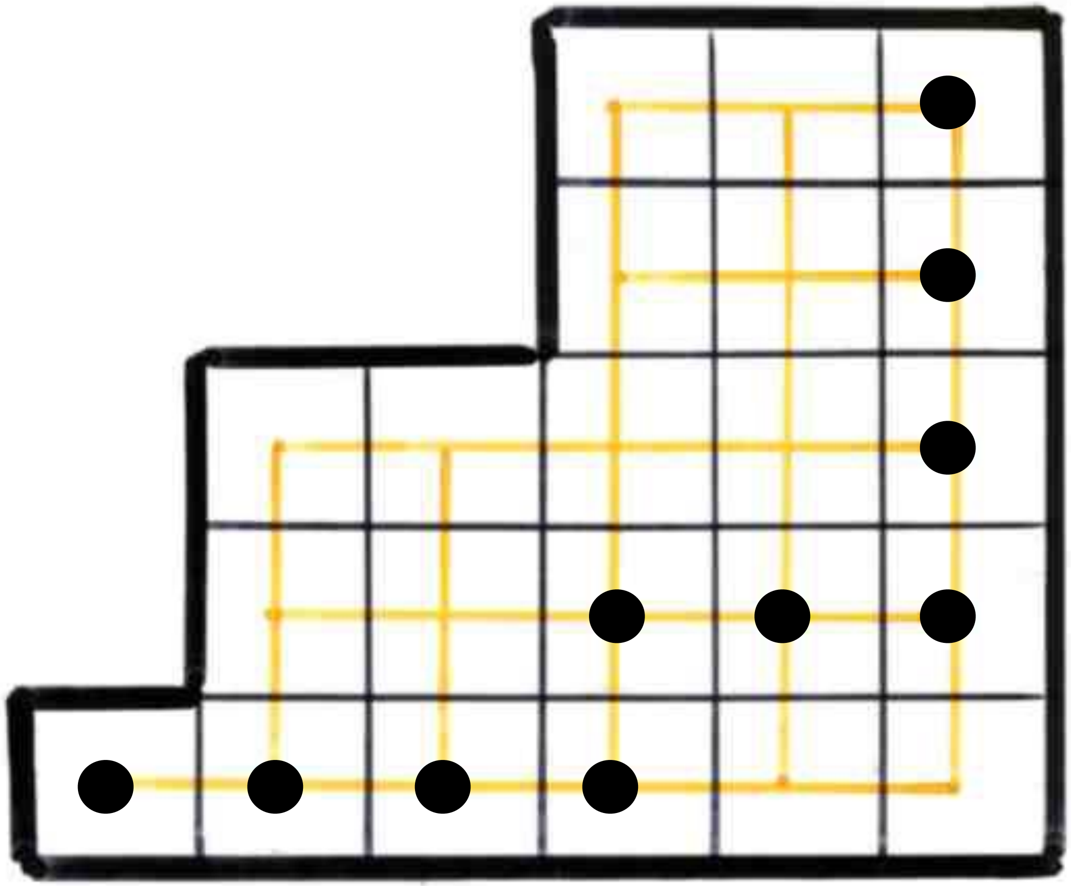


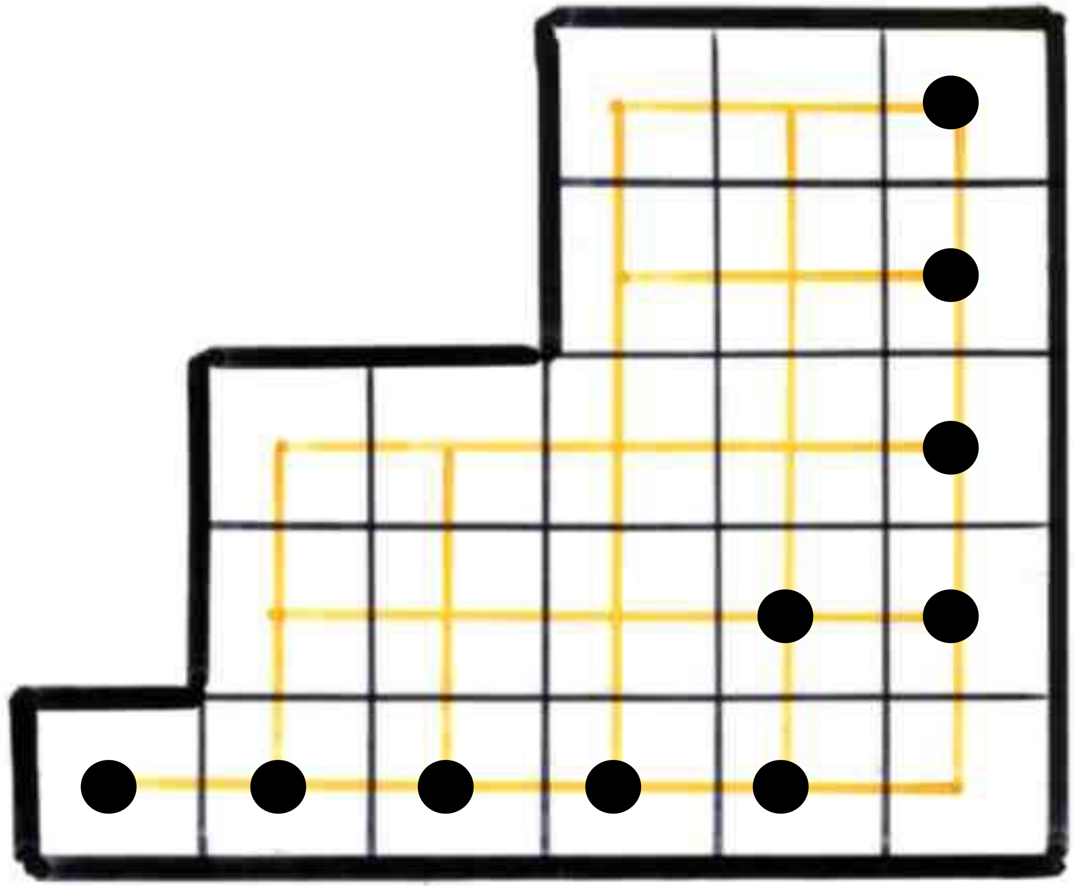


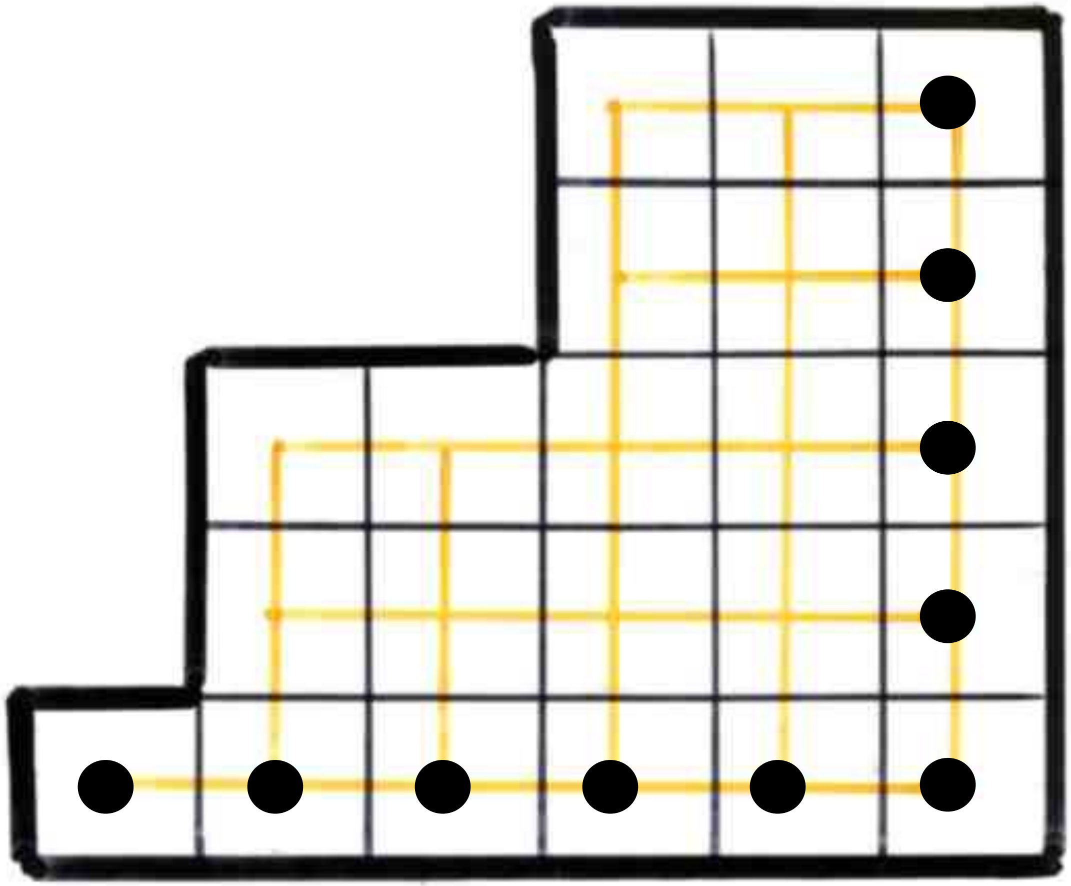




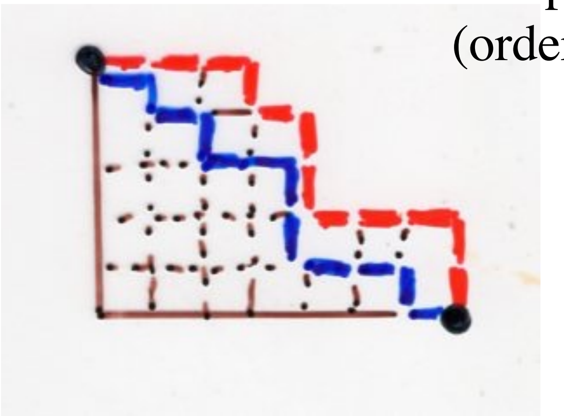




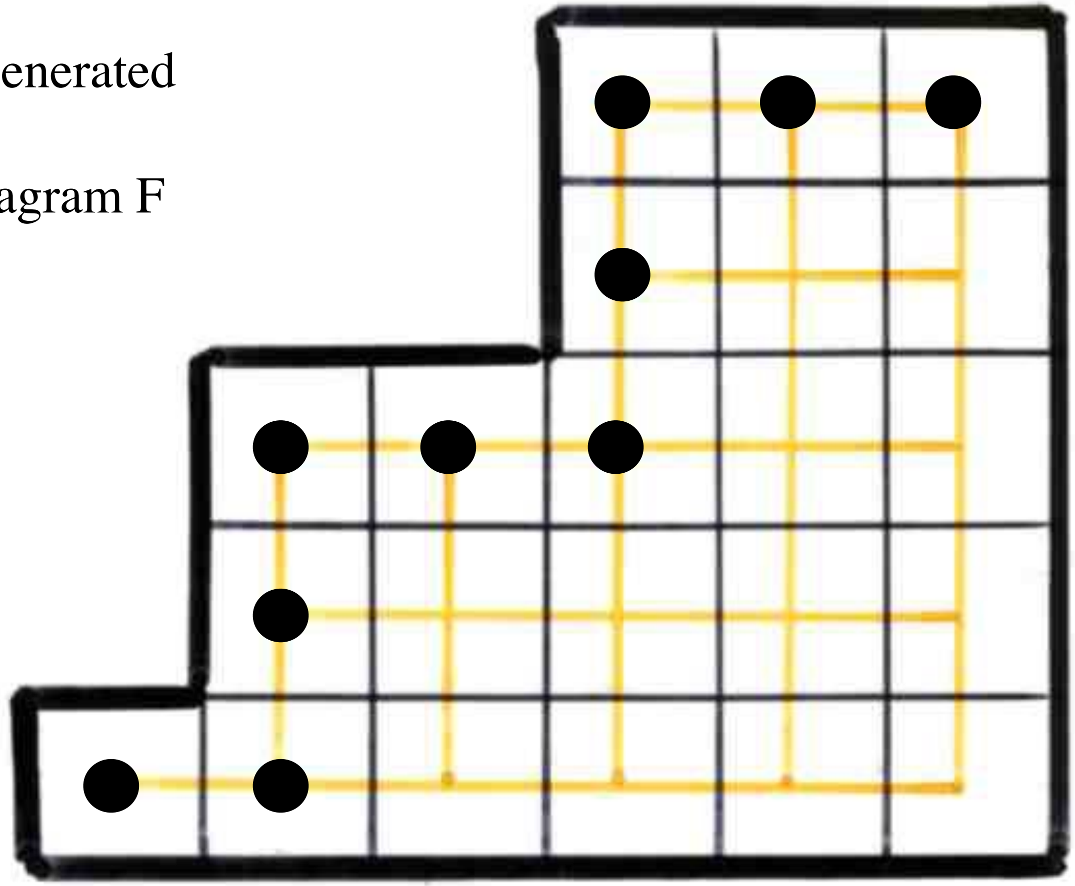




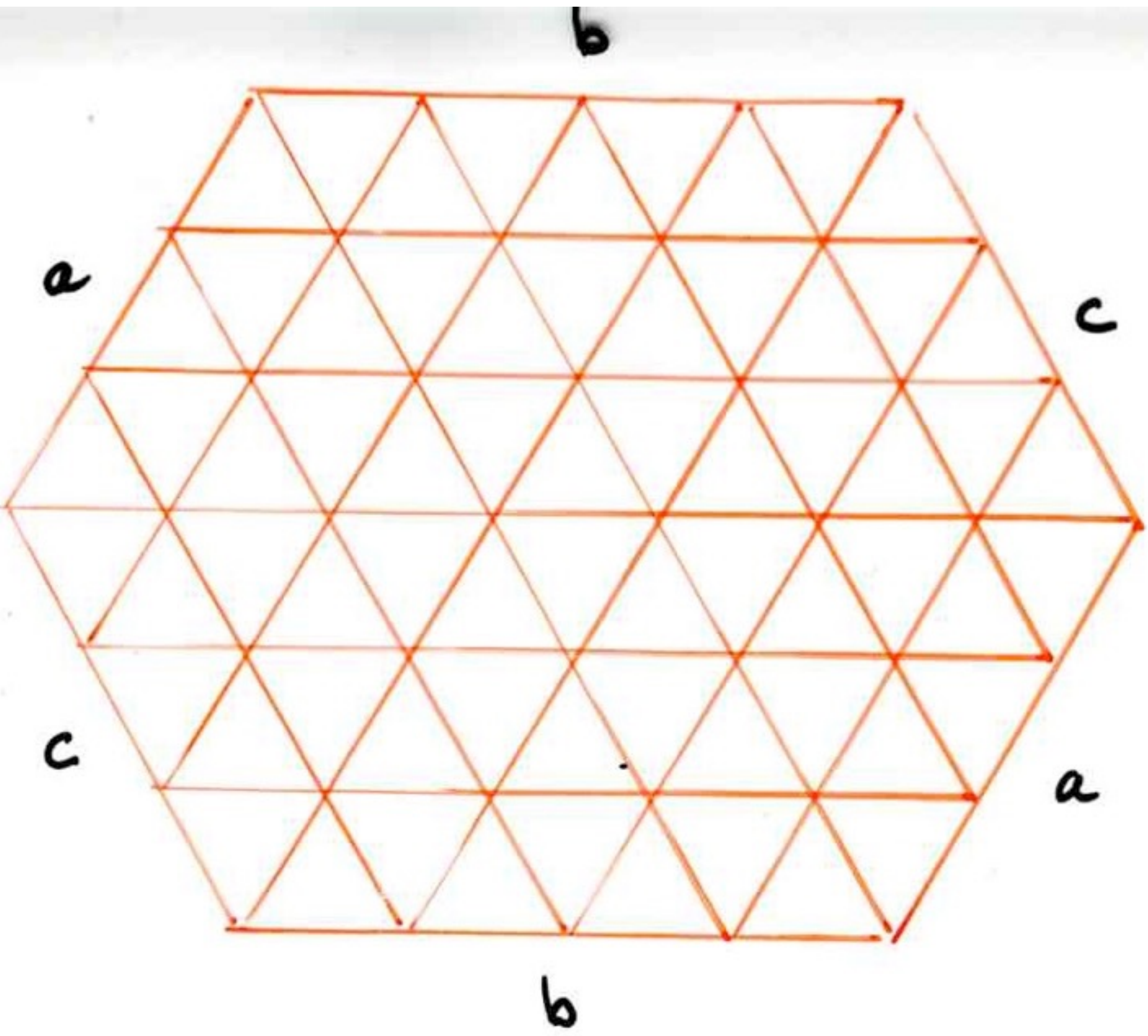
The poset of Ferrers diagrams included in a given Ferrers diagram F
(ordered by inclusion of diagrams)

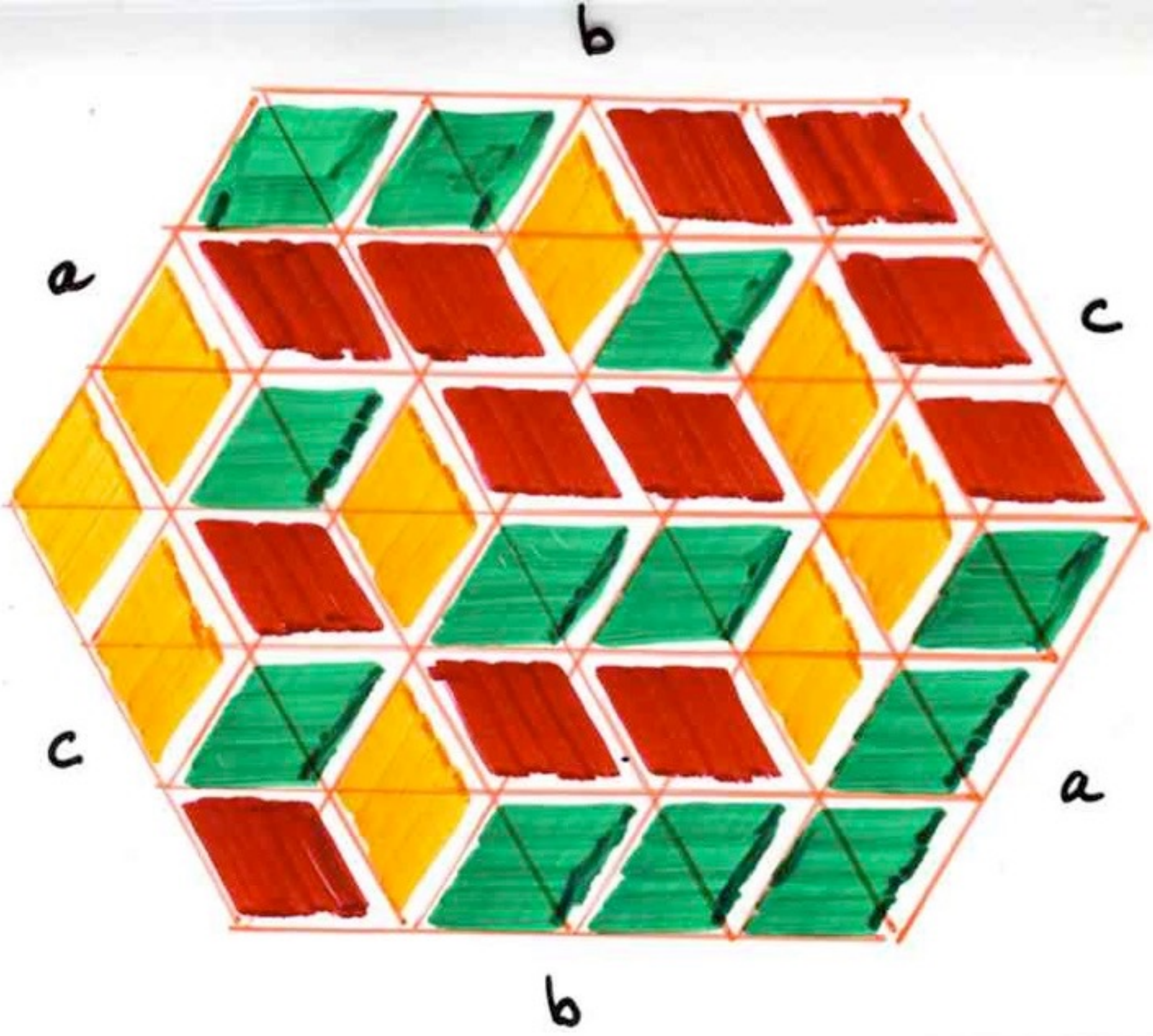


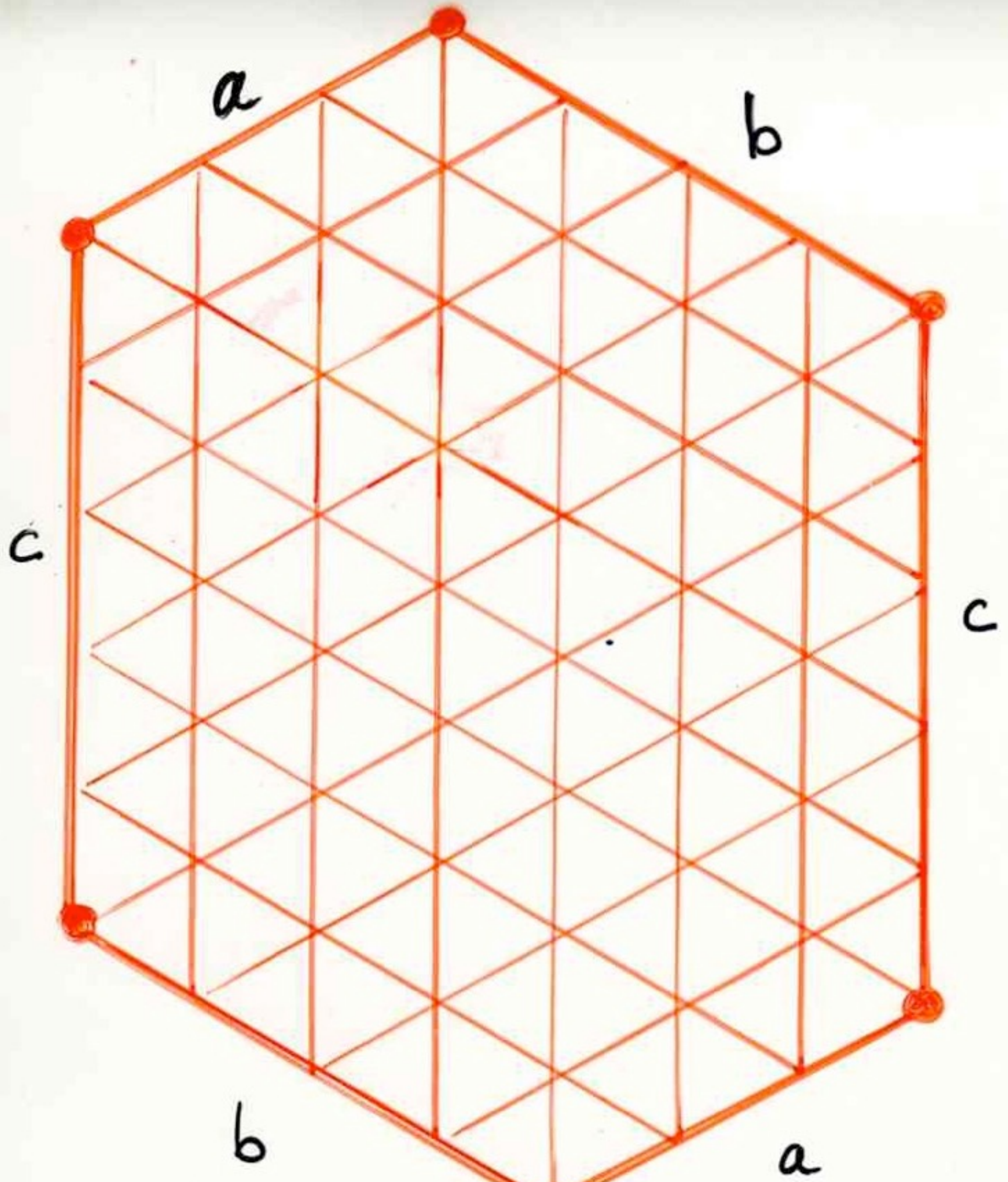
is ismorphic to the maule generated
by the following cloud
associated to the Ferrers diagram F

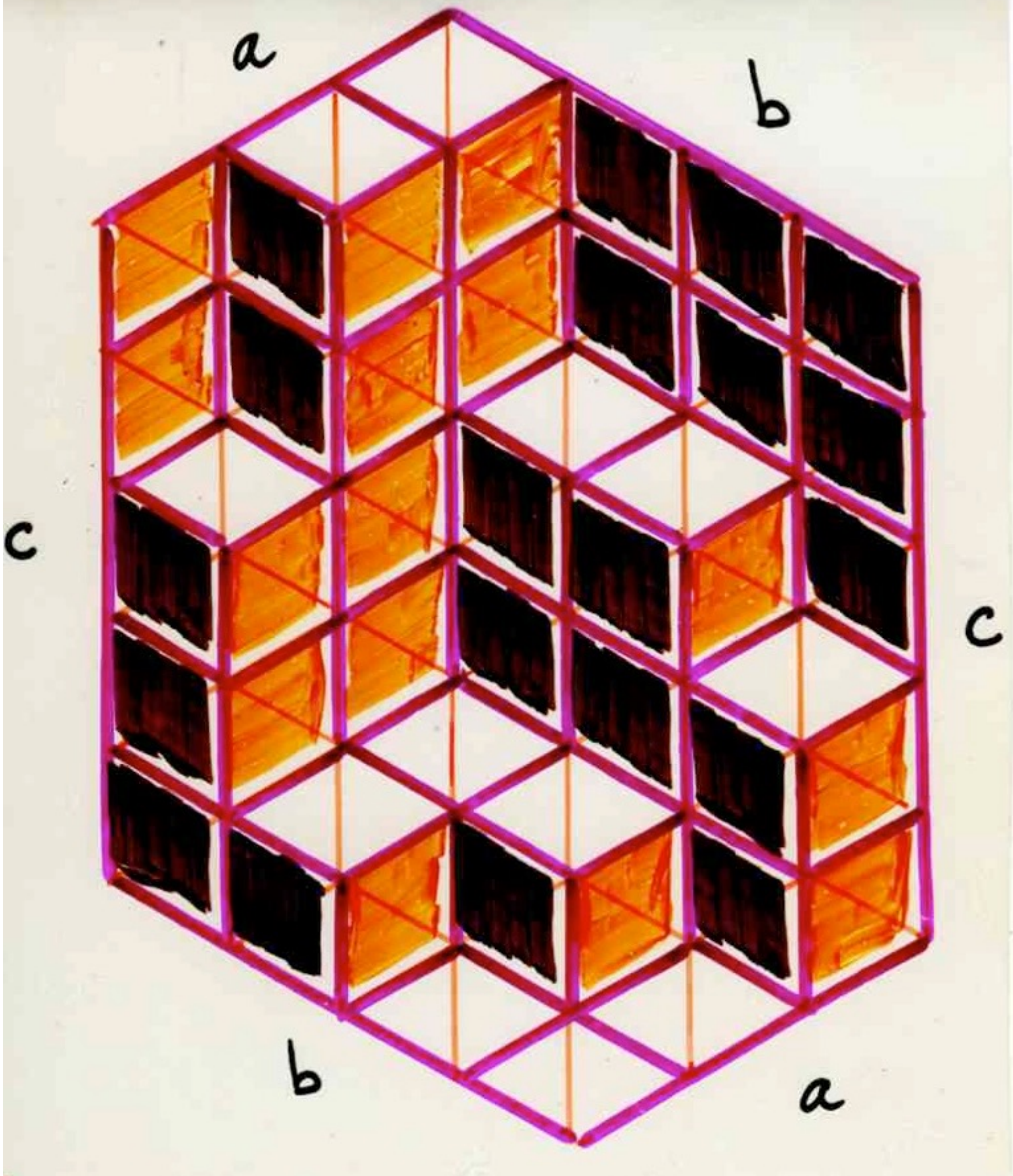


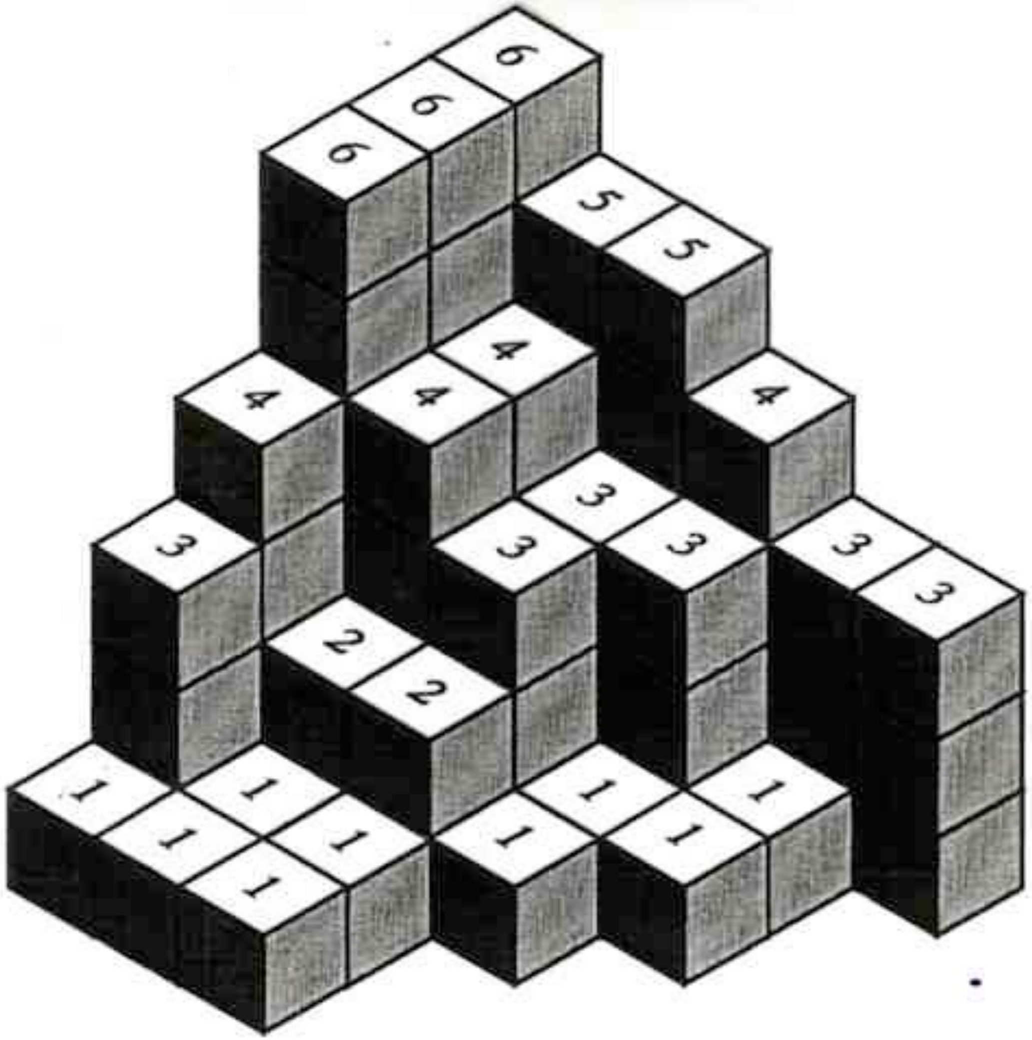
Tilings lattice


b
a
c
c
a
b


b
a
c
c
a
b


a
b
c
c
b
a


a
b
c
c
b
a



6	5	5	4	3	3
6	4	3	3	1	
6	4	3	1	1	
4	2	2	1		
3	1	1			
1	1	1			

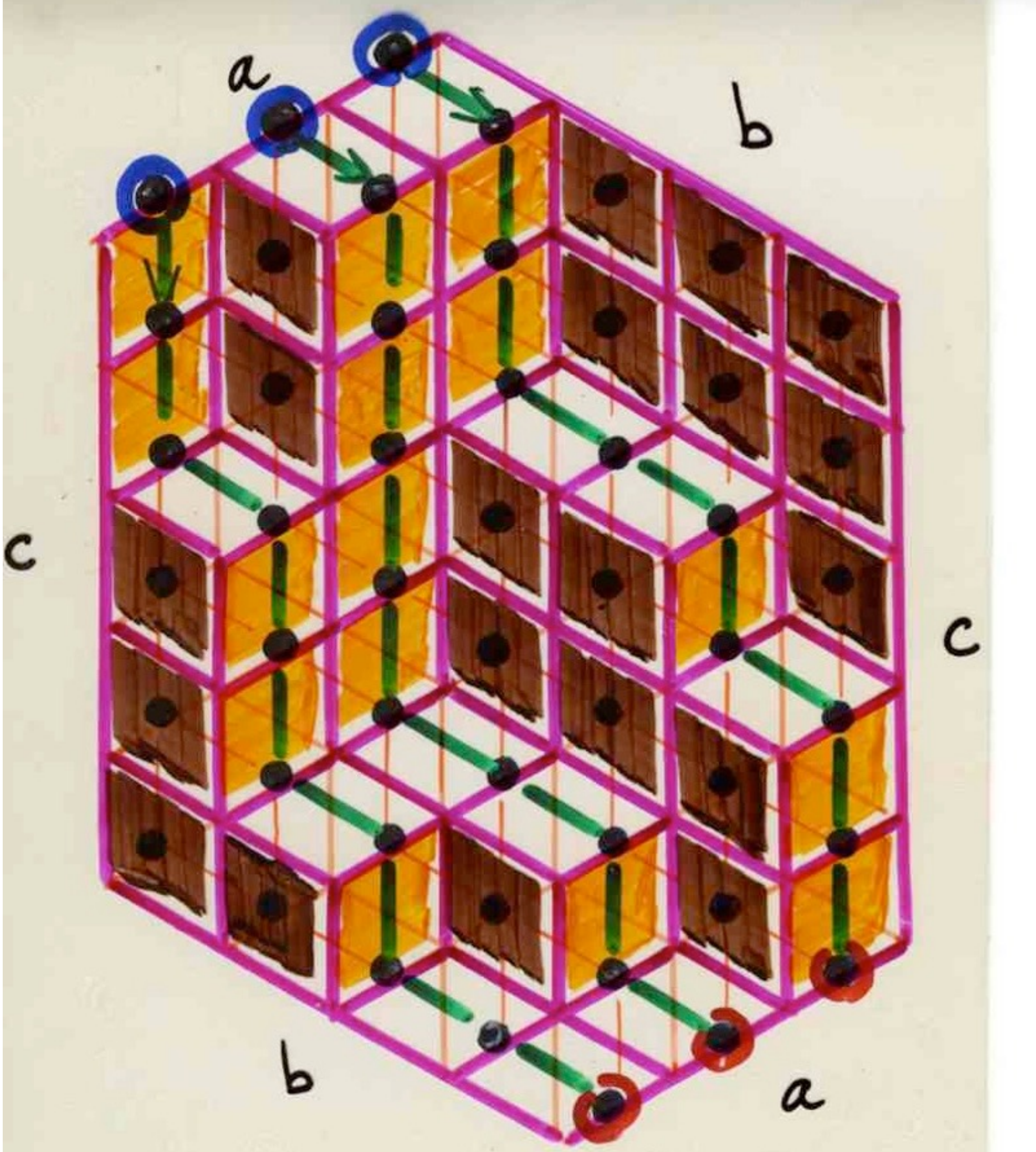
plane partitions

3D Ferrers diagrams

in a box $\mathcal{B}(a,b,c)$

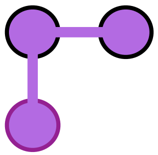
$$\prod_{\substack{1 \le i \le a \\ 1 \le j \le b \\ 1 \le k \le c}} \frac{i+j+k-1}{i+j+k-2}$$

MacMahon famous formula for the number of plane partitions included in a box (a, b, c) can be proved using a coding of the plane partitions with configuration of non-crossing paths


a
b
c
c
b
a

the associated cloud of points are all the vertices of all the paths

As in the case of Young lattice, Γ-moves are only elementary Γ-moves,



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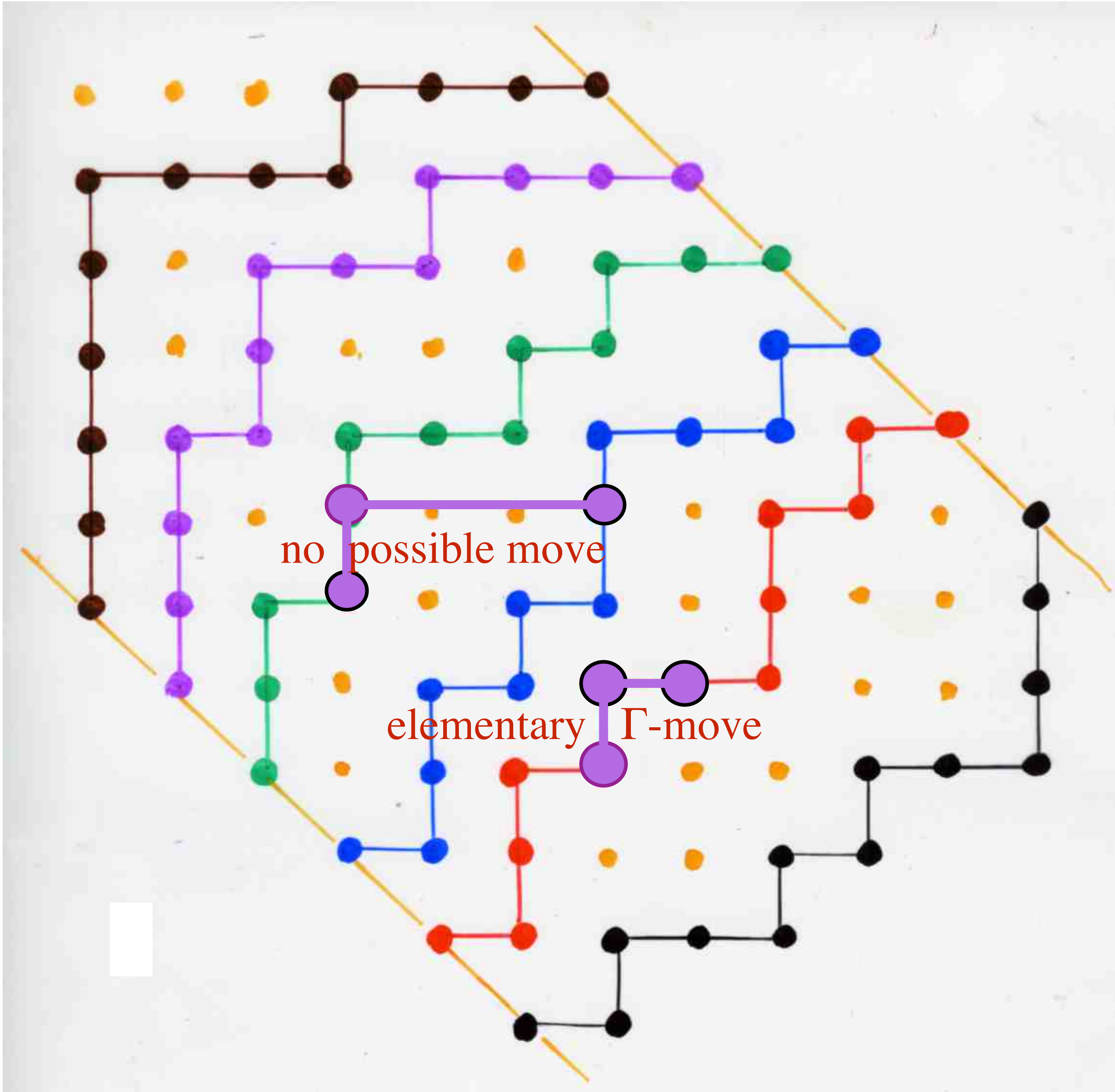
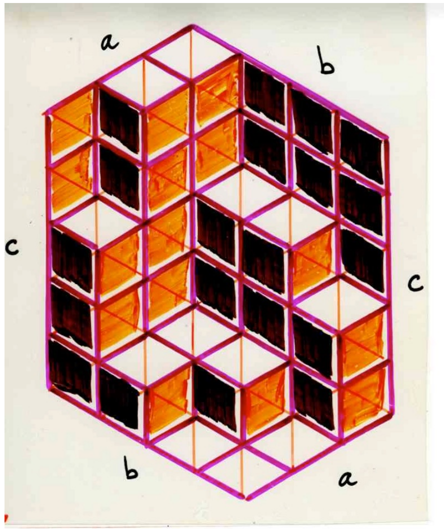
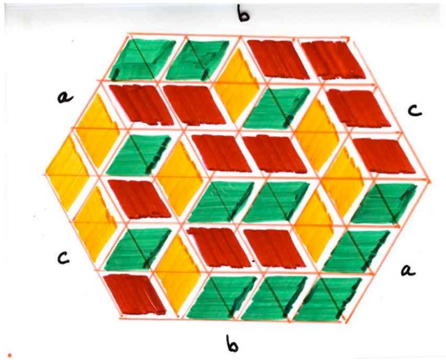


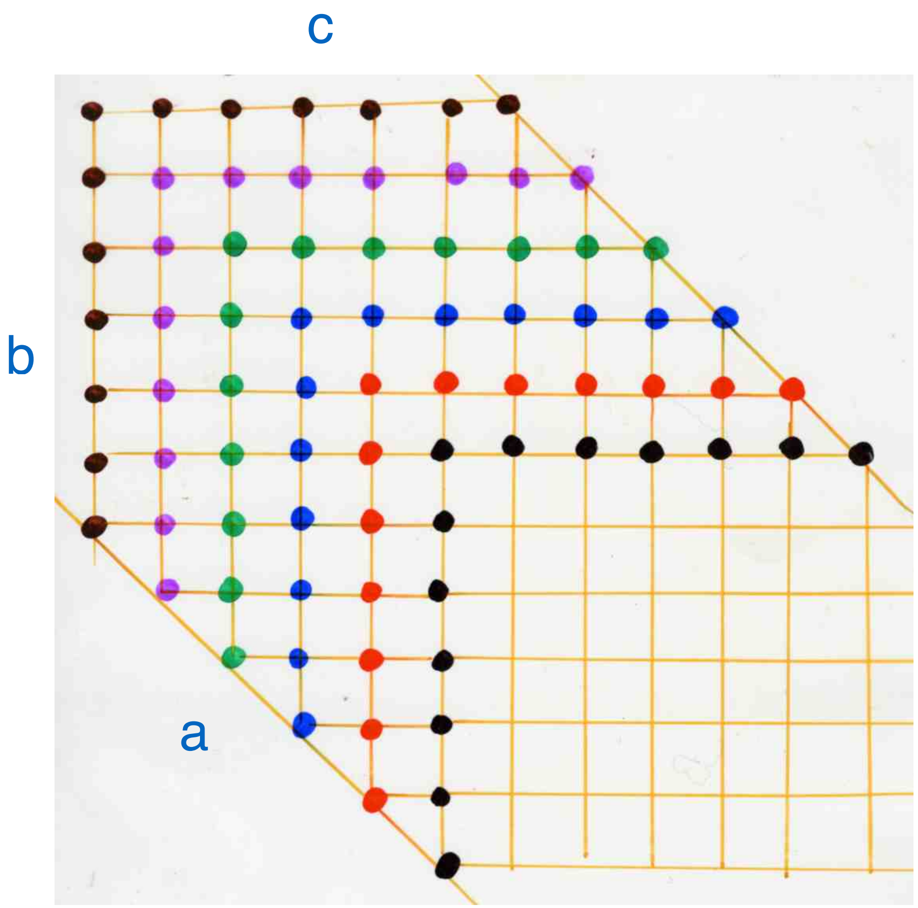
The poset of plane partitions included in a given box of size (a, b, c) (ordered by inclusion of 3D diagrams),



equivalently tilings of an hexagon of size (a,b,c),

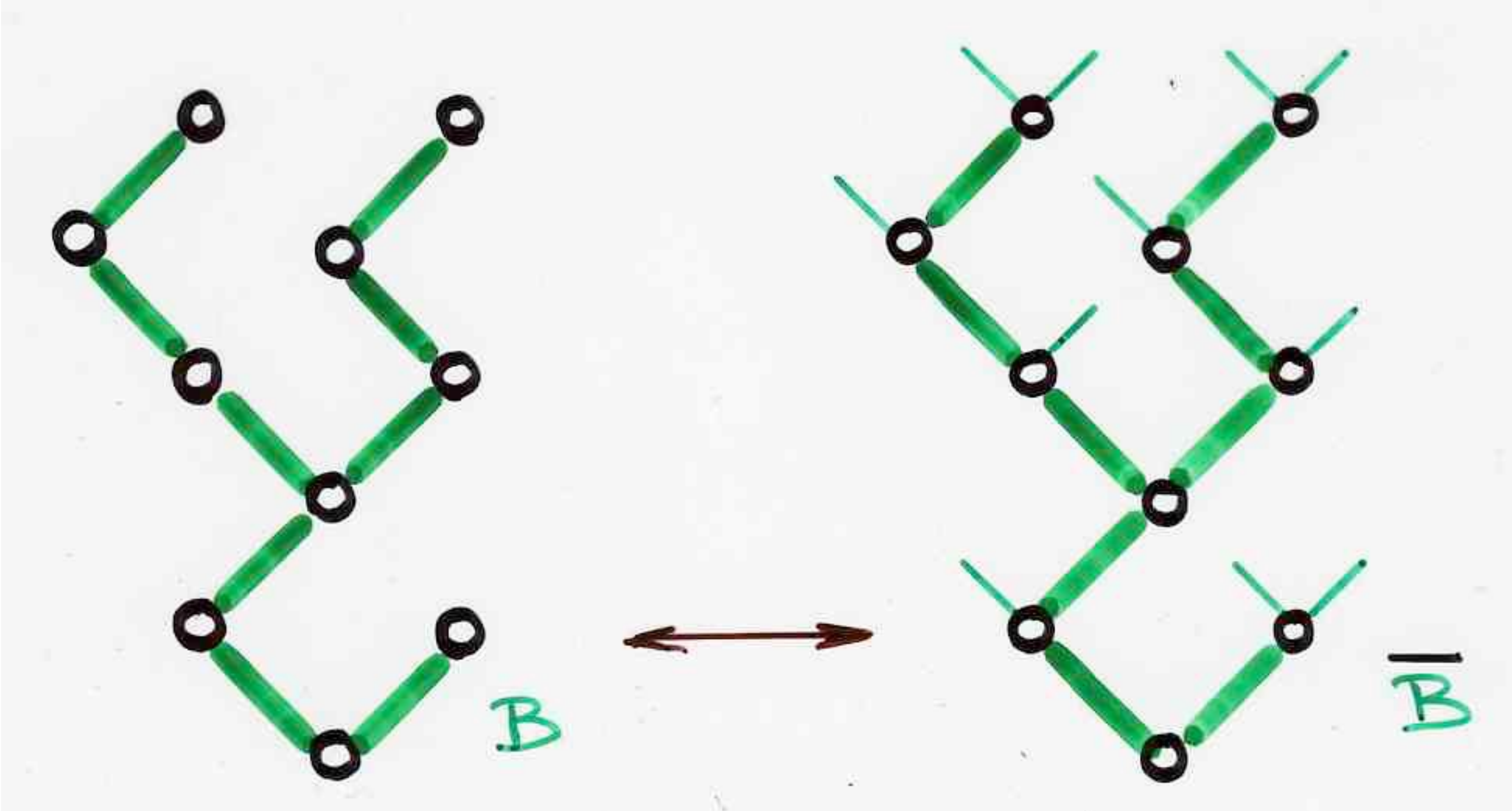


is isomorphic to the simple maule generated by the following cloud of points associated to the triple (a, b, c).

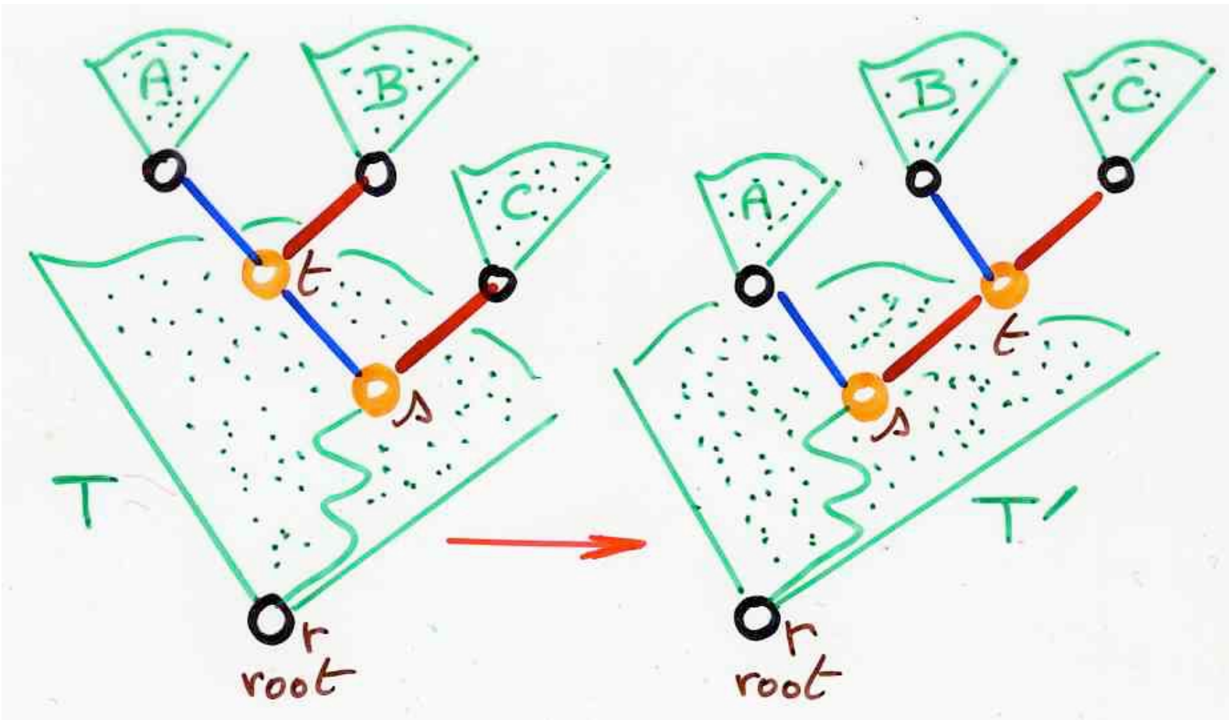


Tamari lattice

definition



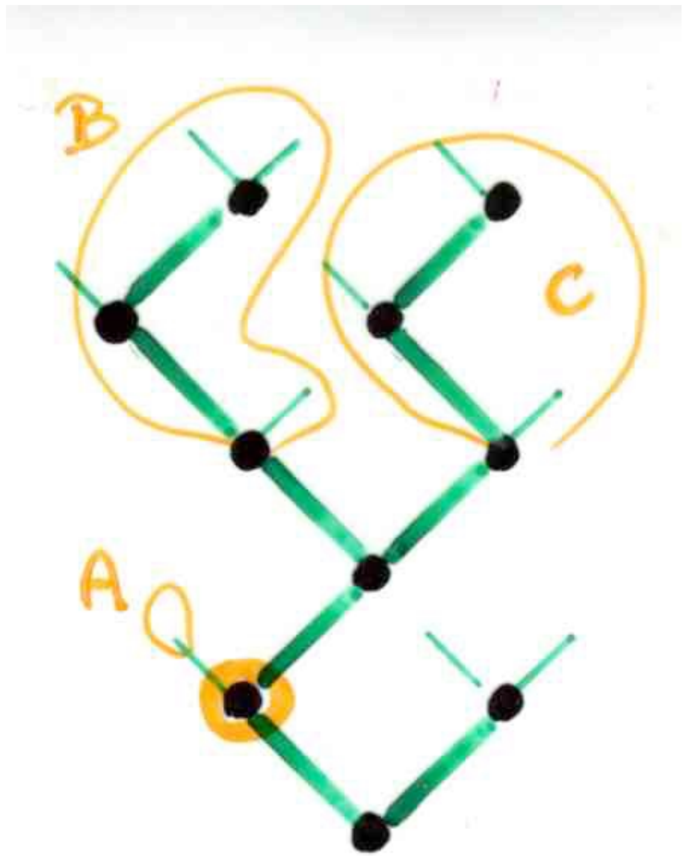
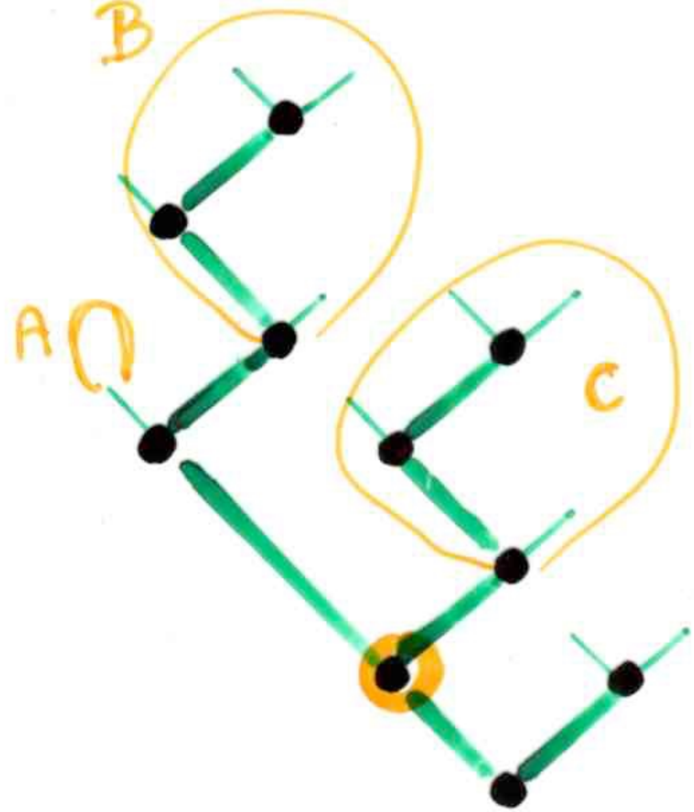
a binary tree B
and its associated complete (full) binary tree $\overline{B}$

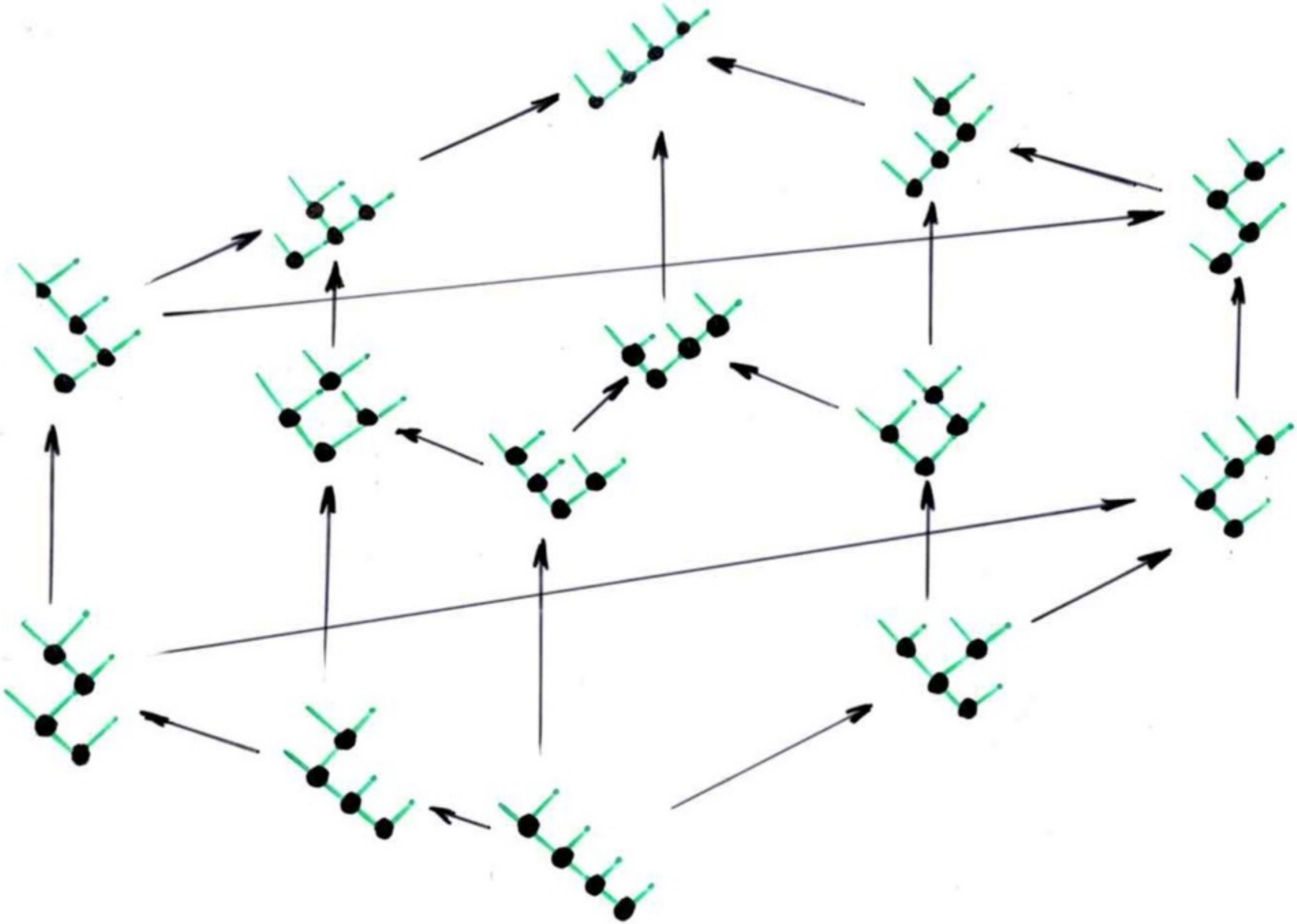


Tamari lattice

Rotation in a binary tree:
the covering relation in the Tamari lattice

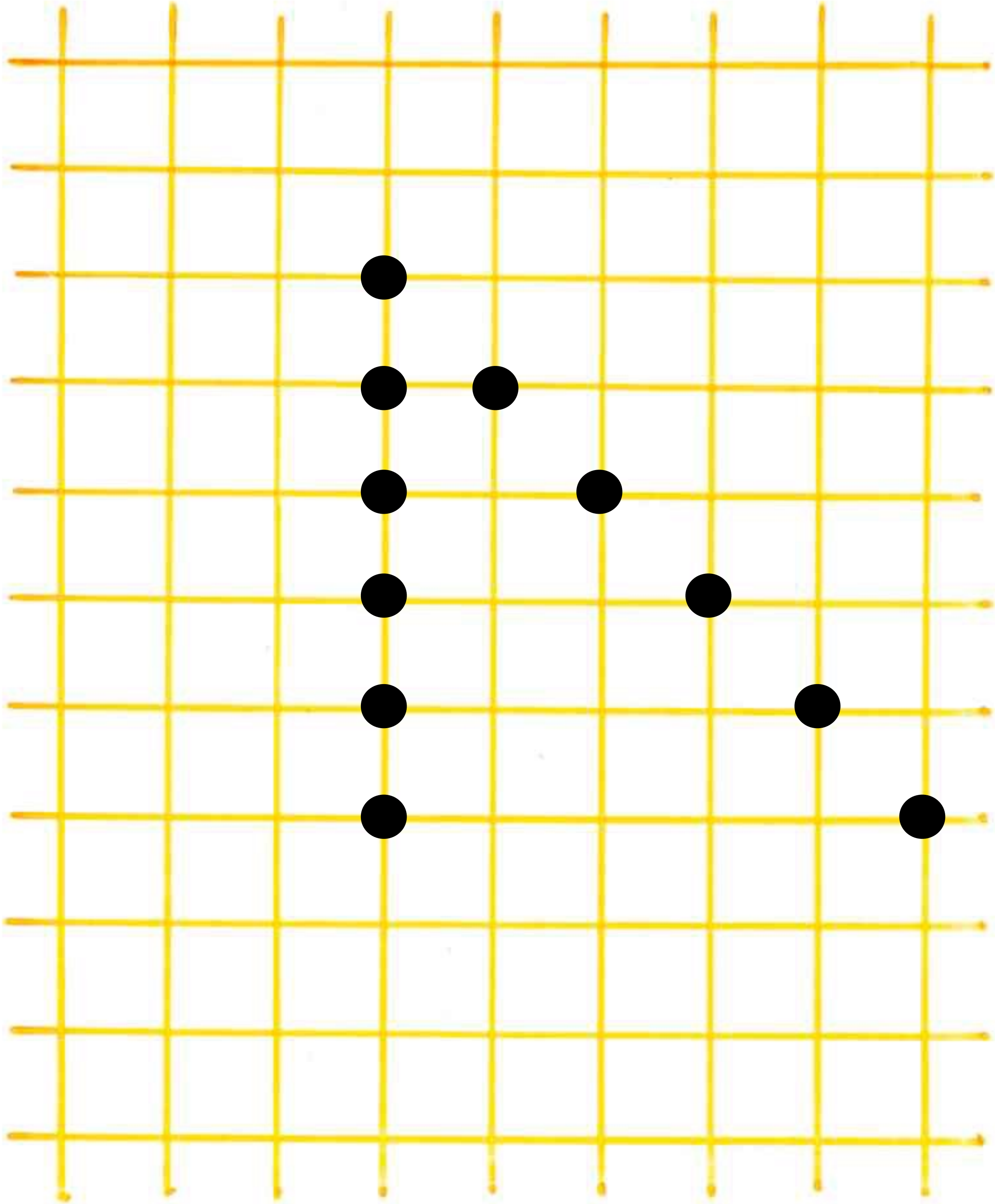
Dov Tamari (1951) thèse Sorbone
"Monoides préordonnés et chaînes de Malcev"

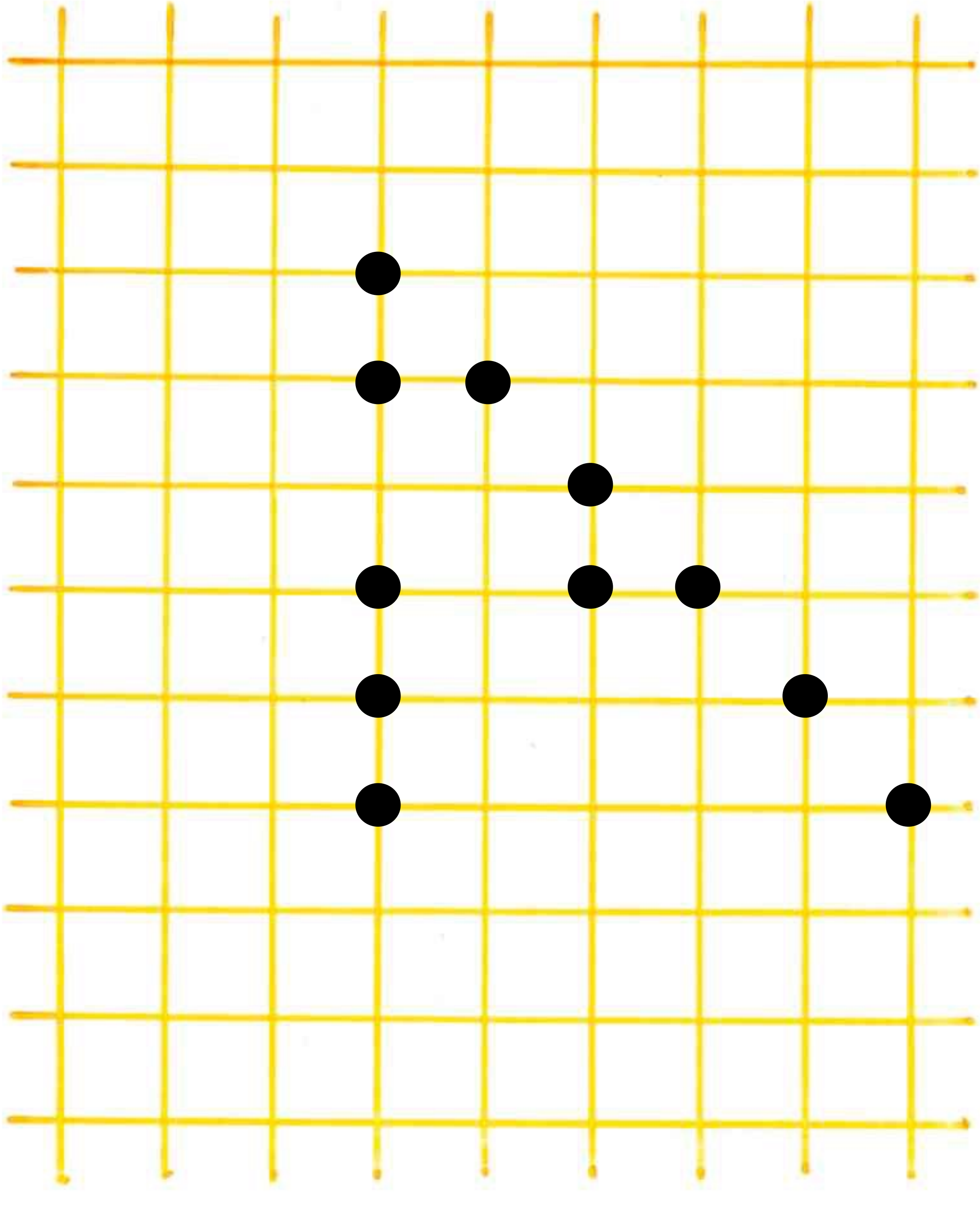

B
C
A

B
A
C

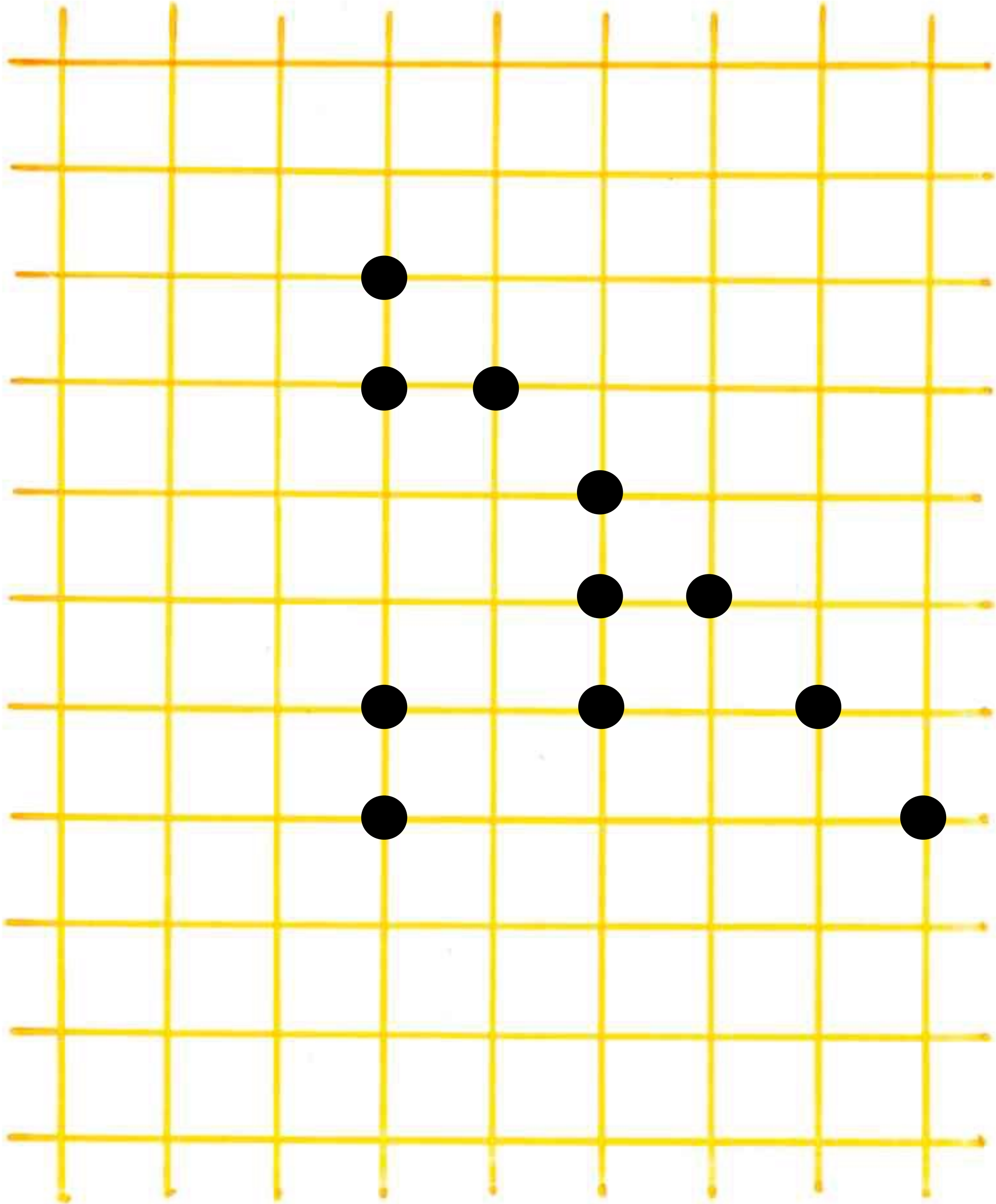


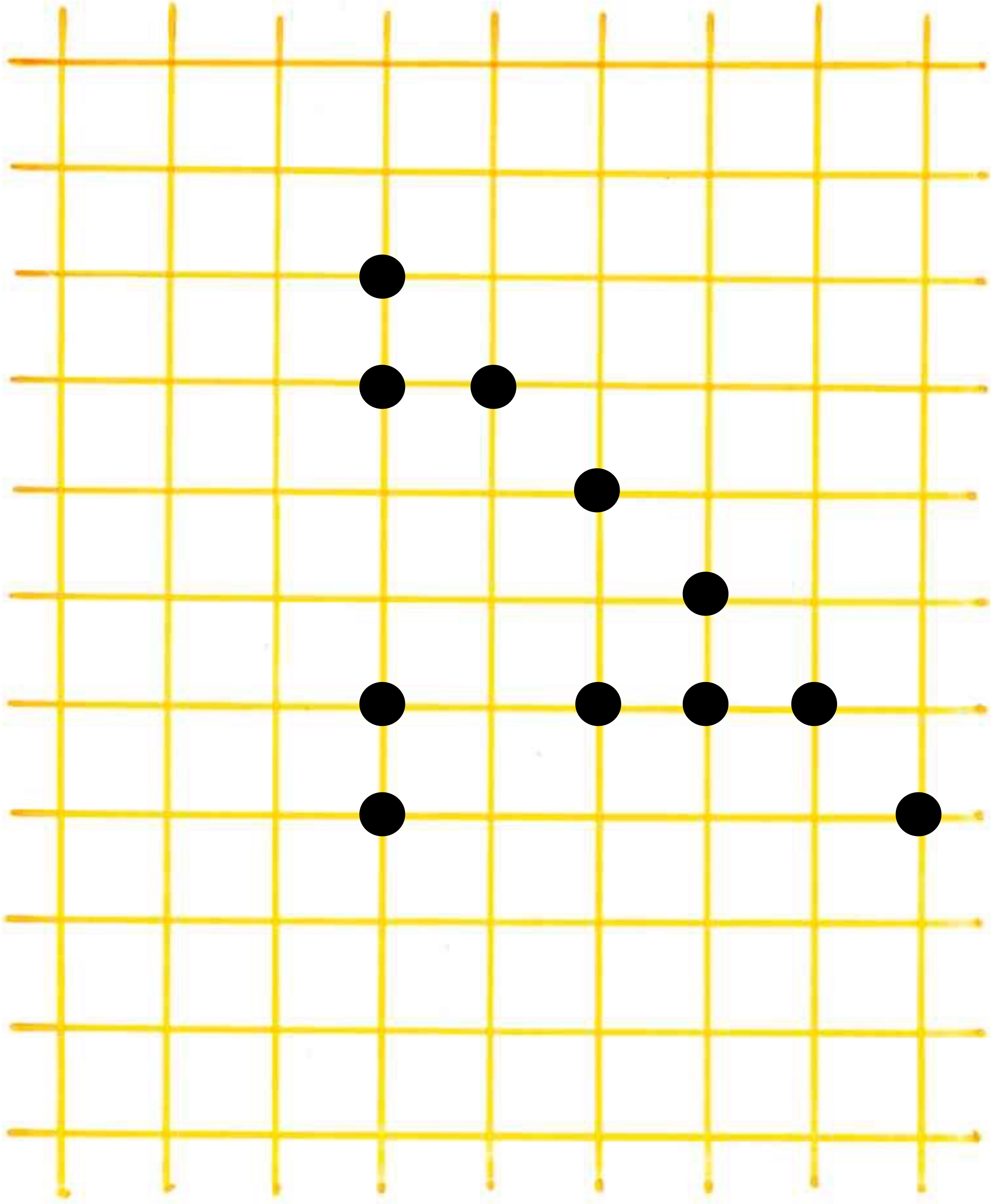
Tamari lattice

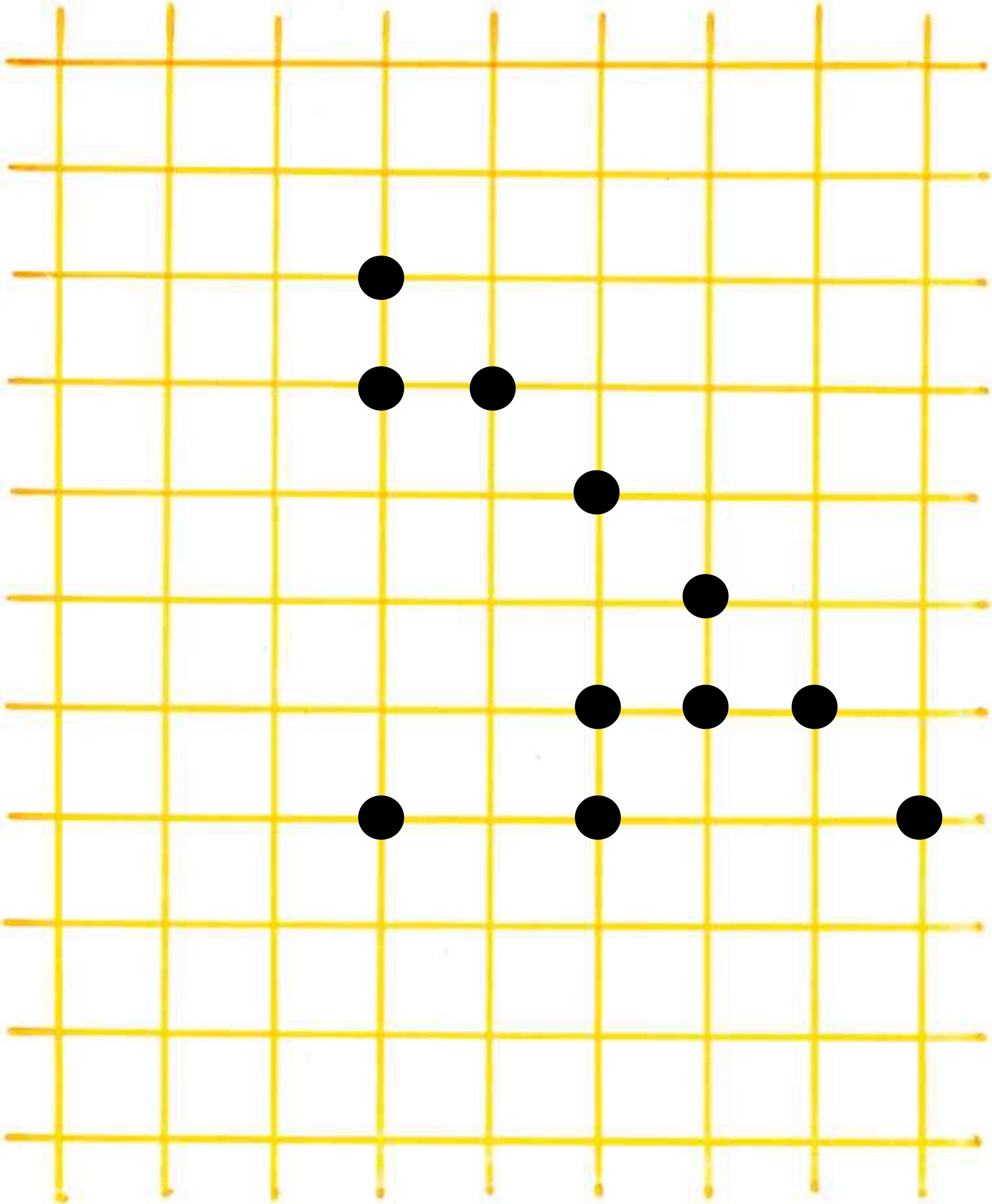
as a maule

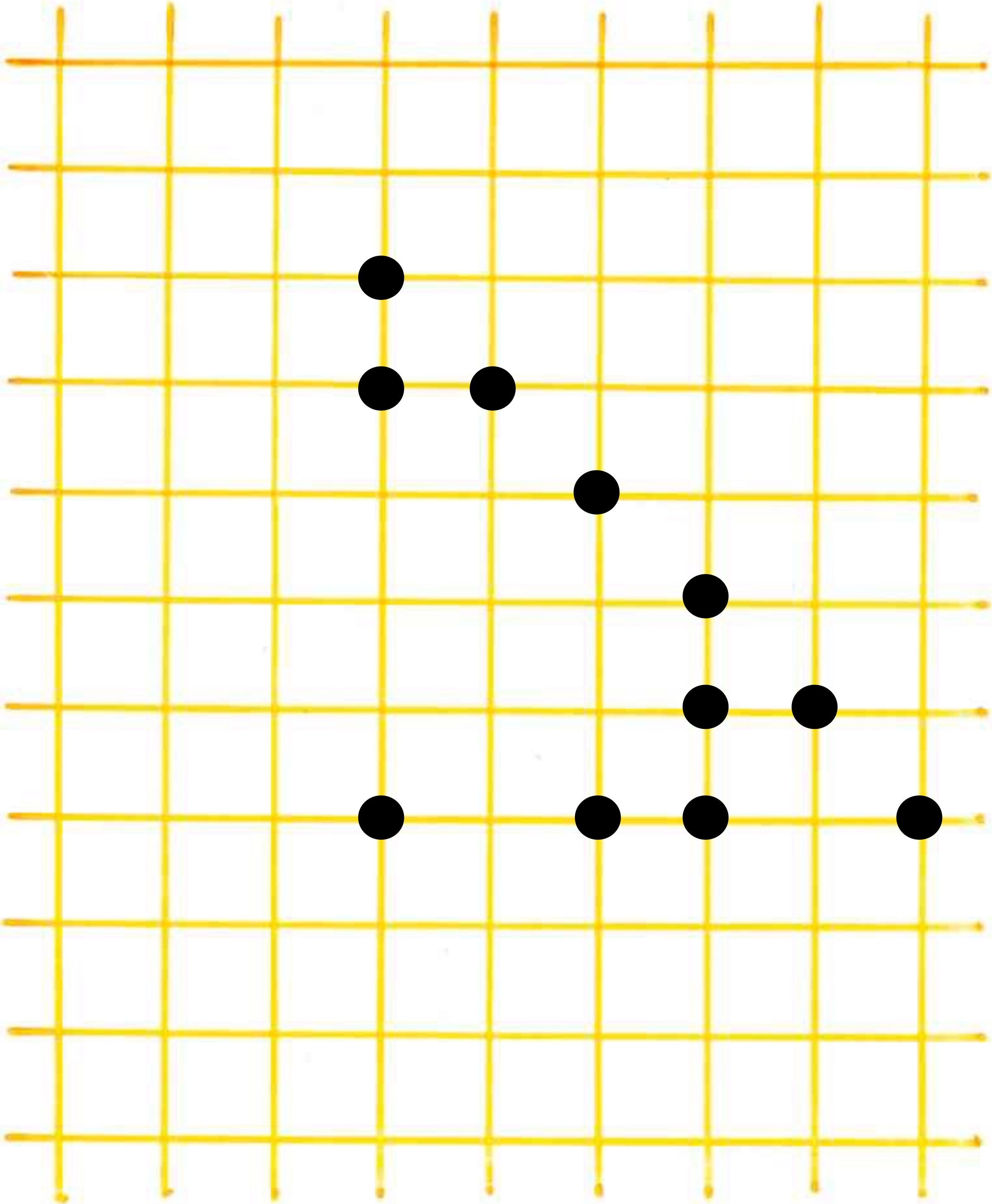


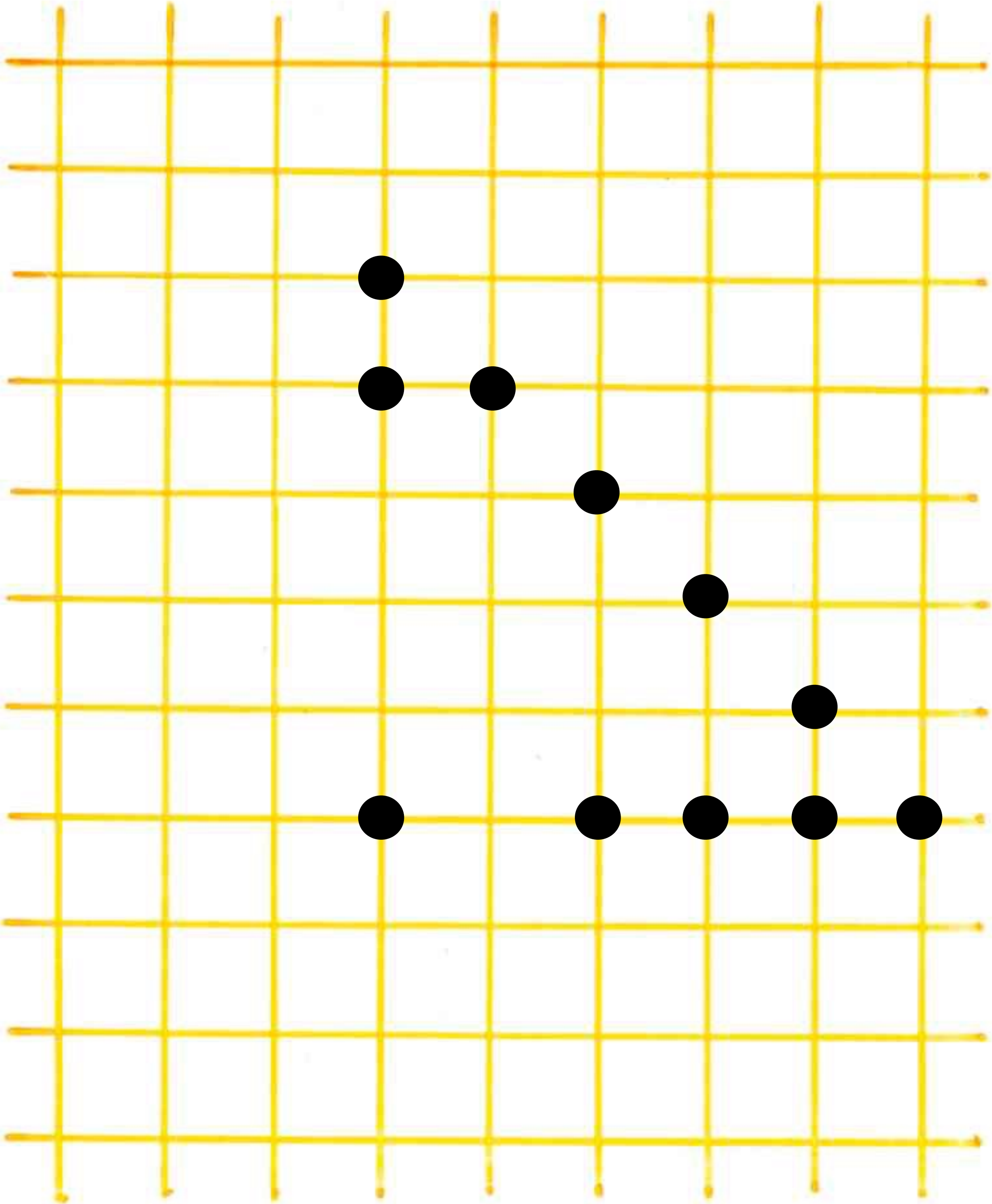


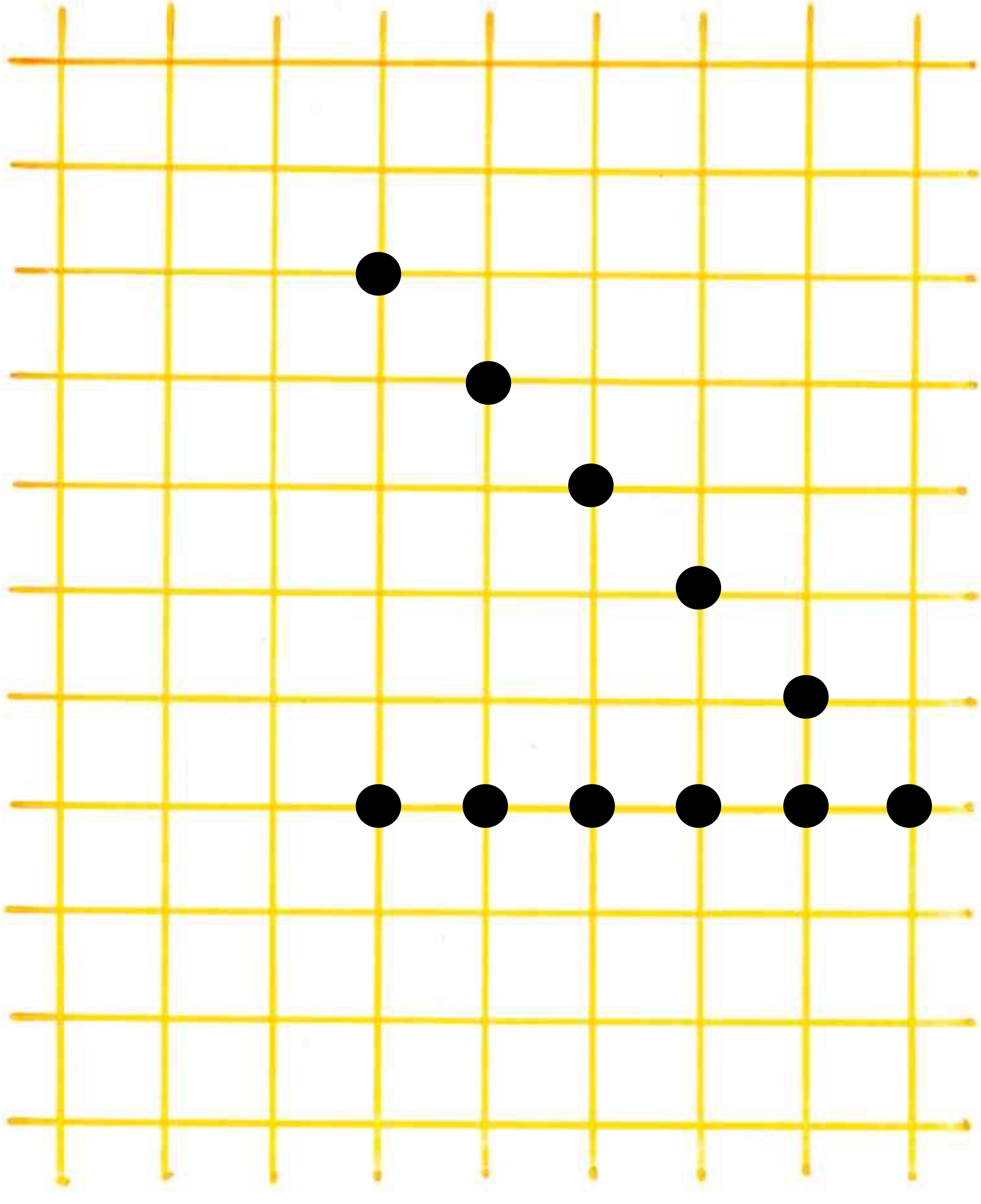




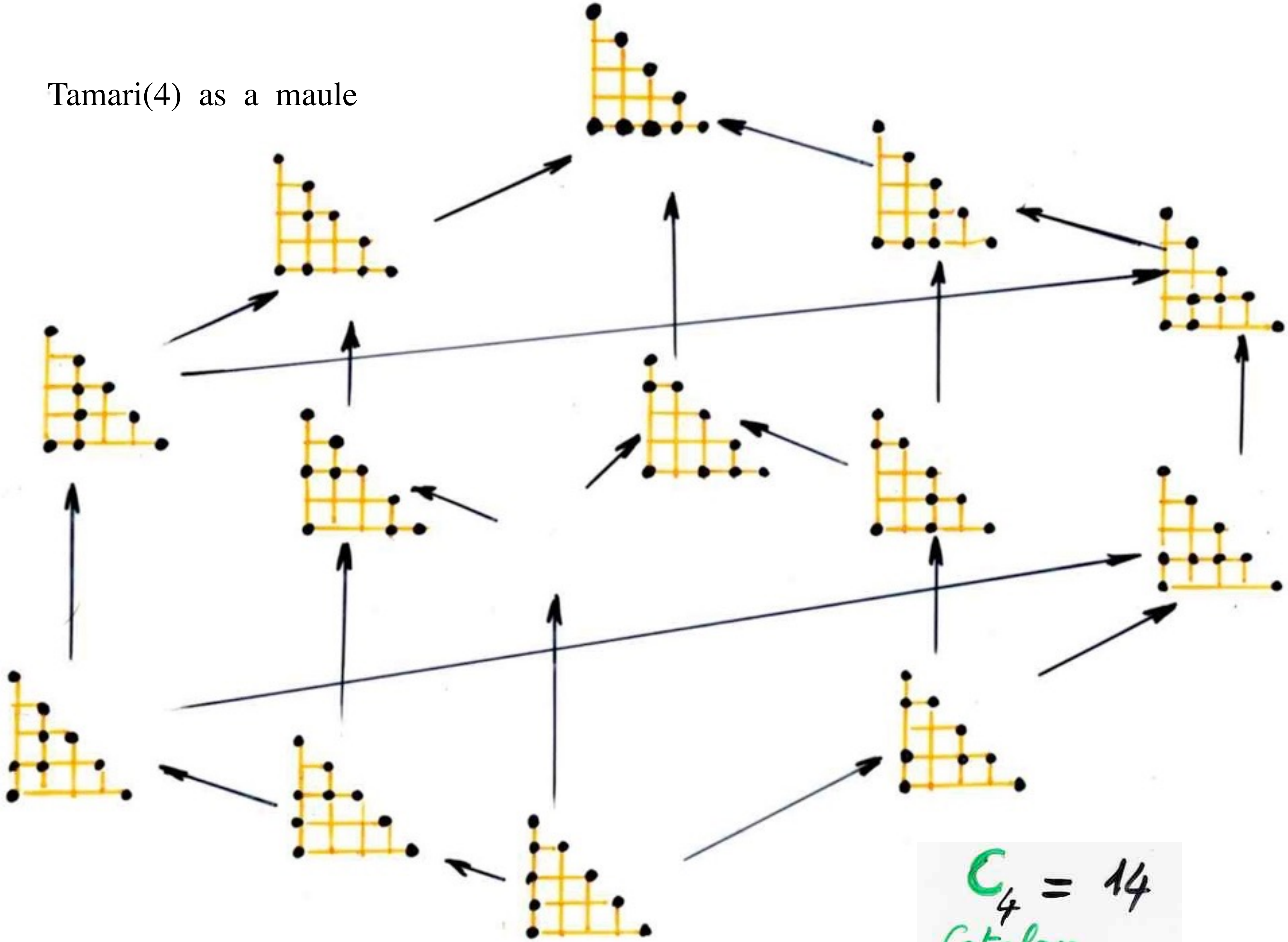








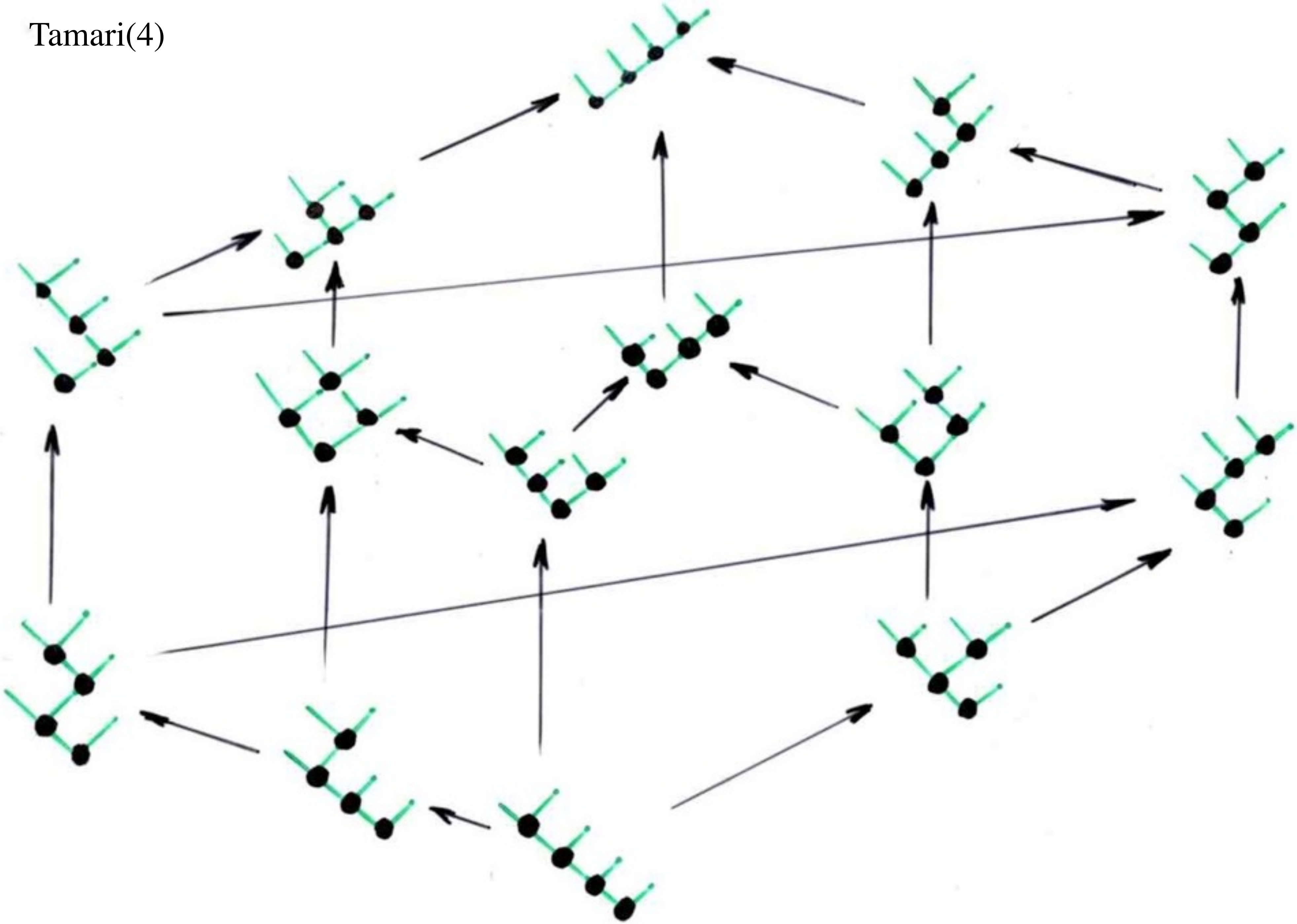
Tamari(4) as a maule



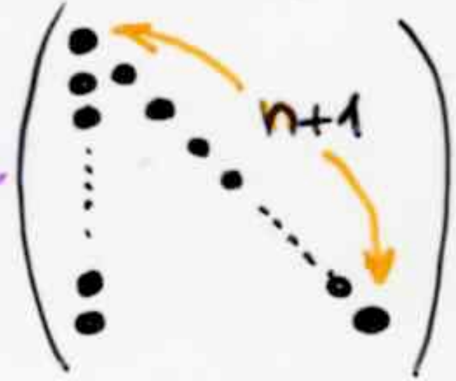
$C_4 = 14$

Catalan

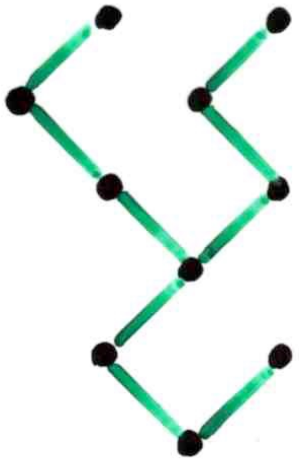
Tamari(4)

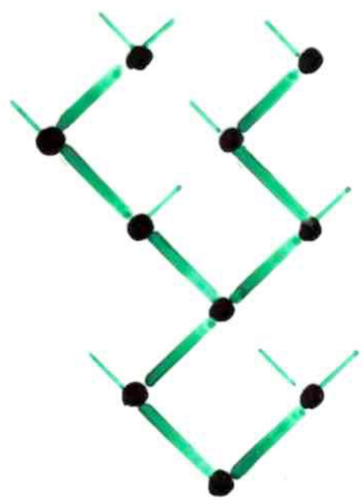


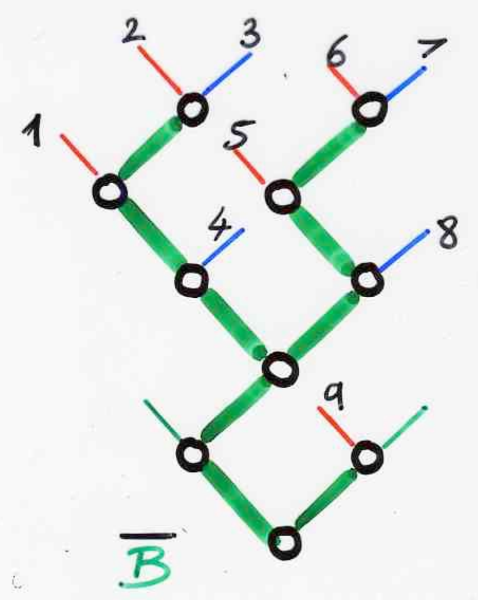
Proposition

Tamari(n) = Maule 

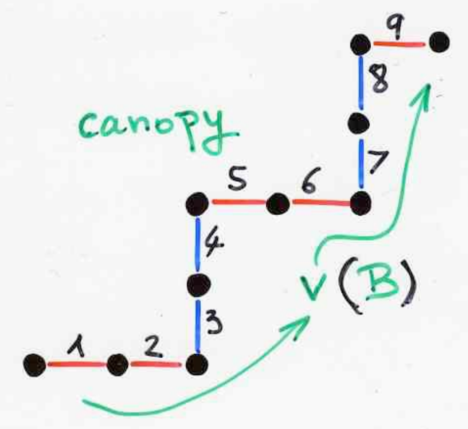
canopy of a binary tree



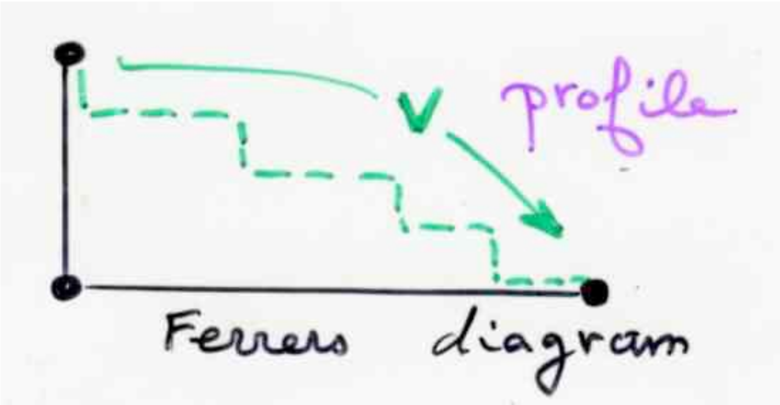


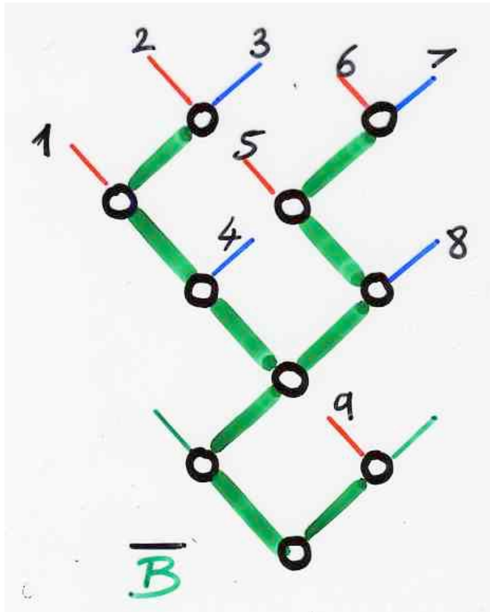


Loday, Ronco (1998) (2012)

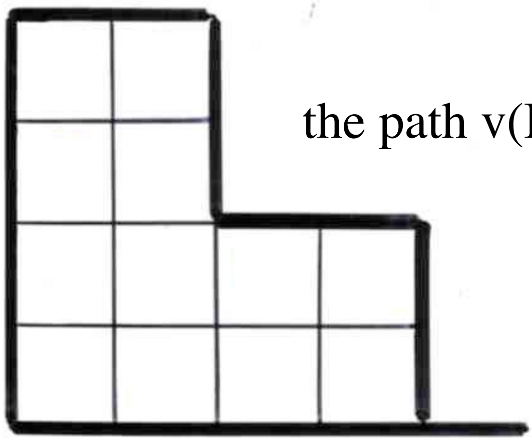


The external edges (except the first and last) of the extended binary tree are ordered from left to right (symmetric order). According to the fact the edge is left (red) or right (blue), this gives a word of length (n-1) in 2 letters, which can be seen as a path with elementary steps East (red) and North (blue).



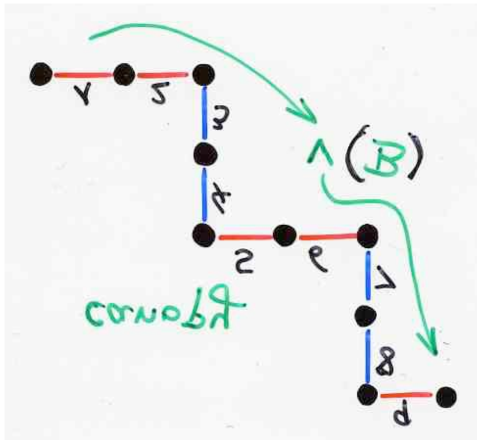


With the french notation for Ferrers diagrams, we will need to see the canopy as a path v(B) with elementary steps East and South, which define a Ferrers diagram F(B) (with possibly empty row or column). The path v, called the **profile** of F(B) is its North-East border.



the path v(B)

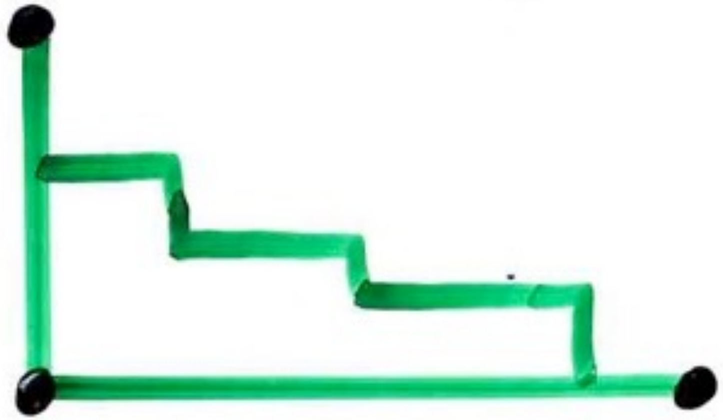
F(B)



alternative tableaux

alternative tableau

- Ferrers diagram F

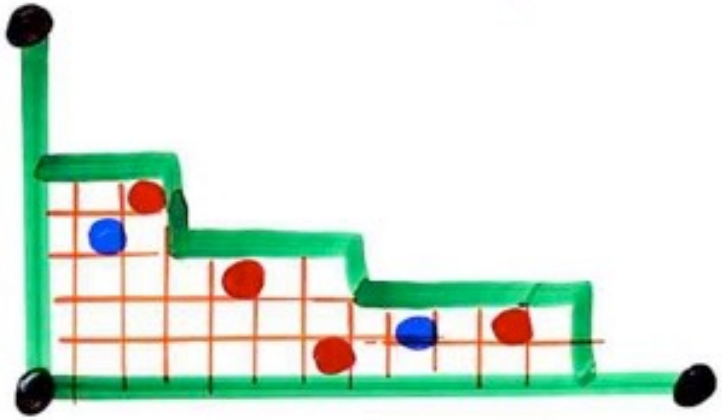


(possibly empty rows or column)

(nb of rows) + (nb of columns) $= n$

alternative tableau

– Ferrers diagram F



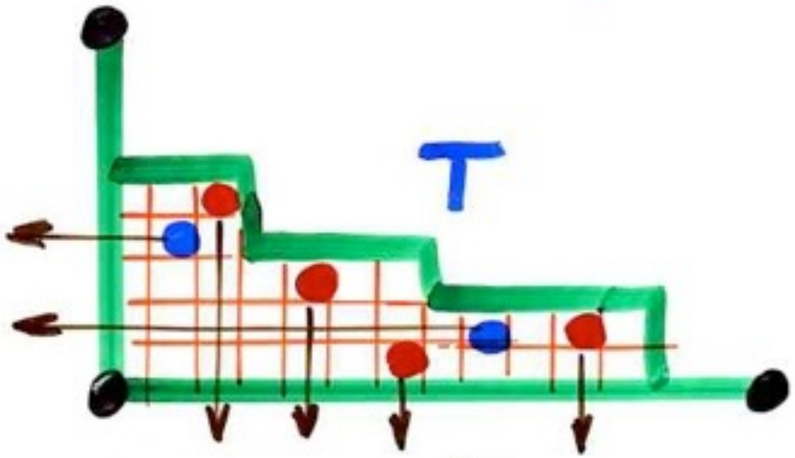
(possibly empty rows or column)

(nb of rows) + (nb of columns) $= n$

– some cells are coloured **red** or **blue**

alternative tableau T

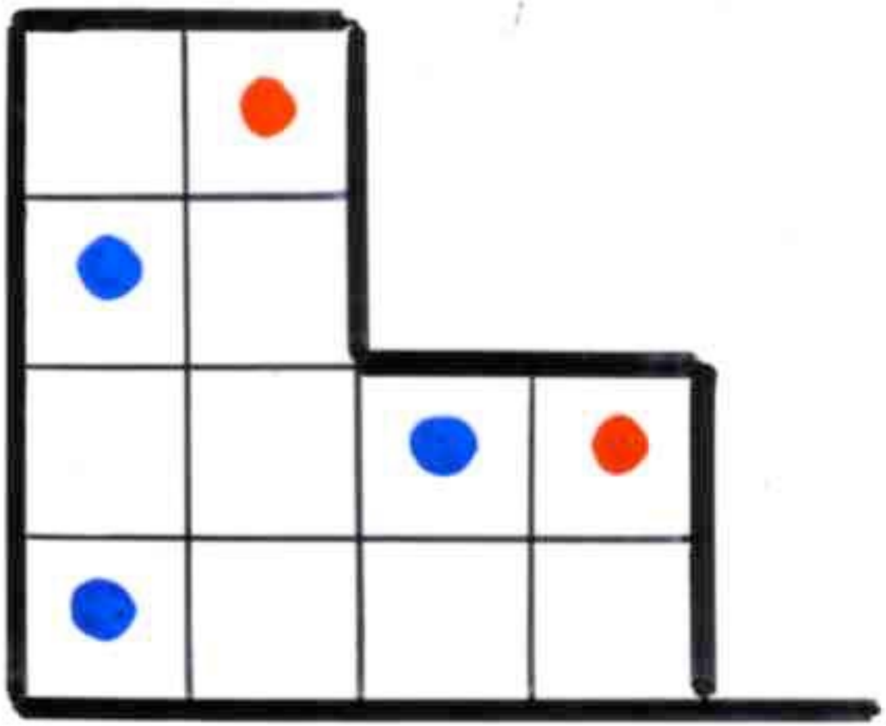
- Ferrers diagram F (possibly empty rows or column)

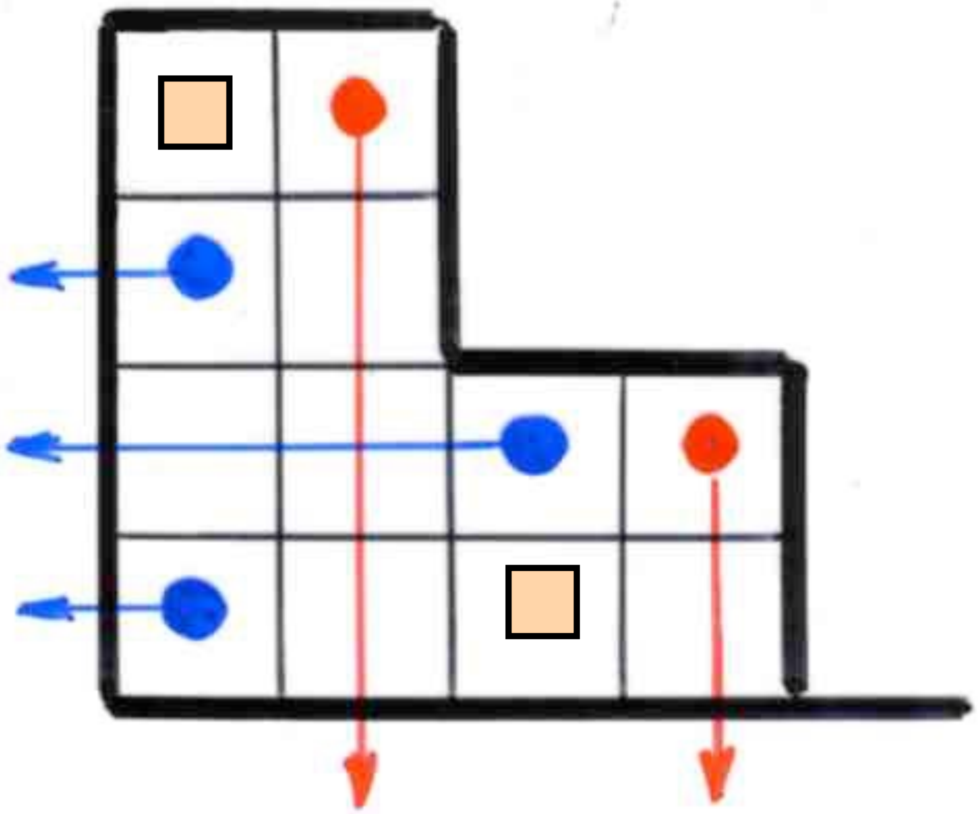


(nb of rows) + (nb of columns) $= n$

- some cells are coloured red or blue
- $\left\{ \begin{array}{l} \text{no coloured cell at the left of} \\ \text{no coloured cell below} \end{array} \right.$

n size of T

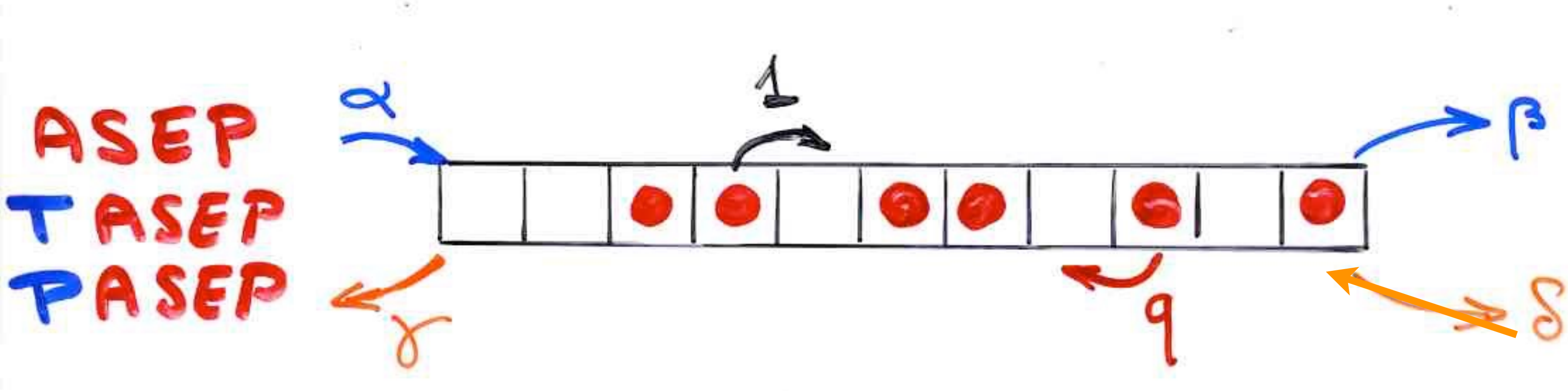




Prop. The number of alternative tableaux of size n is

$$(n+1)!$$

The general PASEP model in physics with its 5 parameters.
(partially asymmetric exclusion model)



There is at most one particle per cell. Particles are moving one step forward (with probability one) and backward with probability q. The parameters α, β, γ, δ are probabilities for a particle to get in or out of the strip.

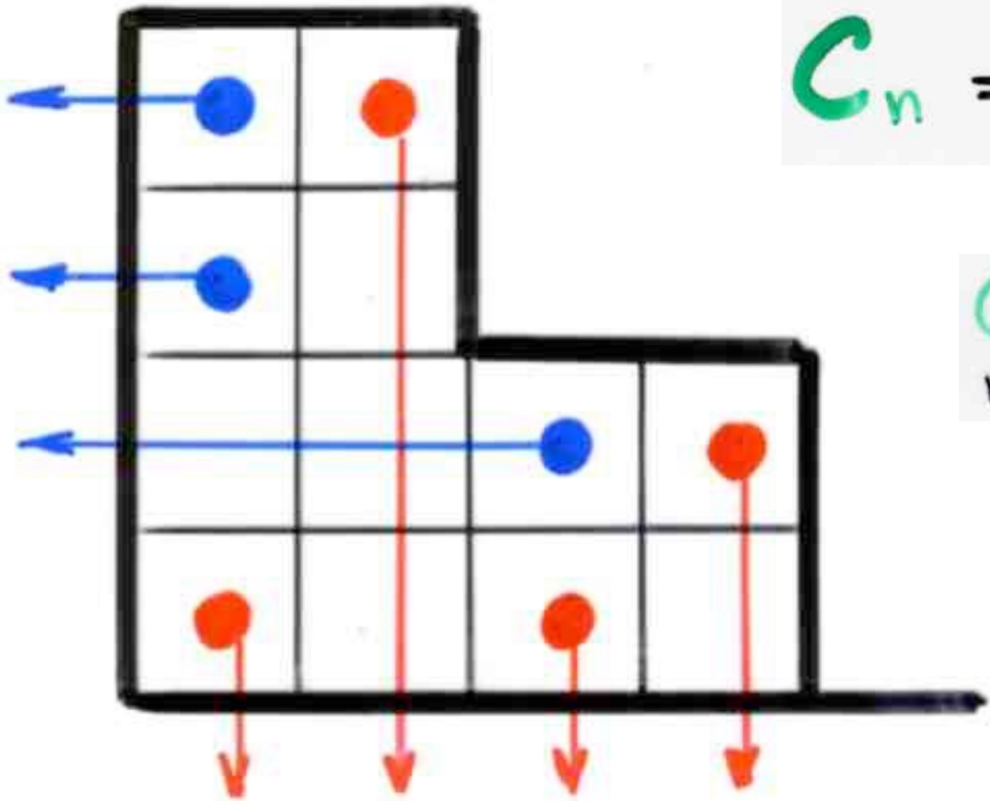
Alternating tableaux give an interpretation of the stationary probabilities for the PASEP model with 3 parameters α, β and q. Catalan alternative tableaux correspond to the TASEP (totally asymmetric exclusion model) where q=0.

Catalan alternative tableaux

Def Catalan alternative tableau T

alt. tab. without cells $\square$

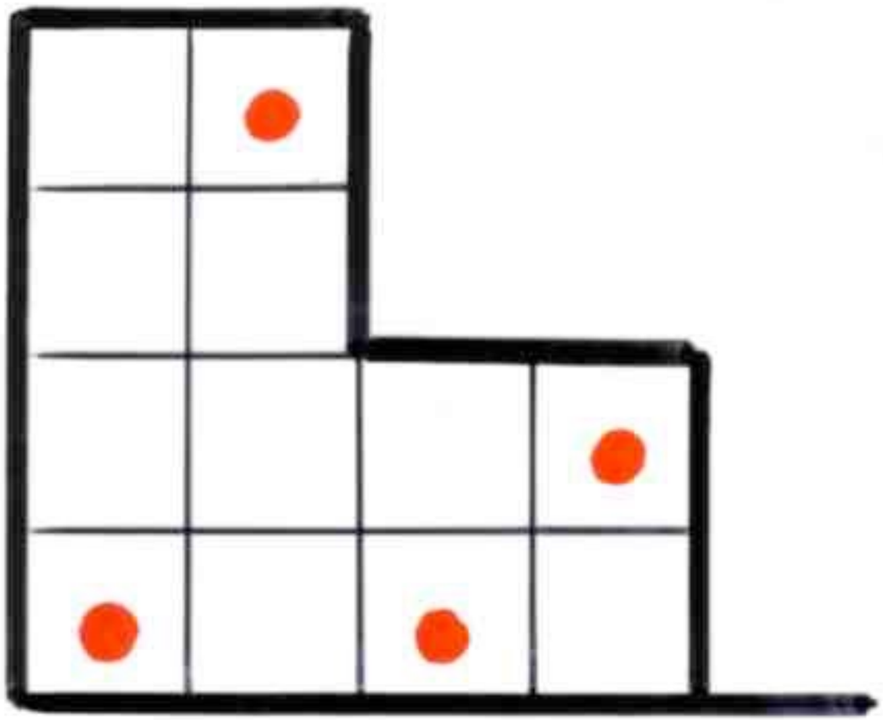
i.e: every empty cell is below a red cell or on the left of a blue cell



$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$

Catalan numbers

Characterisation of alternative Catalan tableaux

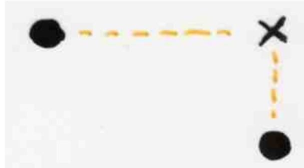


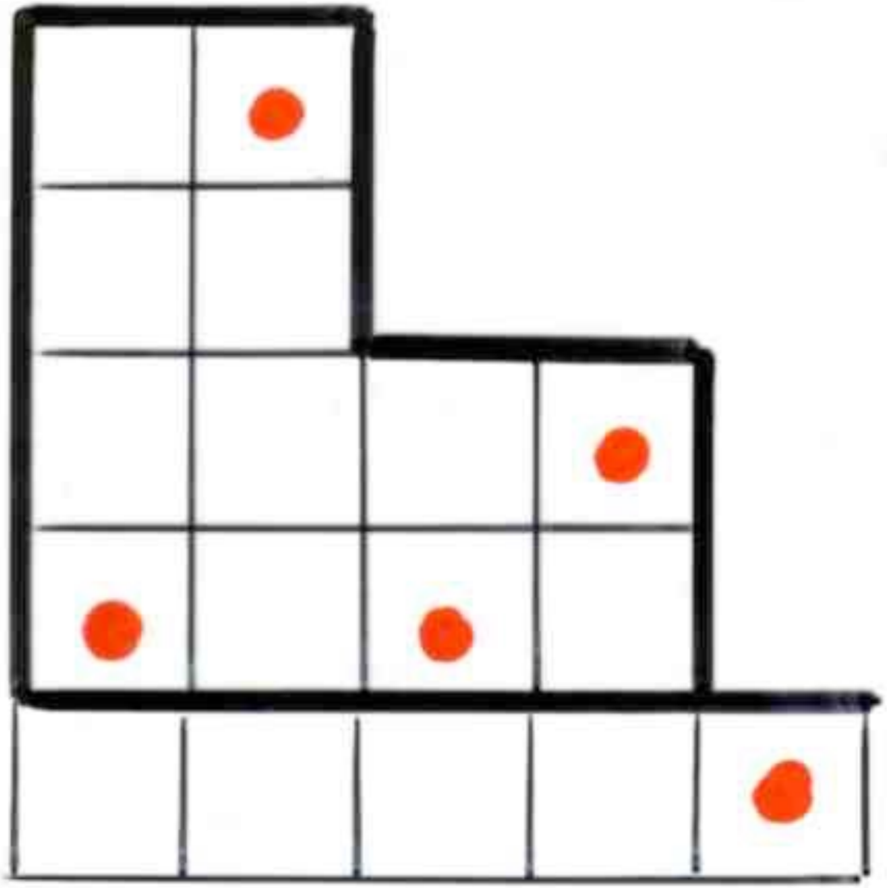
taking only the red points of a Catalan alternative tableau

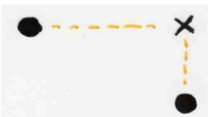
one can reconstruct the original tableau from the knowledge of the red part

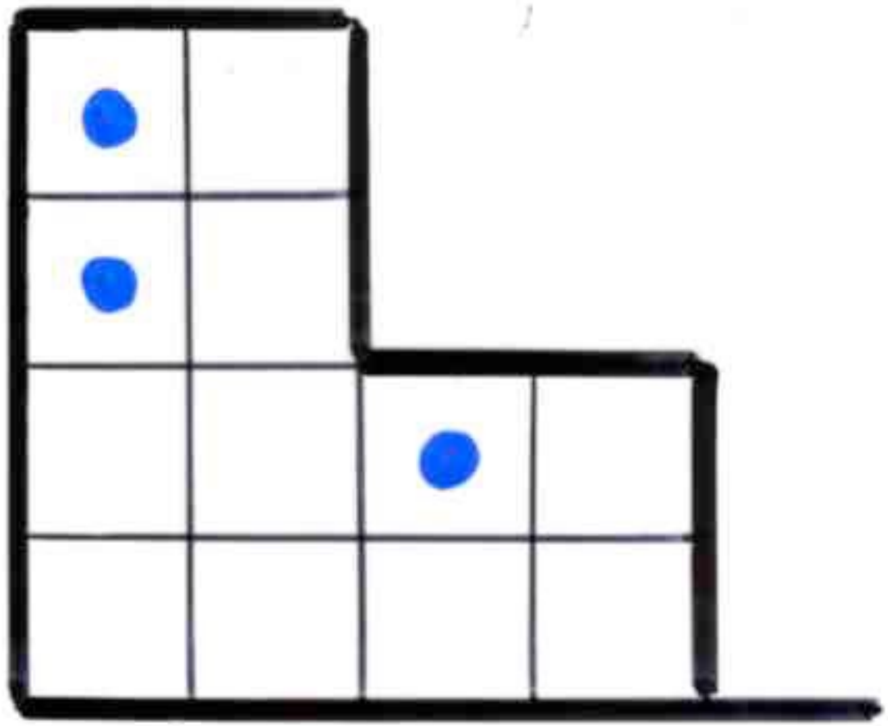
the augmented red part to the Catalan alternative tableau:
adding a red point in the new first row for each empty column of the red tableau

the original tableau is a Catalan alternative tableau if and only if the pattern is forbidden

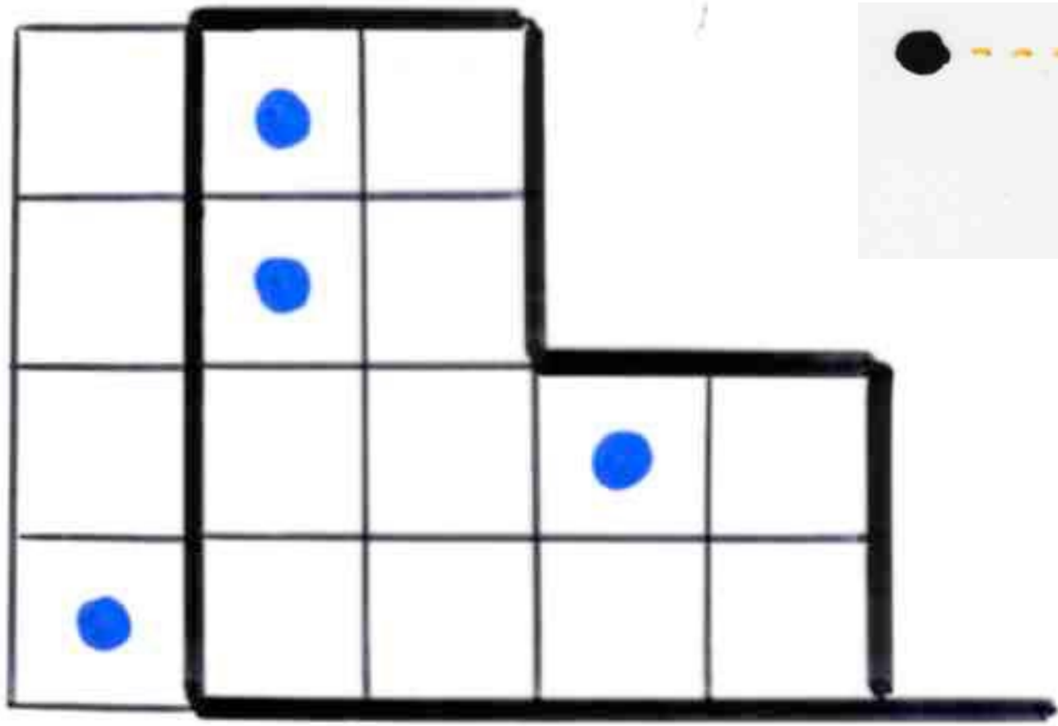


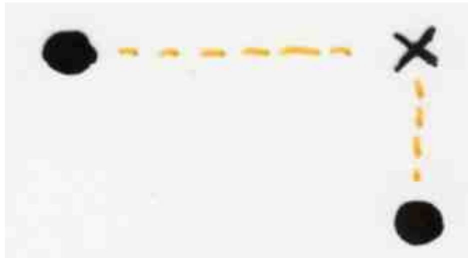


Such tableaux are the so-called « Catalan permutation tableaux », that is a tableau where the pattern  is forbidden and where in each column there is one and only one (red) point)

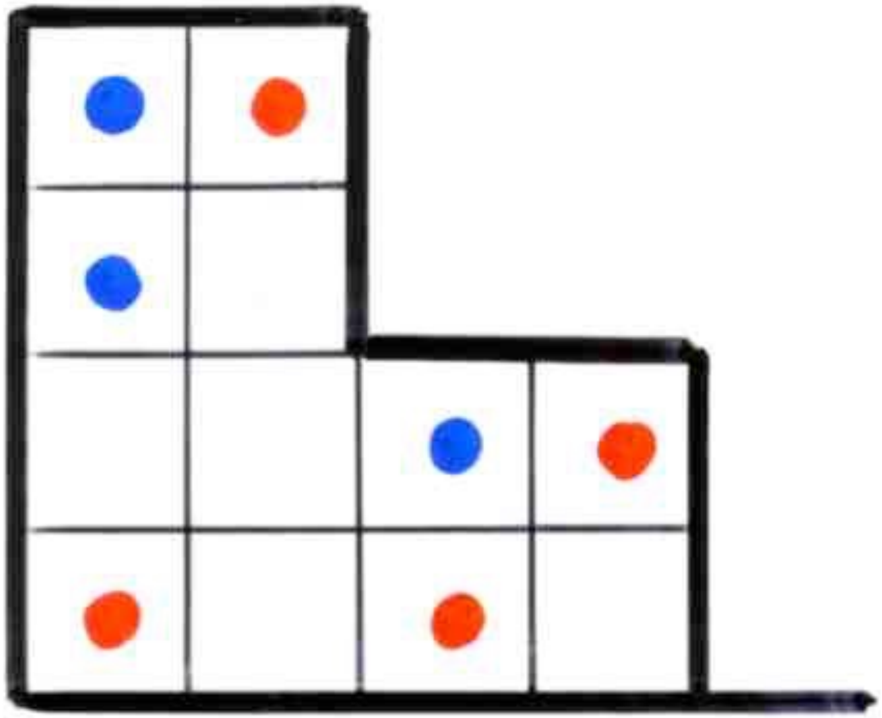


same with the blue points

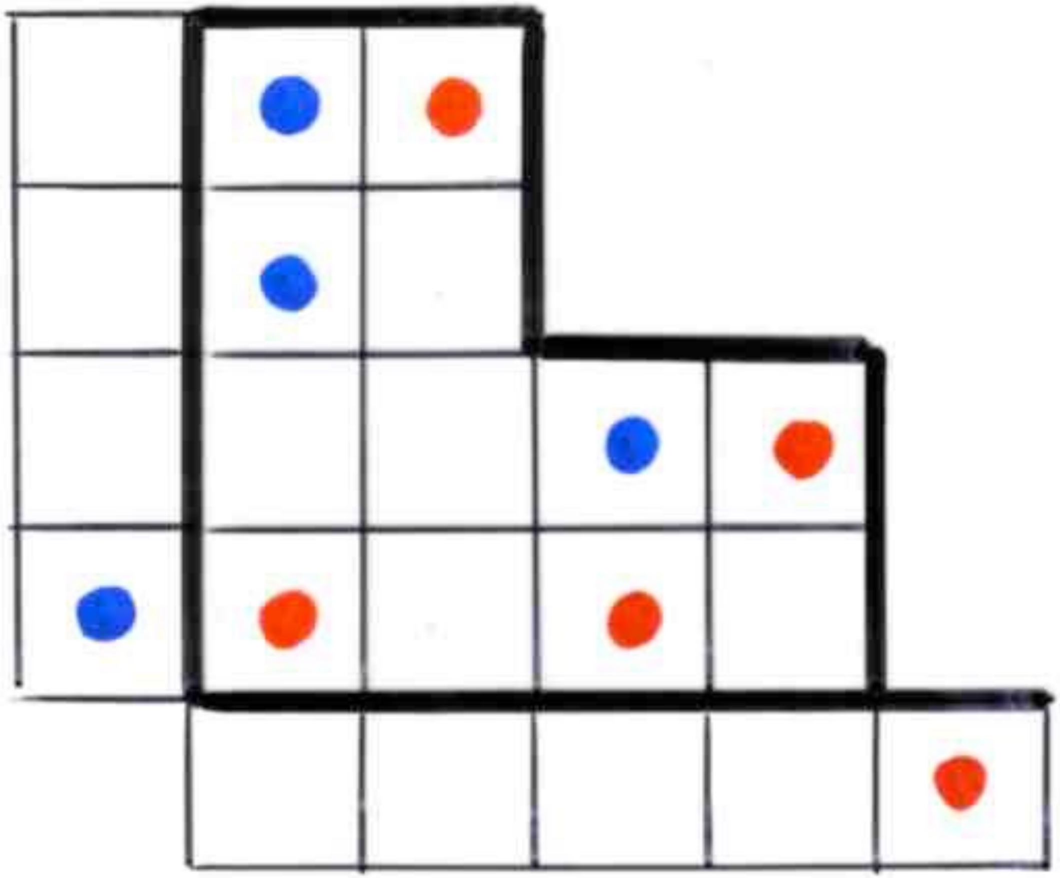




same with the blue points

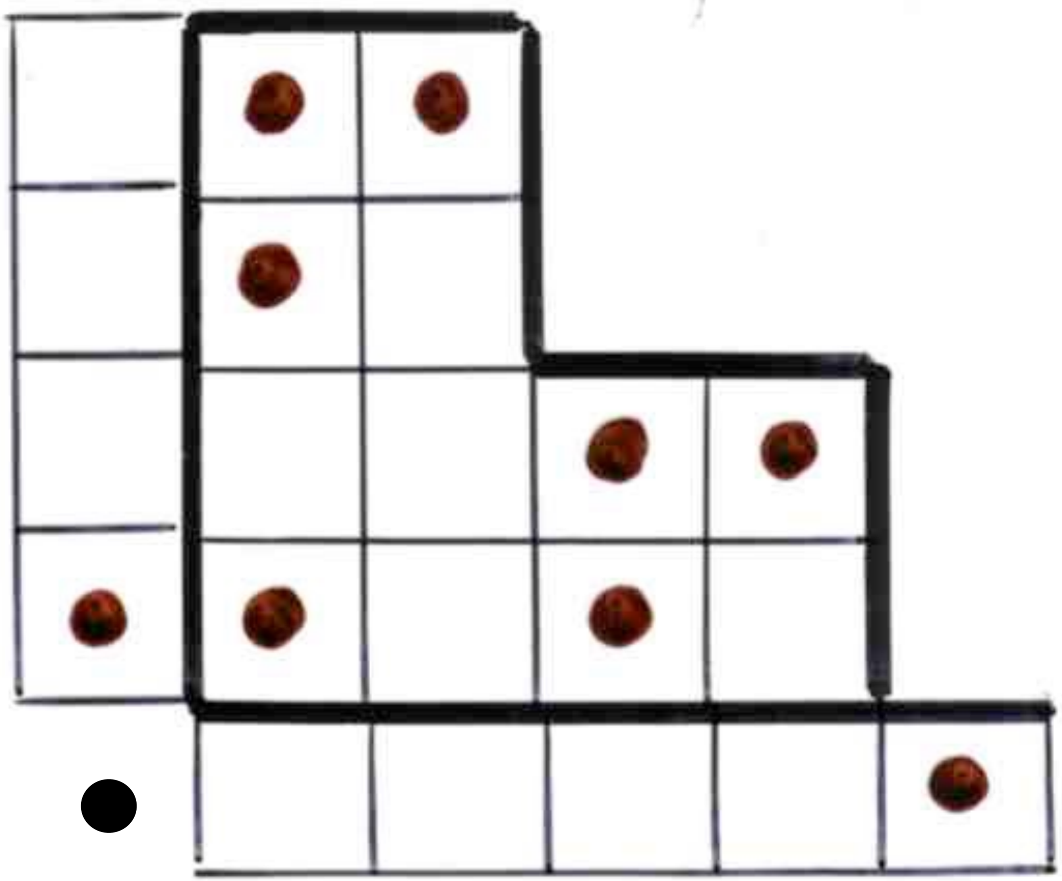


back to the original Catalan alternative tableau

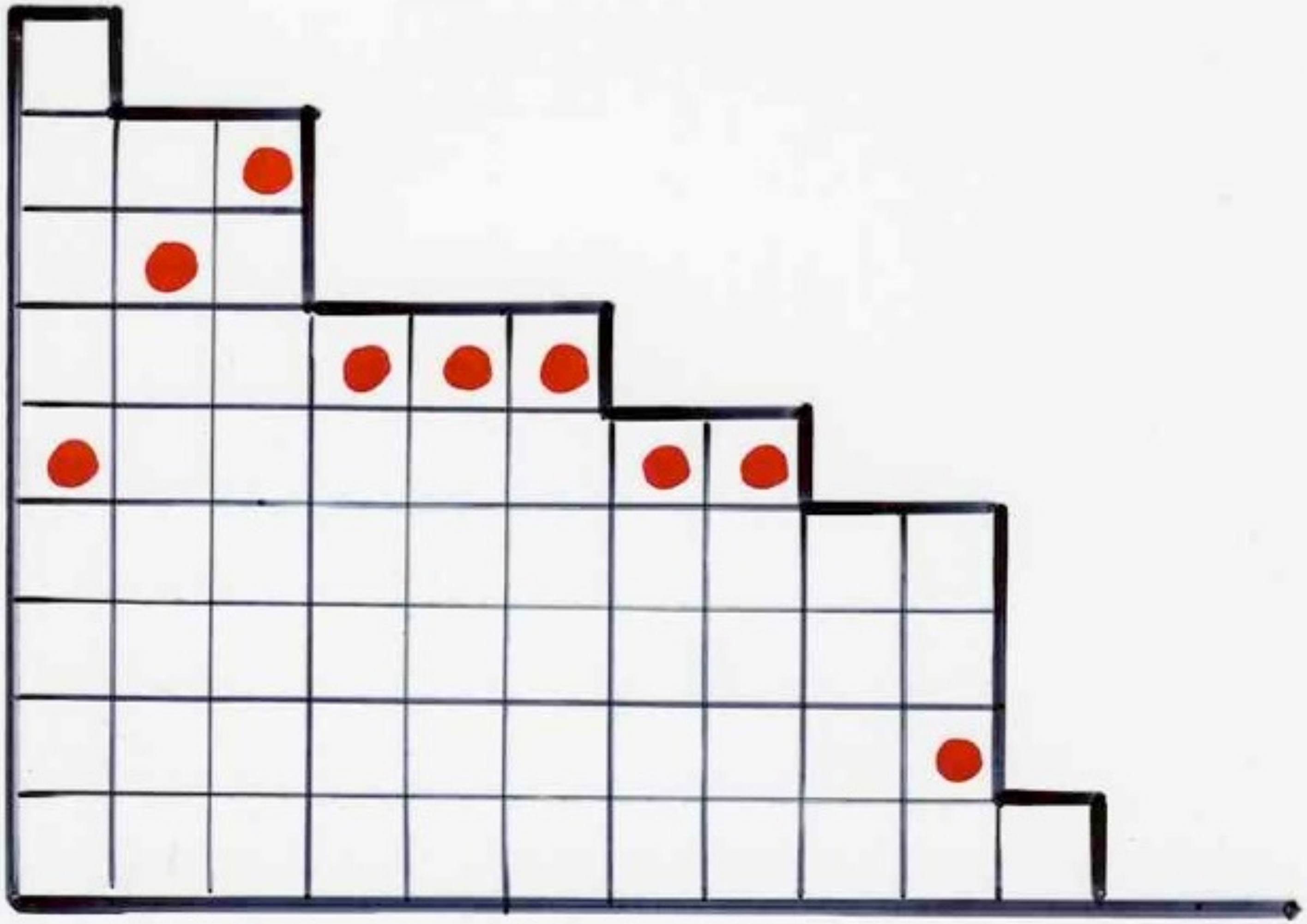


the augmented Catalan alternative tableau

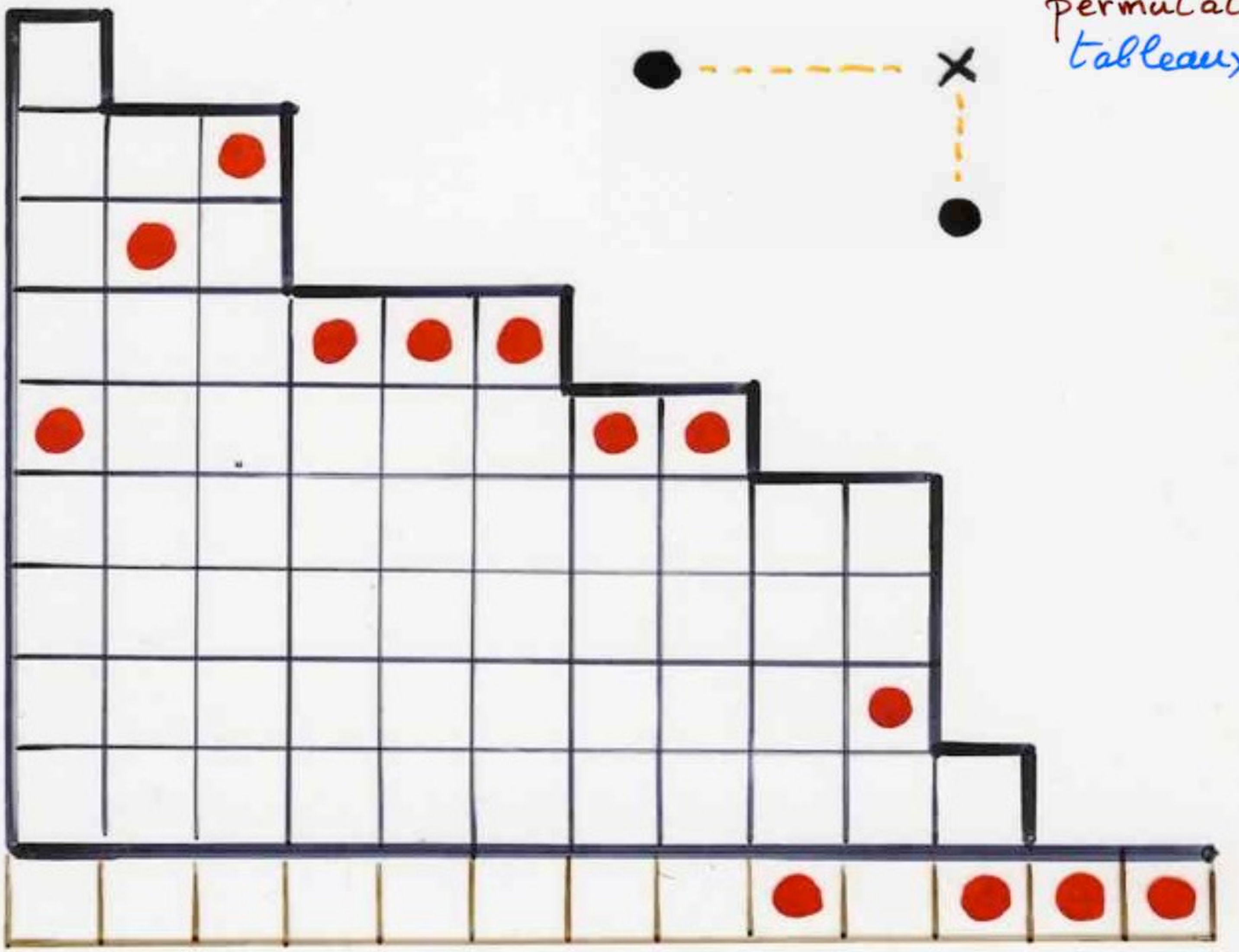
Catalan tree-like tableaux



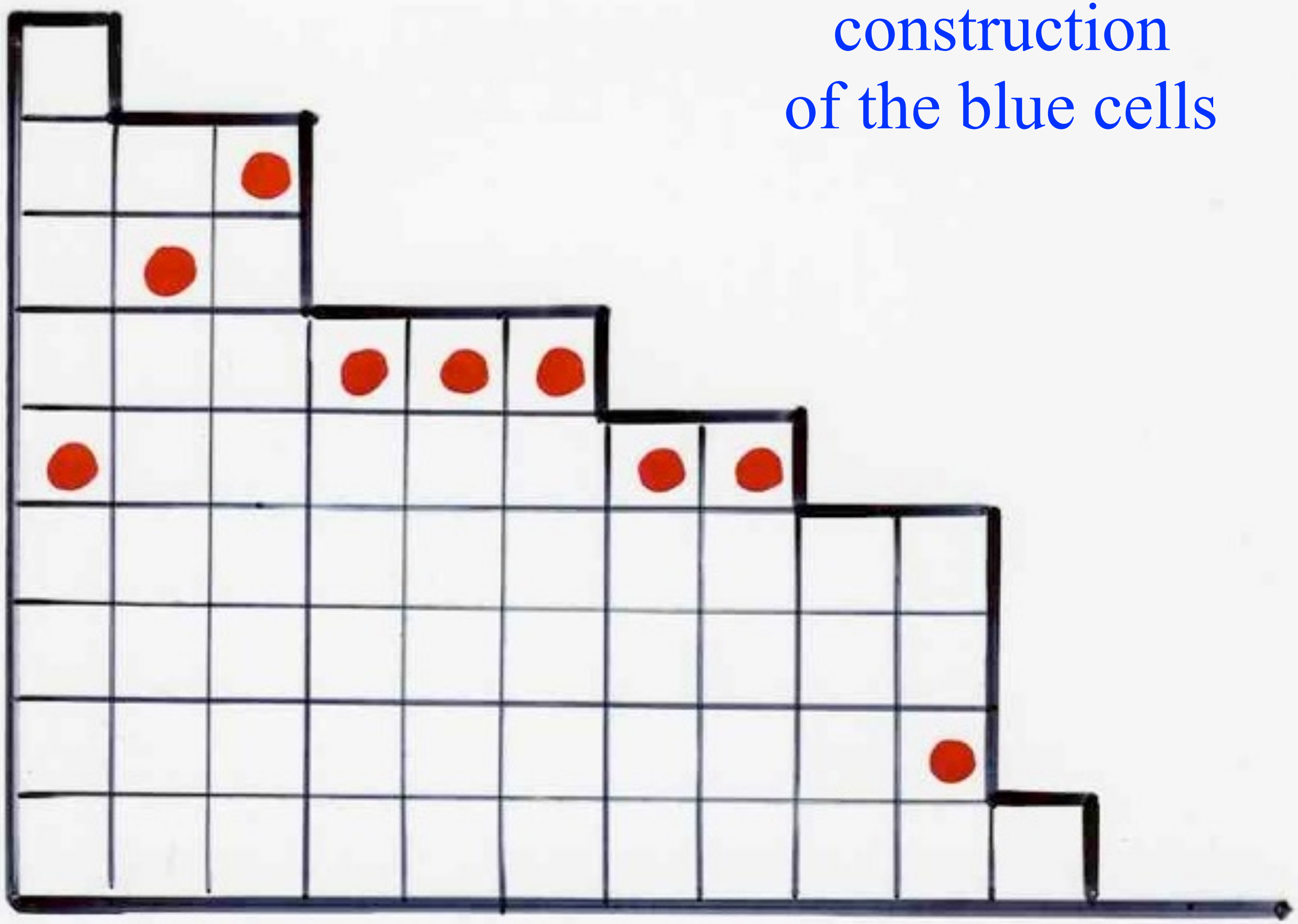
If one forgets the colors of the augmented Catalan alternative tableau, one can reconstruct the original tableau. Adding a point in the SW corner, one get a Catalan tree-like tableau. (see references in part II and slide 109, part II) example

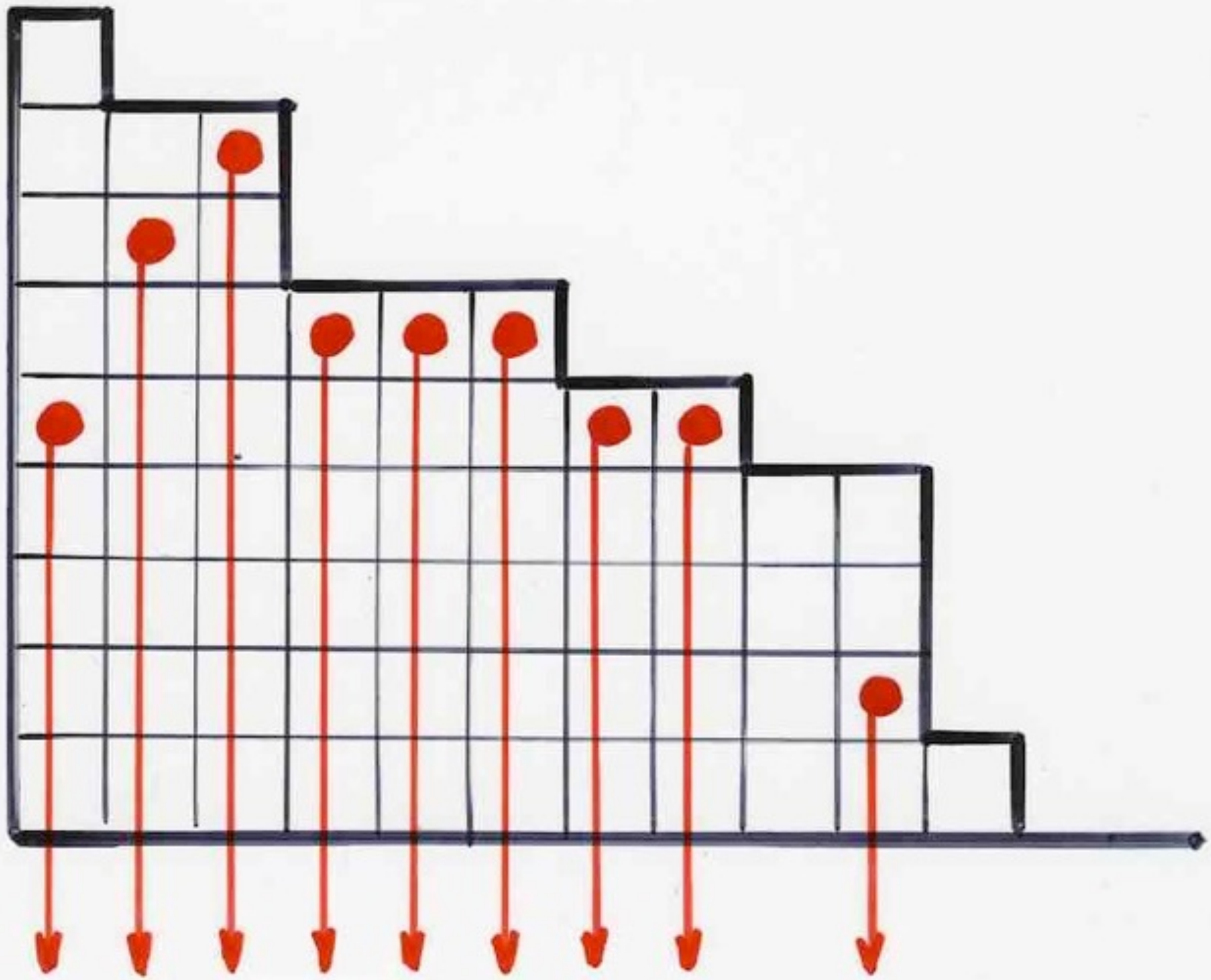


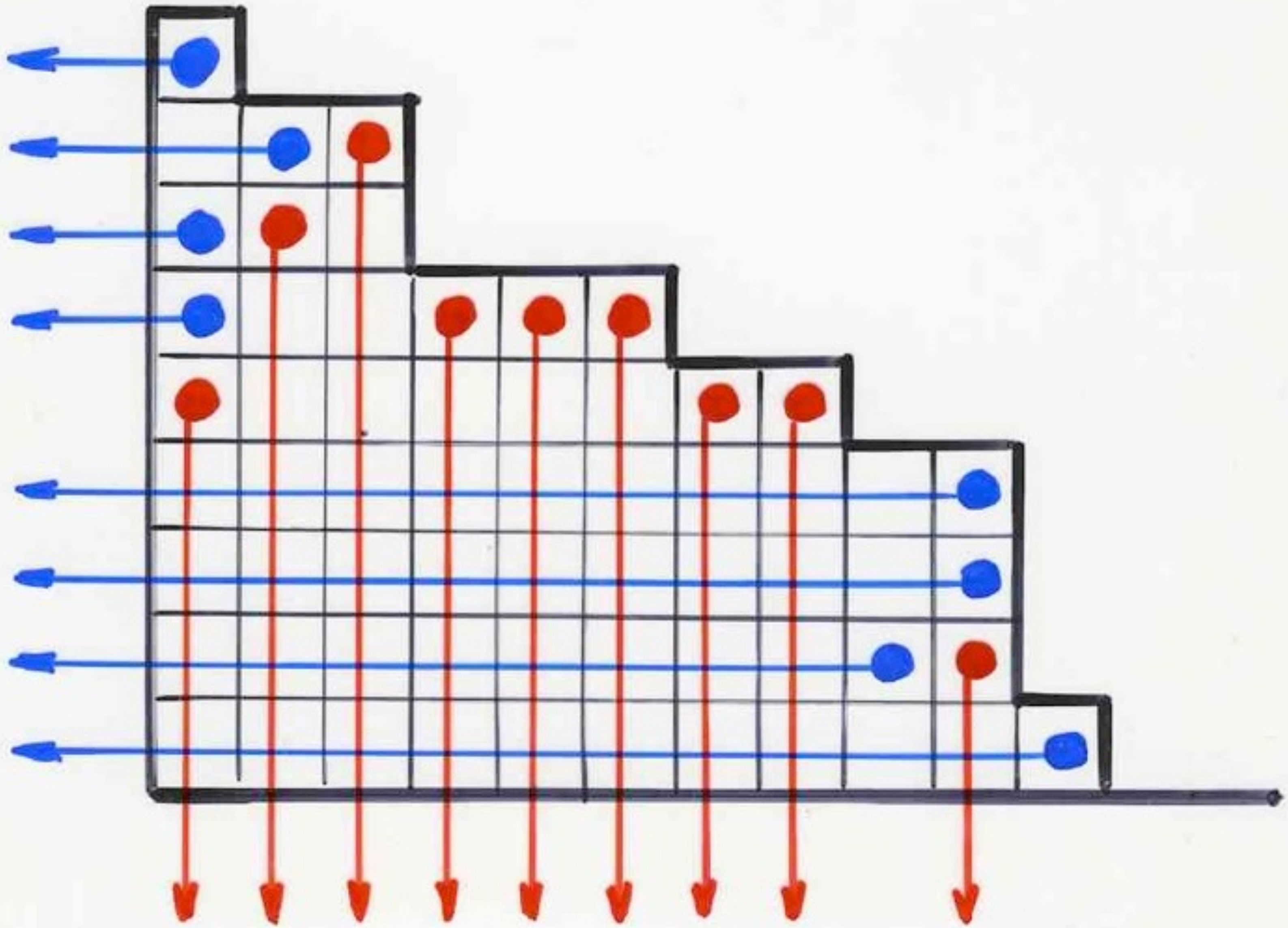
Catalan
permutation
tableaux

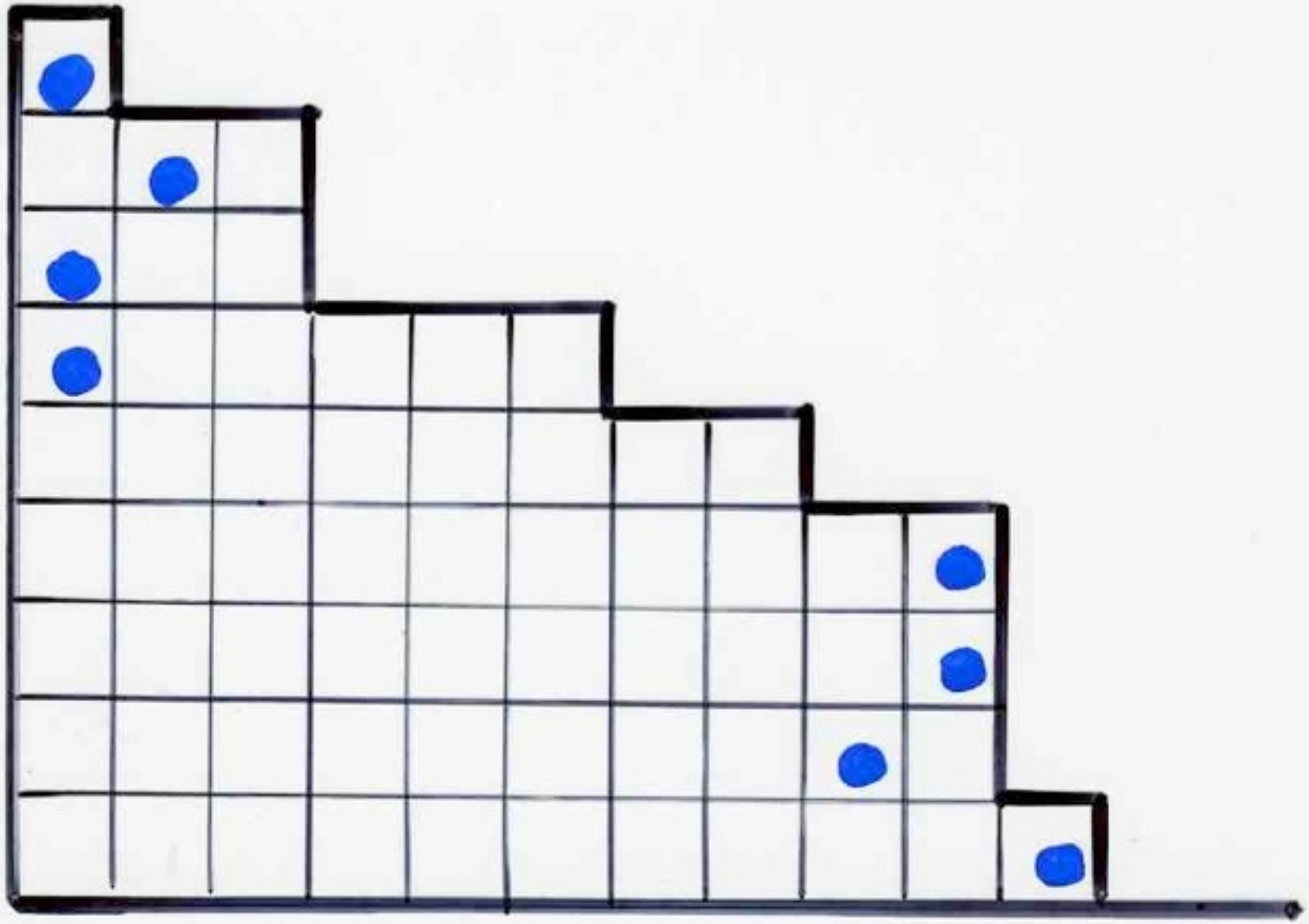


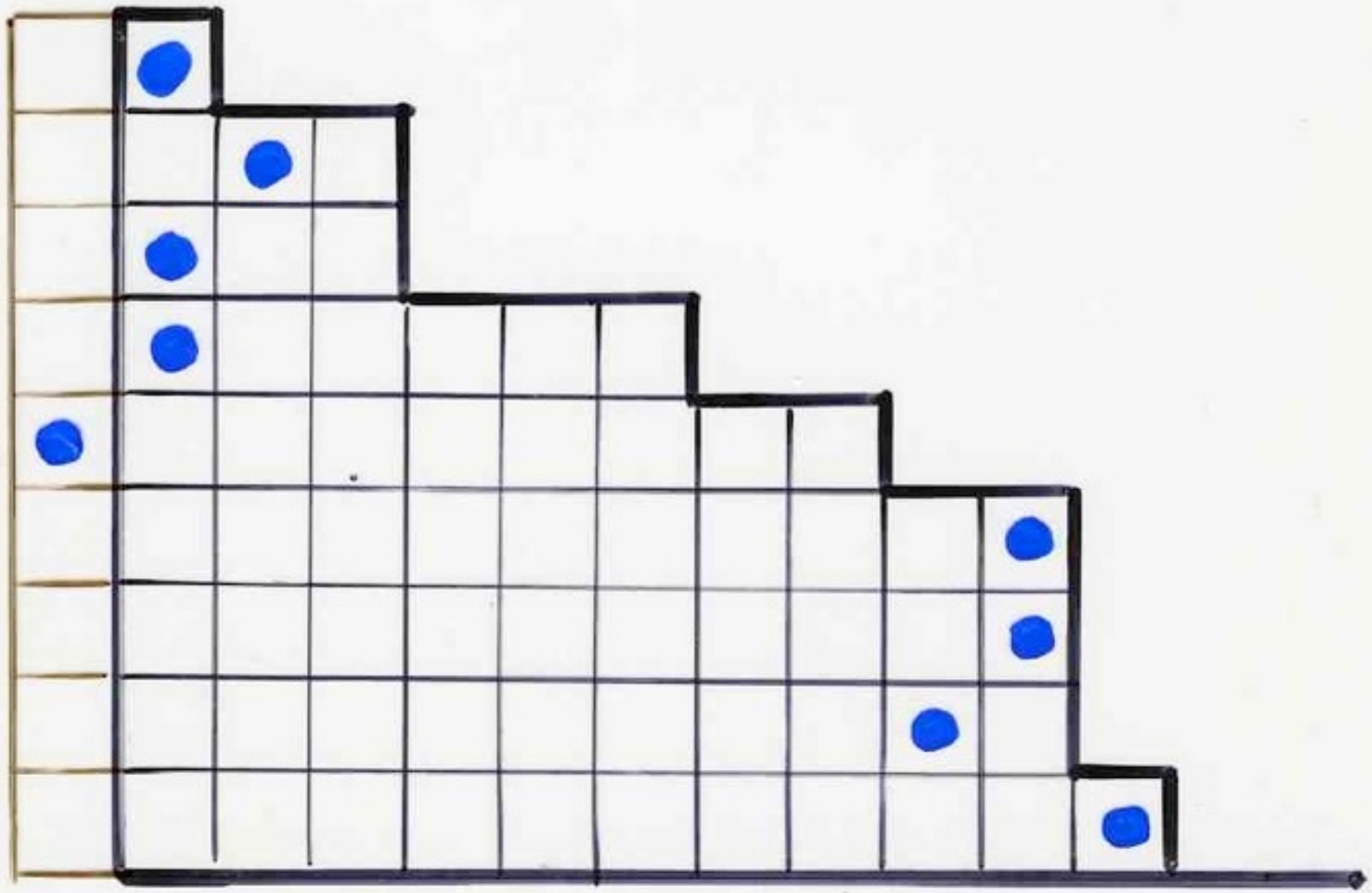
construction
of the blue cells

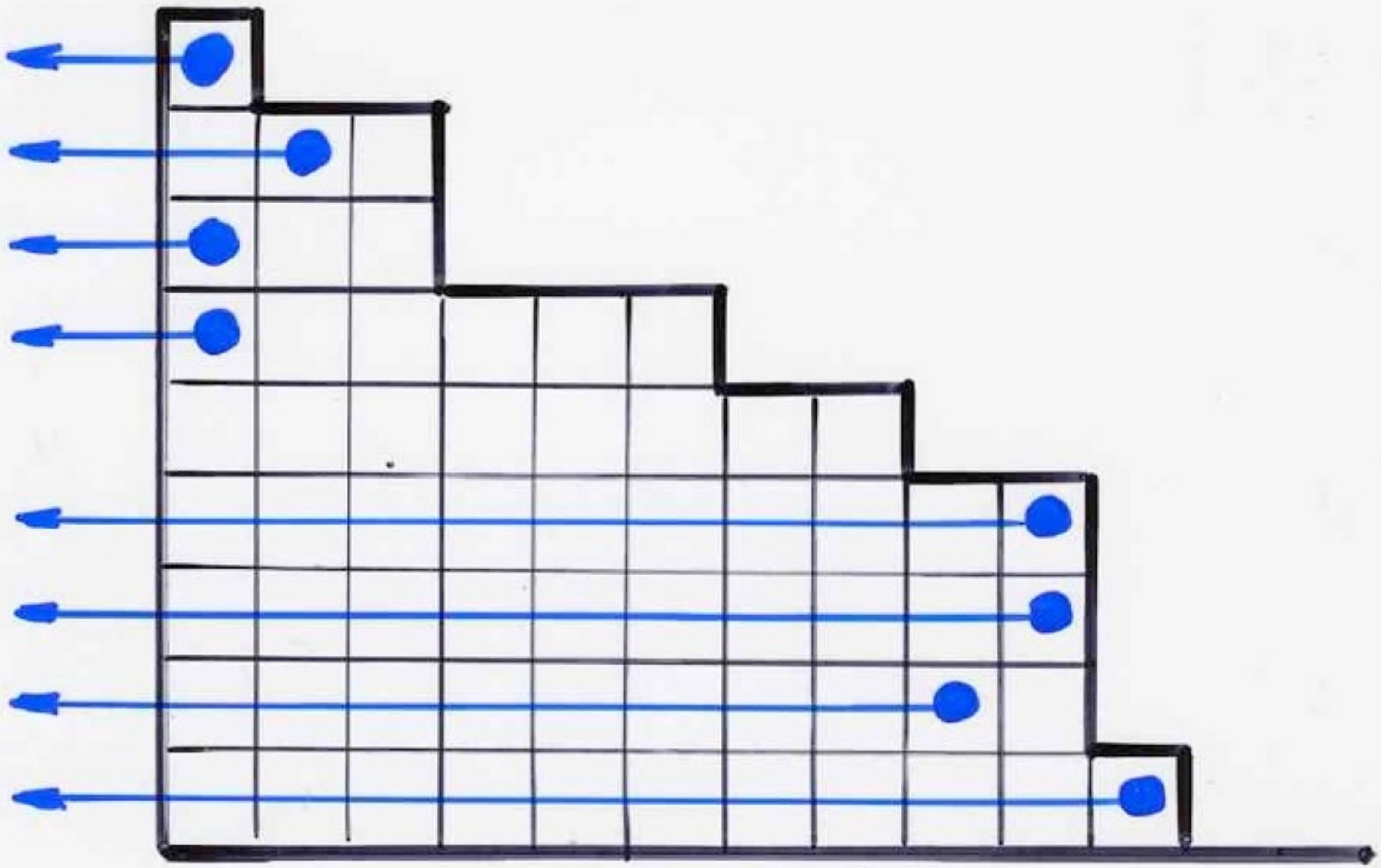




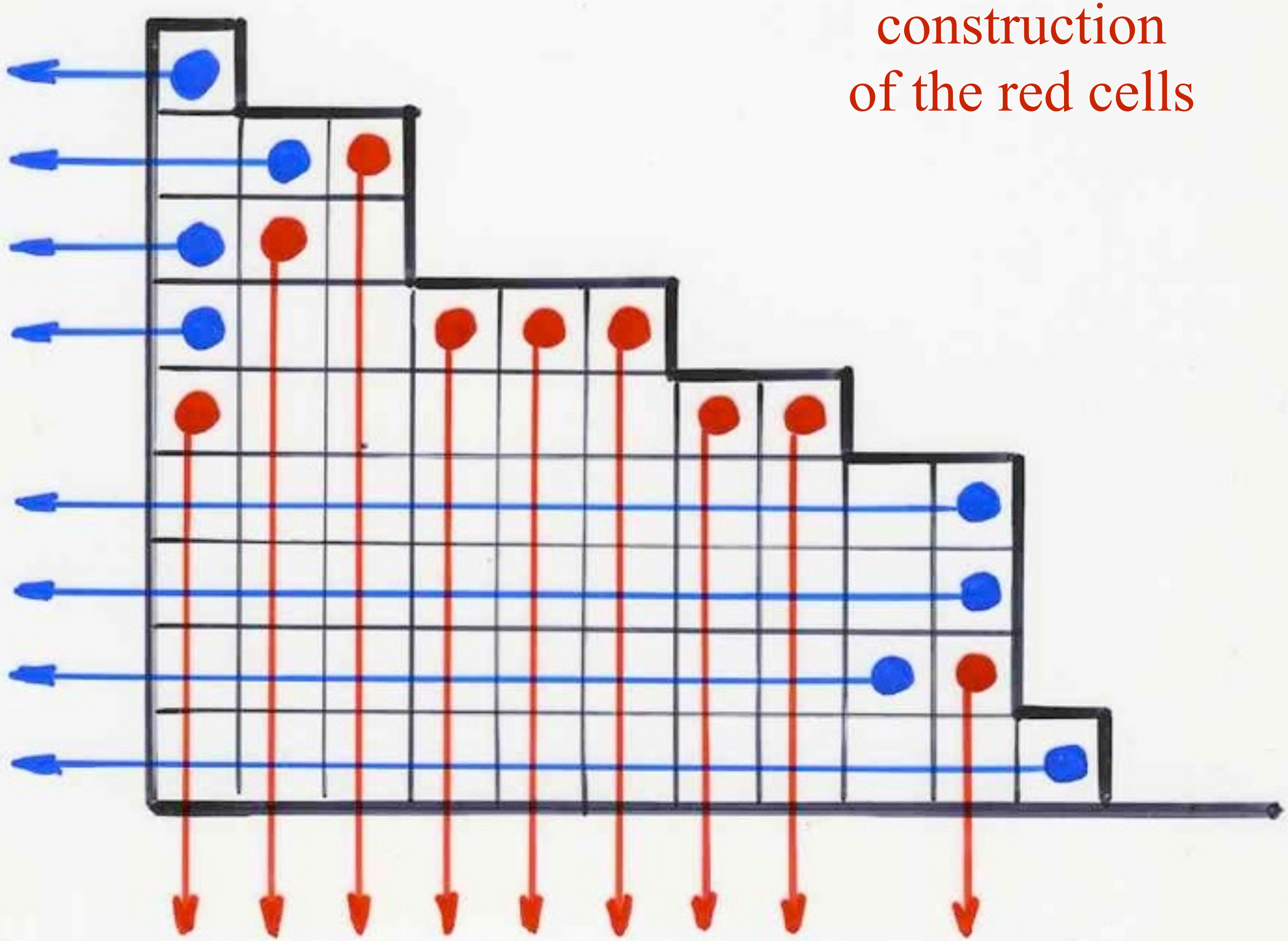








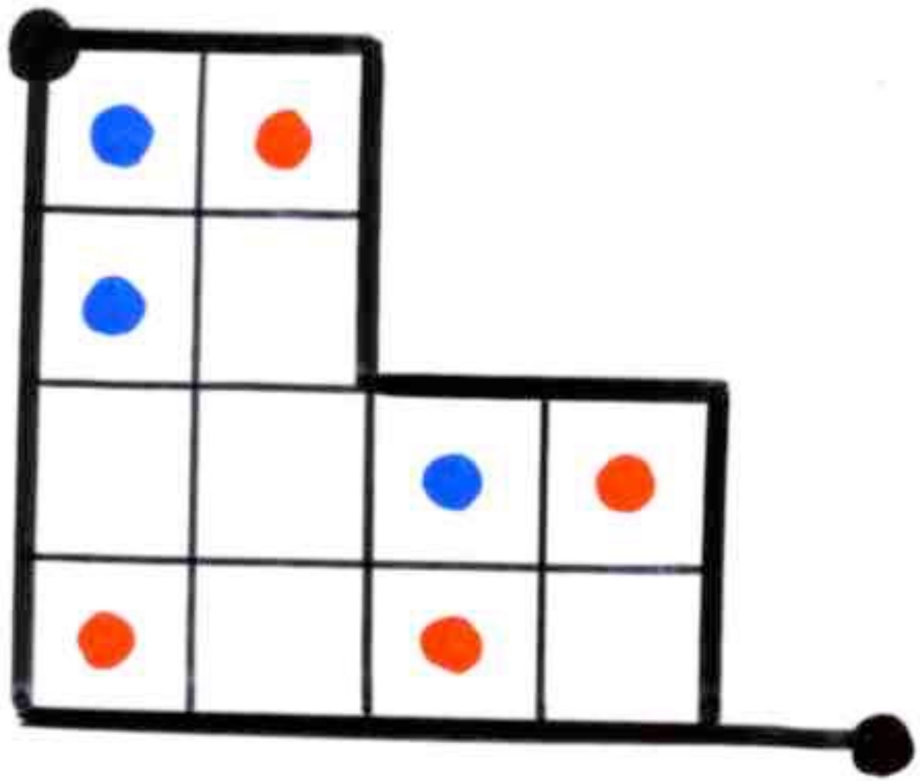
construction
of the red cells



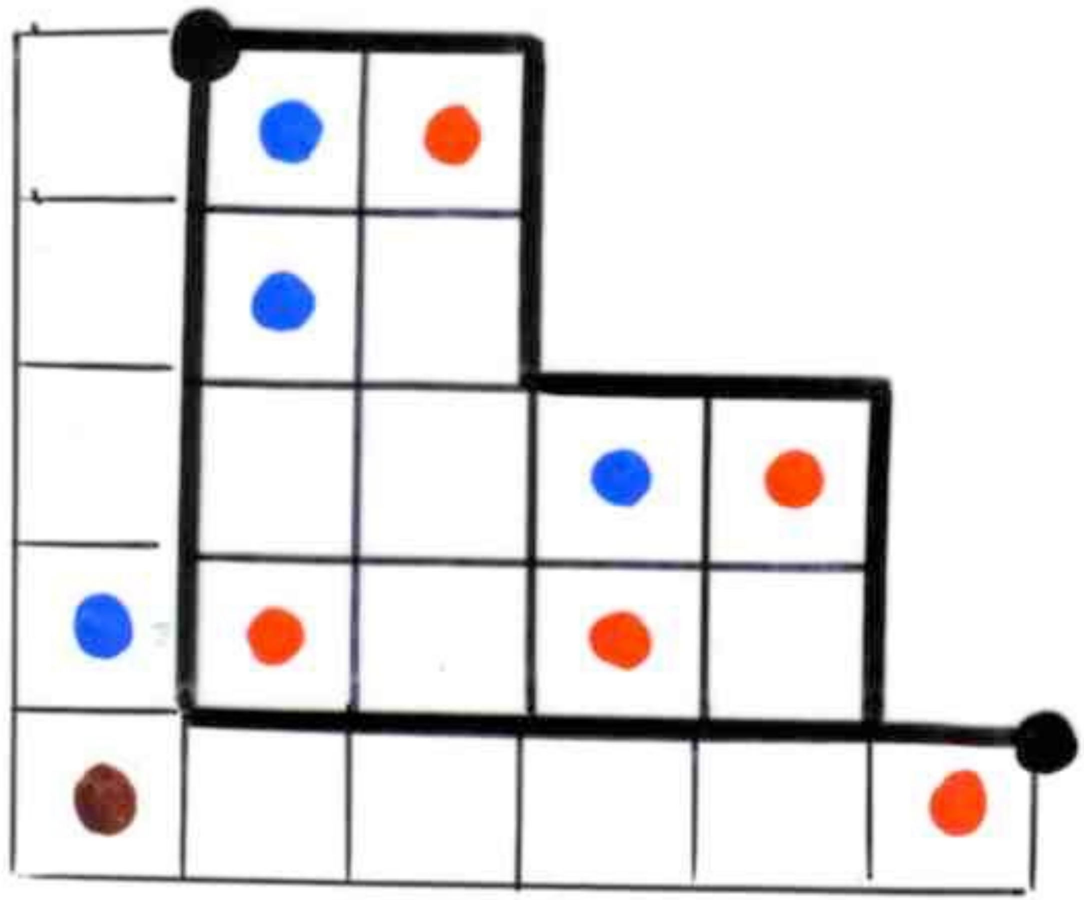
bijection

Catalan alternative tableaux

binary trees

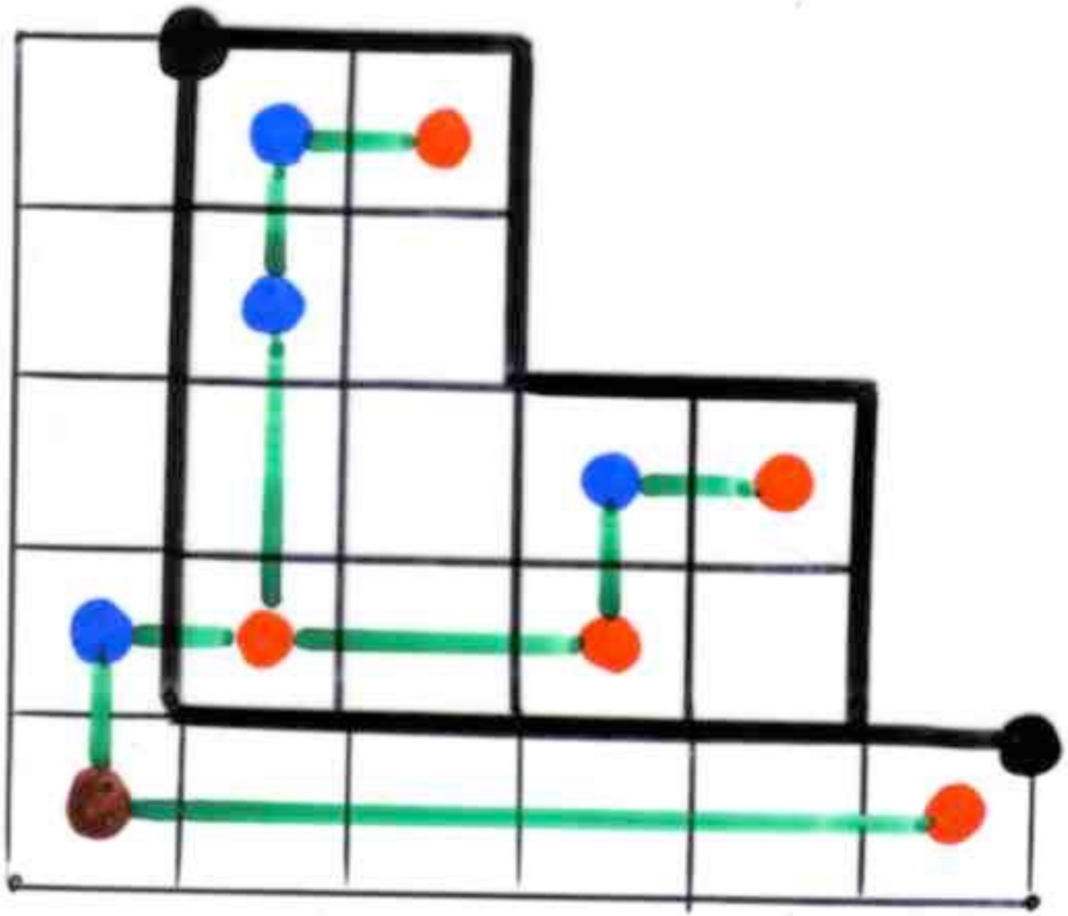


a Catalan alternative tableau

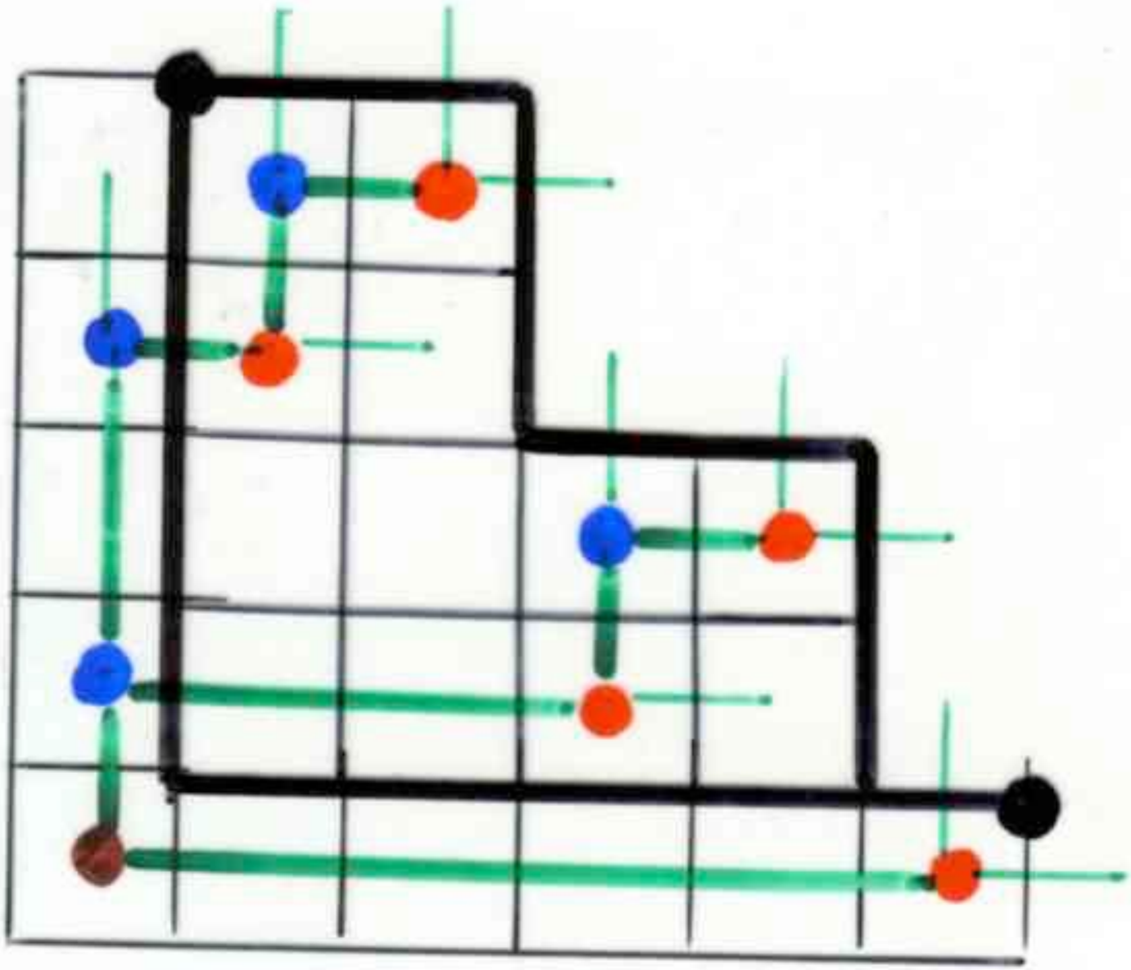


the extended Catalan alternative tableau

for each blue point add a vertical (green) edge below the point
for each red point add an horizontal (green) edge at the left of he point

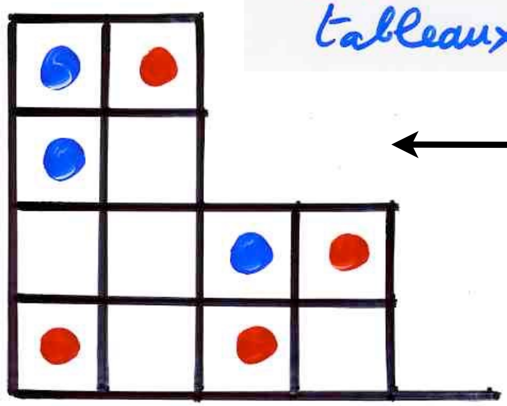


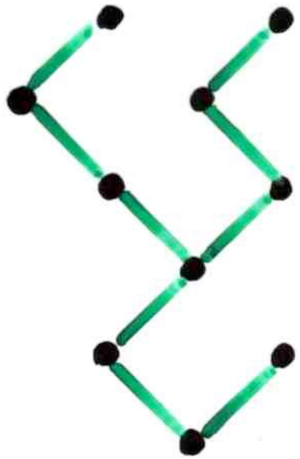
one get a binary tree

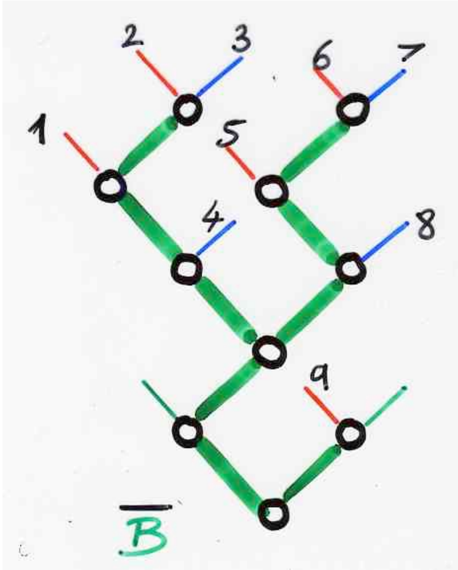


the associated extended (also called complete) binary tree

Catalan alternative tableaux

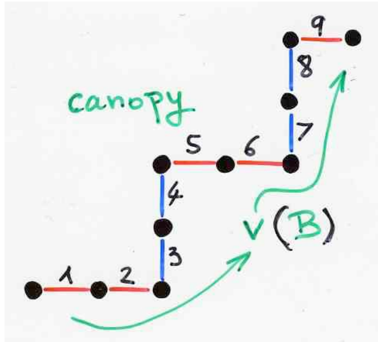


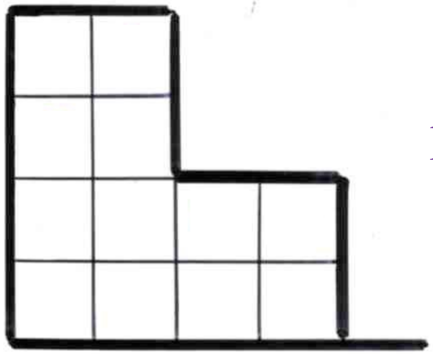




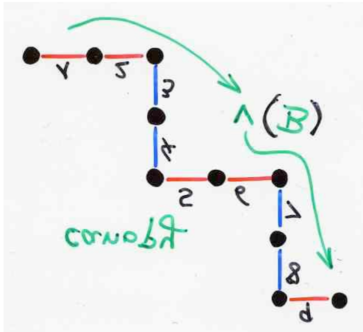
Proposition

The map defined above is a bijection between alternative tableaux with profile v and binary trees with canopy v

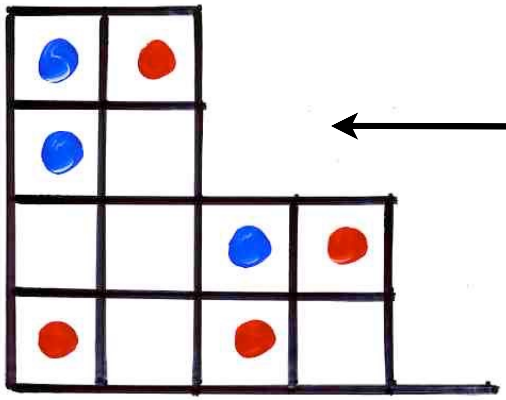


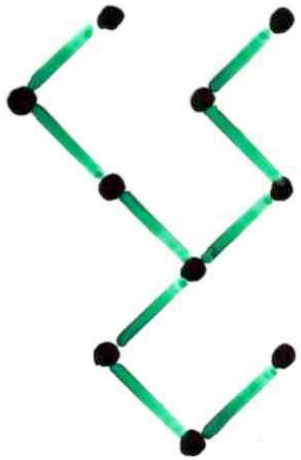


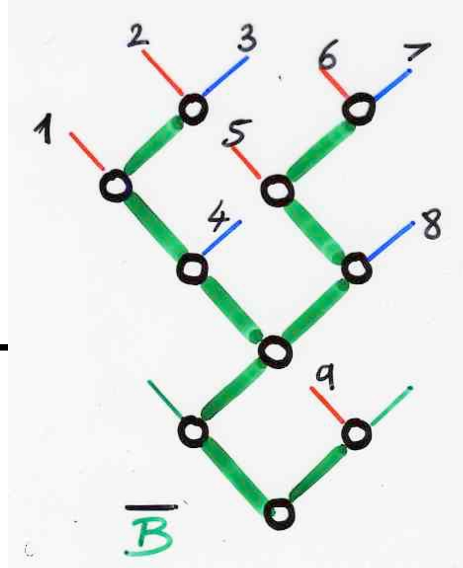
profile of a Ferrers diagram is defined slide 67

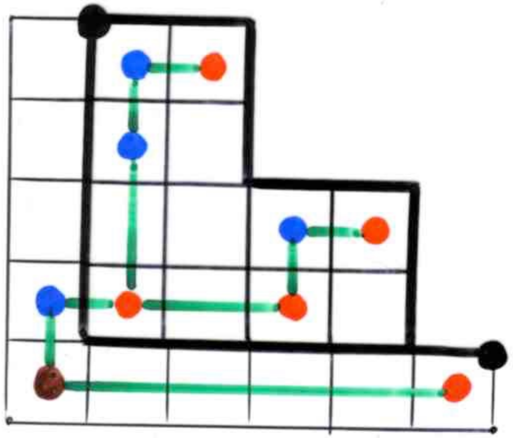


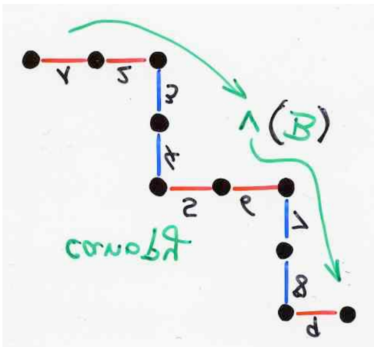
Catalan
alternative
tableaux

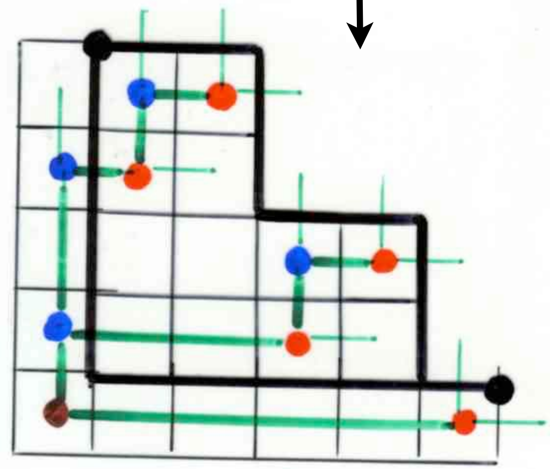


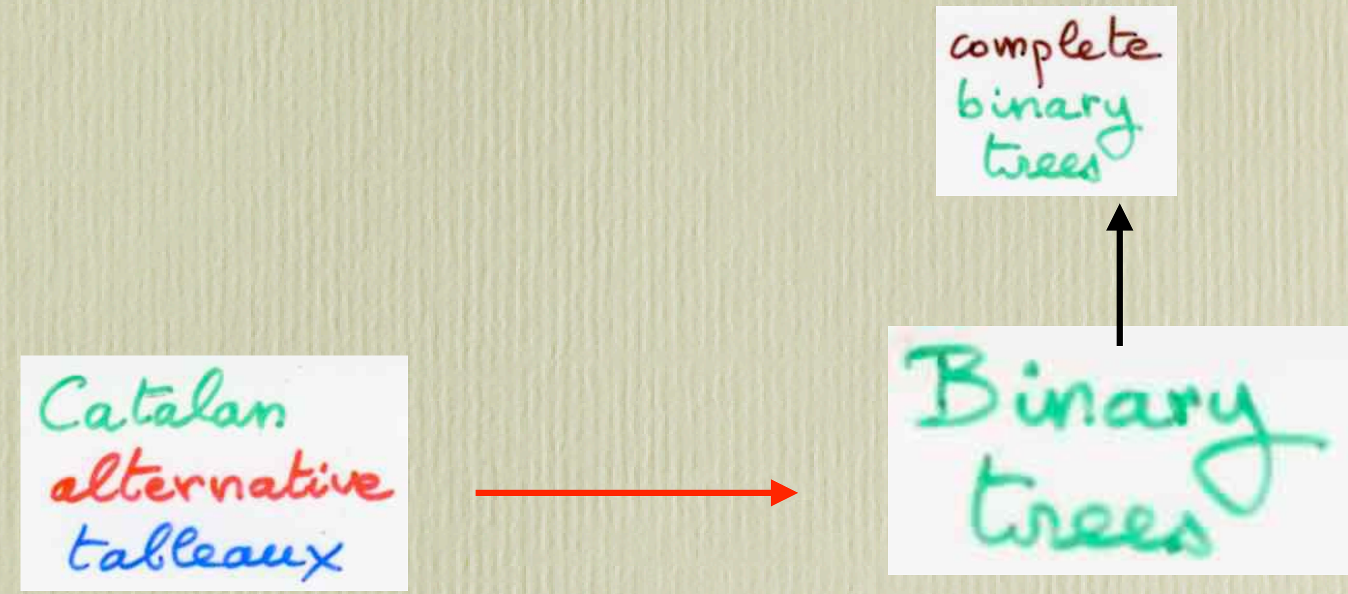




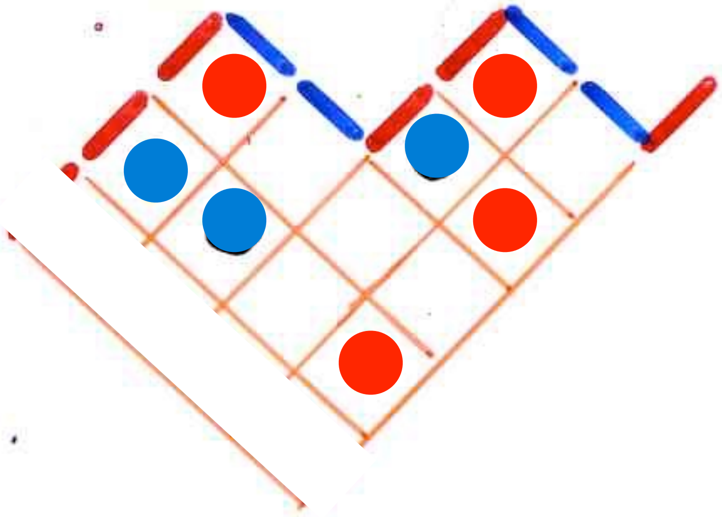


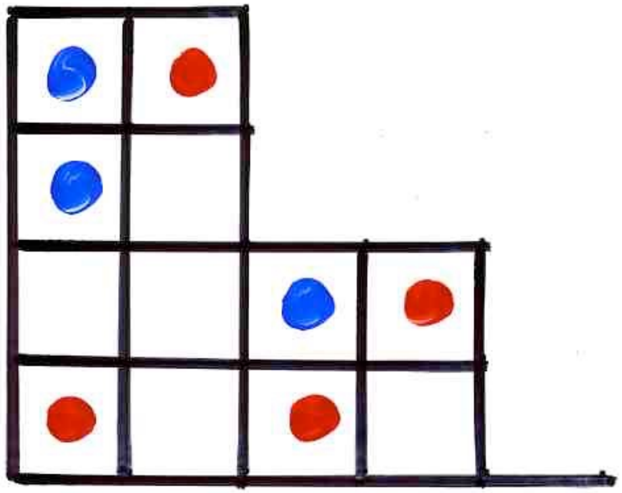


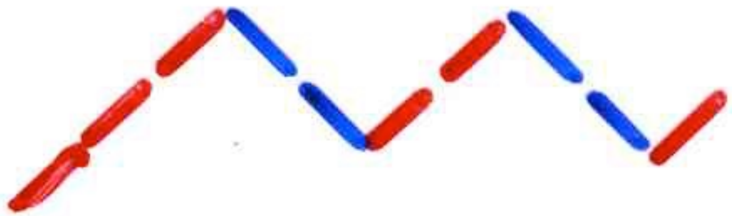


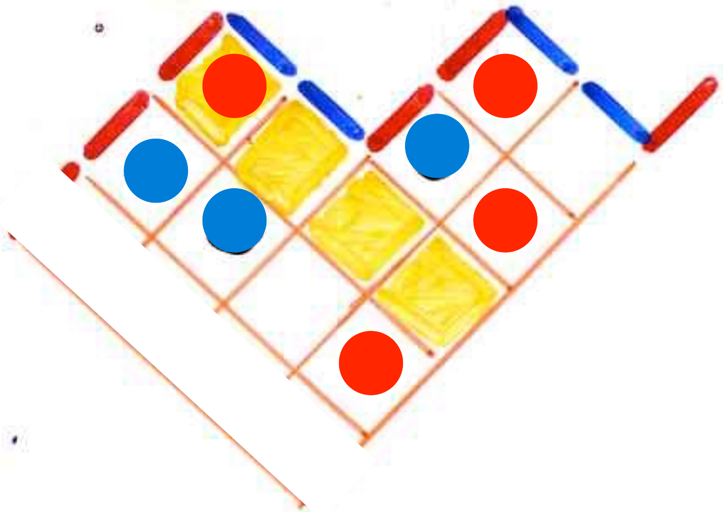

Catalan alternative tableaux
Binary trees
complete binary trees

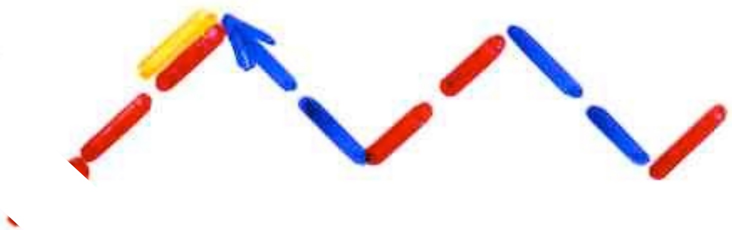
2nd bijection
Catalan alternative tableaux
binary trees

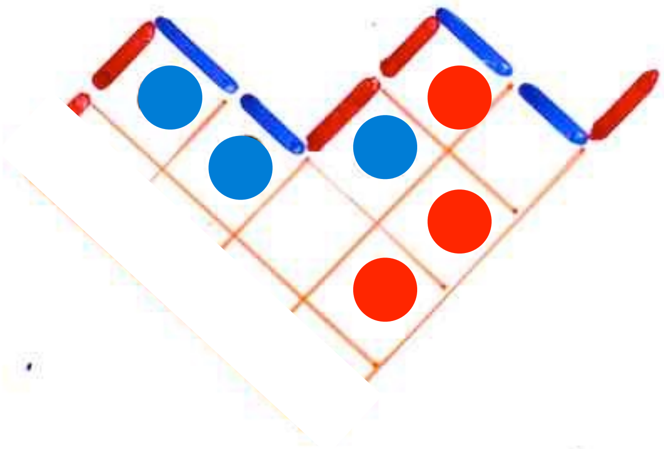


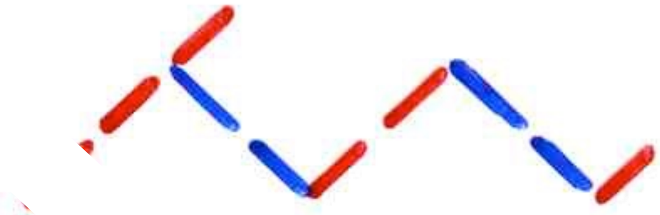


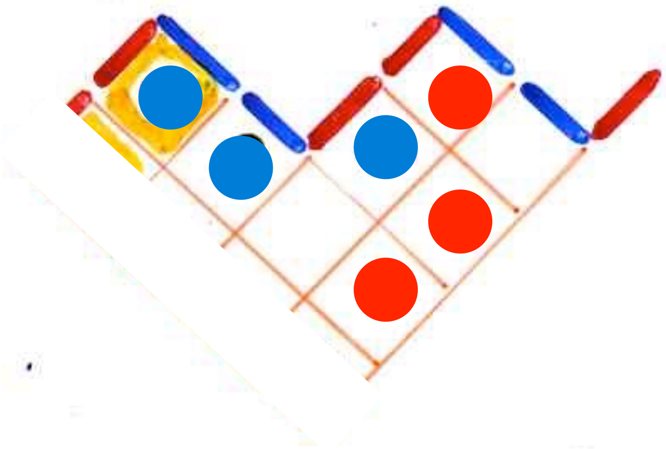


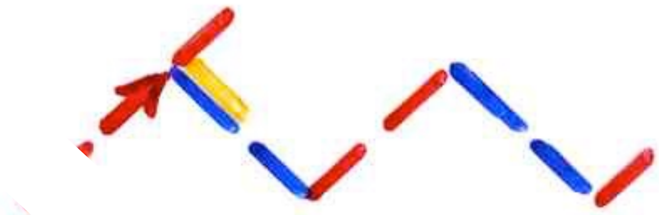


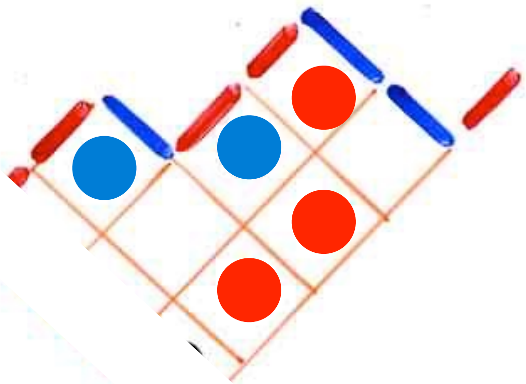


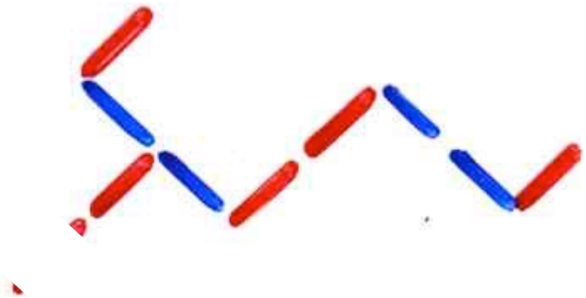


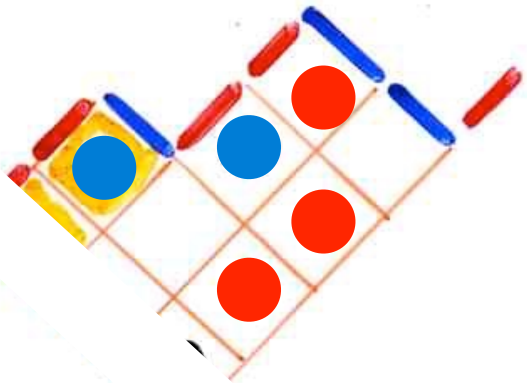


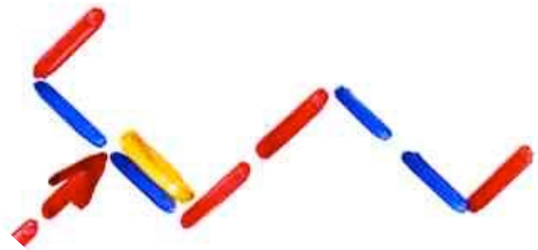


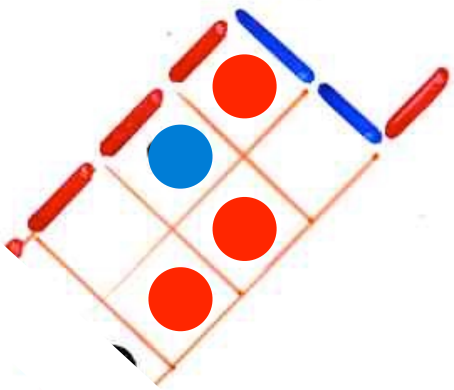


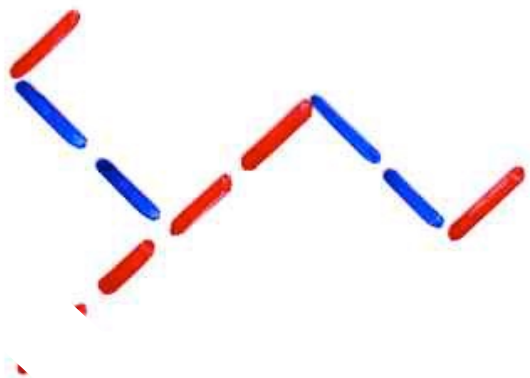


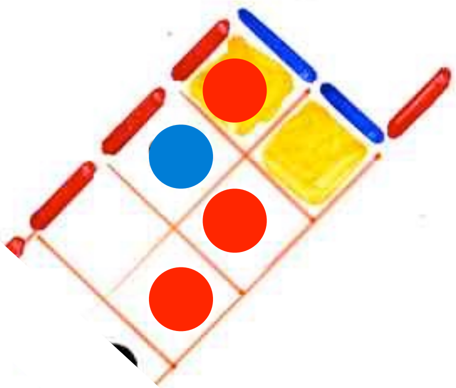


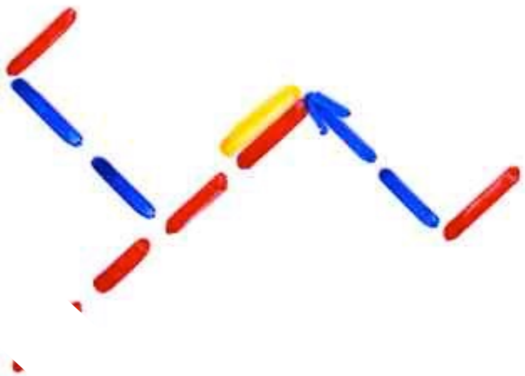


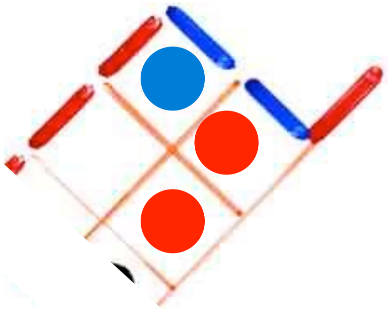


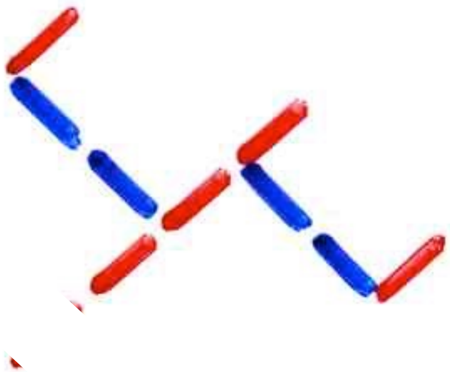


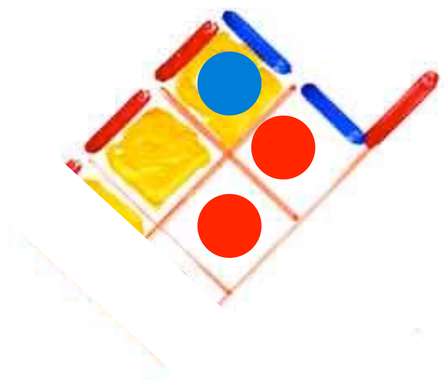


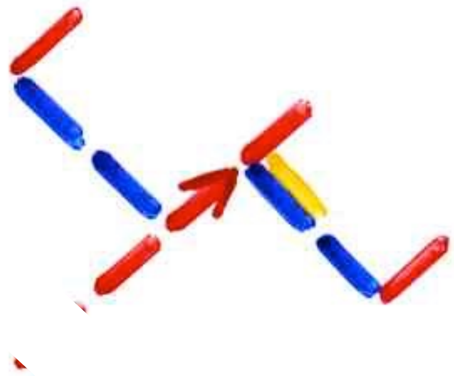


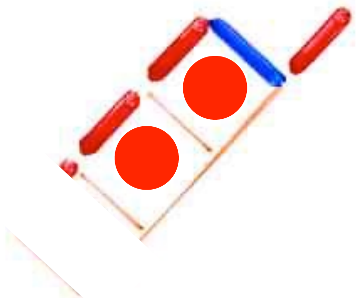


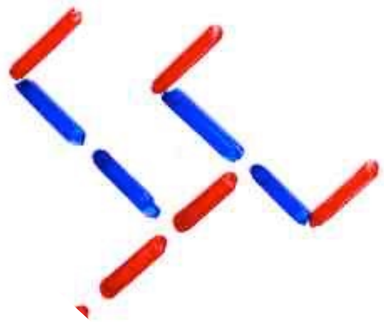


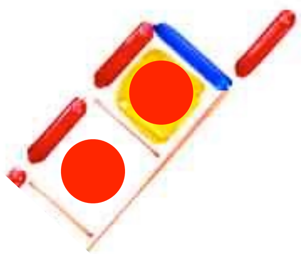


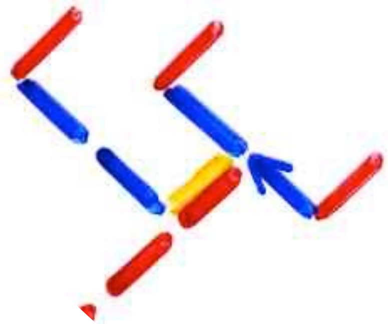


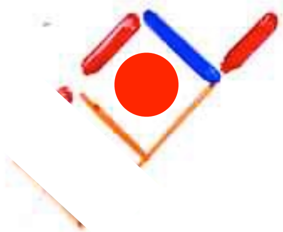


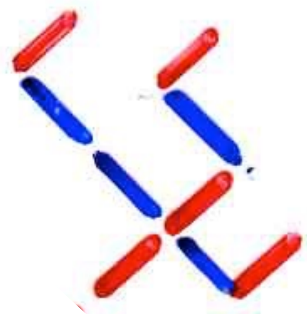


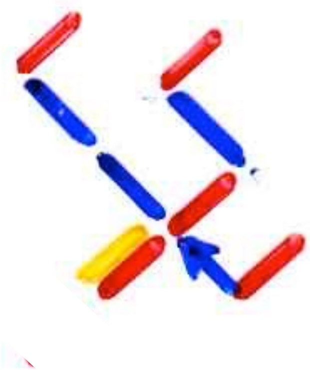


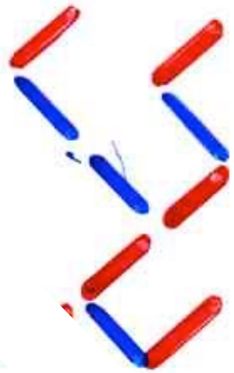






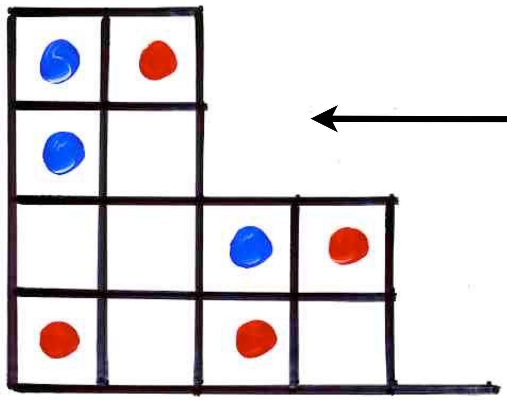


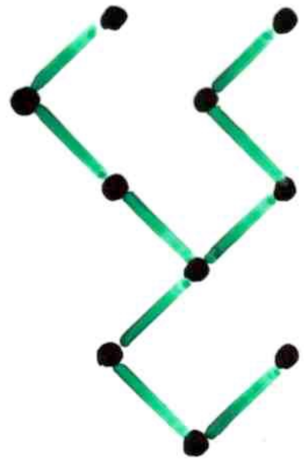


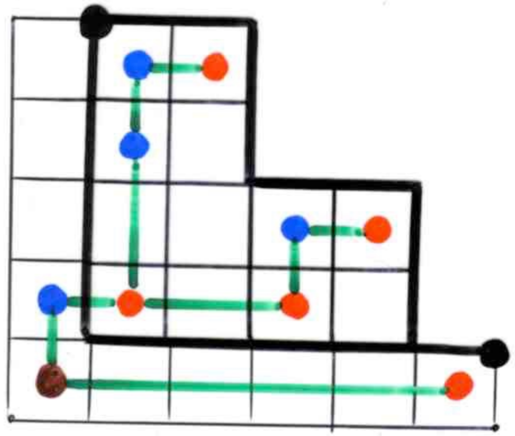


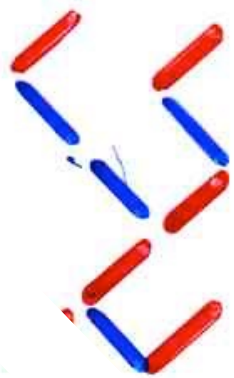
This algorithm based on a kind of « jeu de taquin » on « tableaux and trees » is reversible. One get a bijection between Catalan alternative tableaux and binary trees, which is the same as the one described on slide 97.

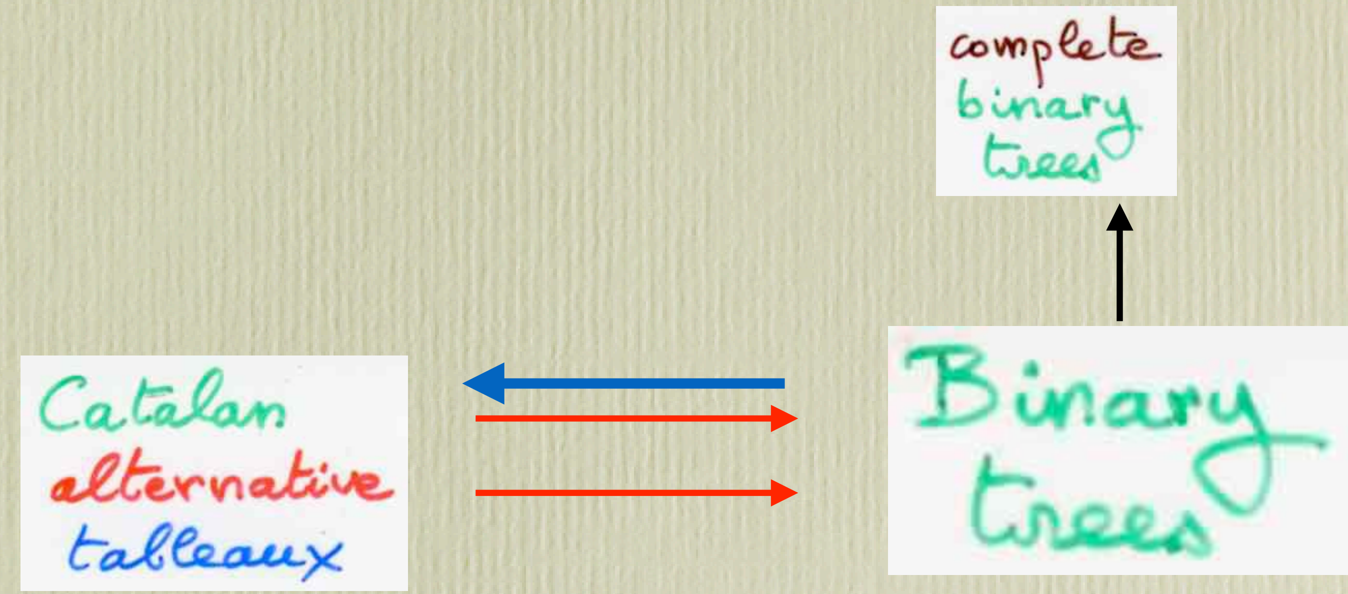
Catalan alternative tableaux







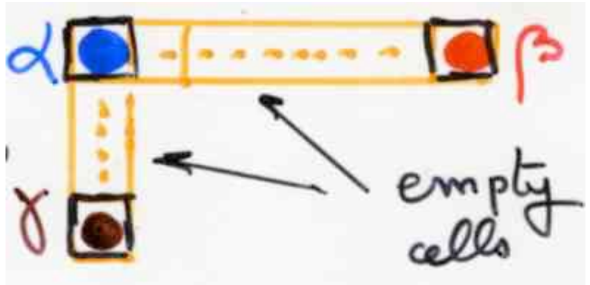



complete binary trees
Catalan alternative tableaux
Binary trees

Tamari and alternative tableaux

Main Lemma

In a Catalan alternative tableau let α, β, γ be 3 colored cells in a Γ position (α is necessarily blue and γ red)



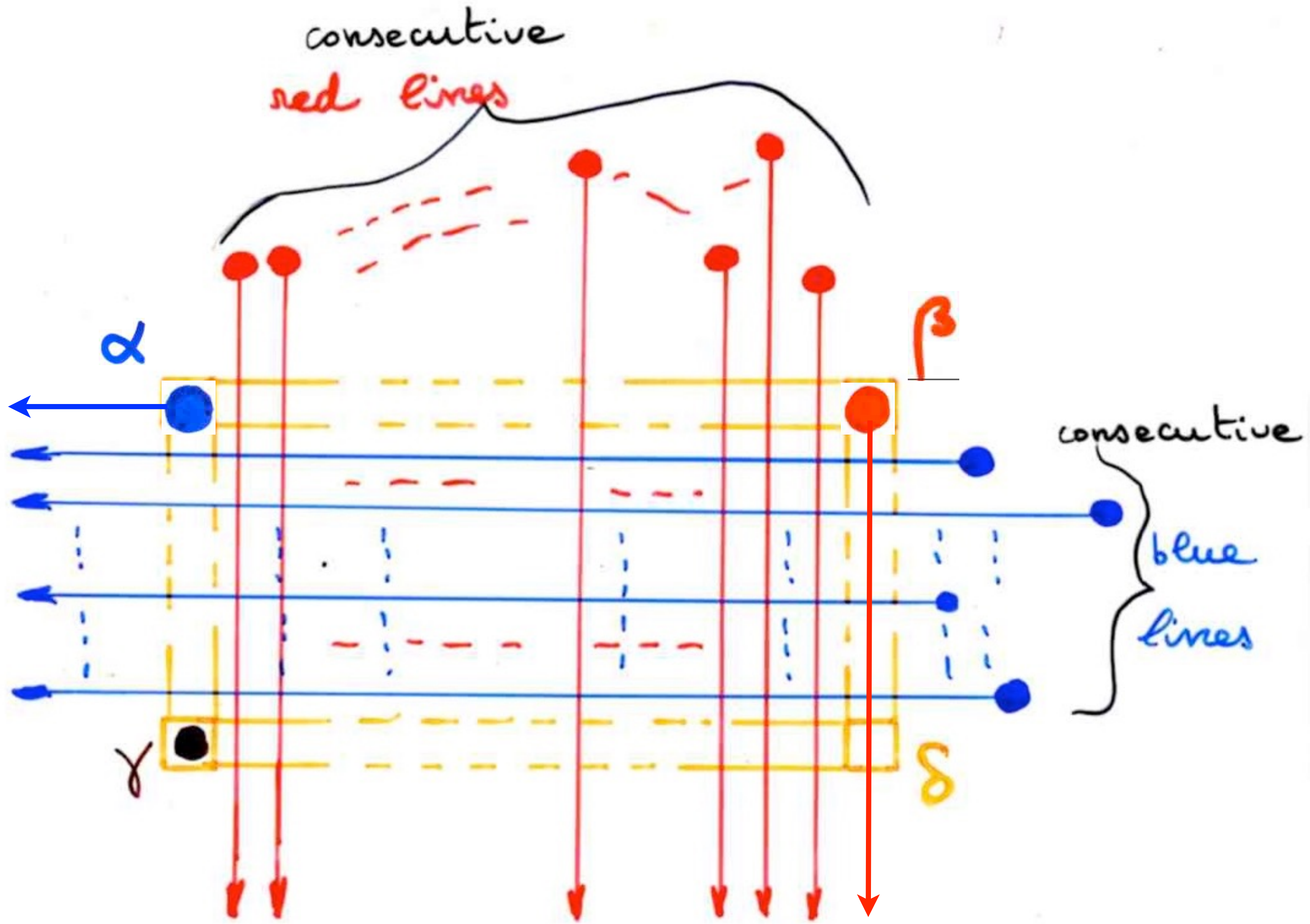
such that there is no colored cell between α and β and between α and γ.

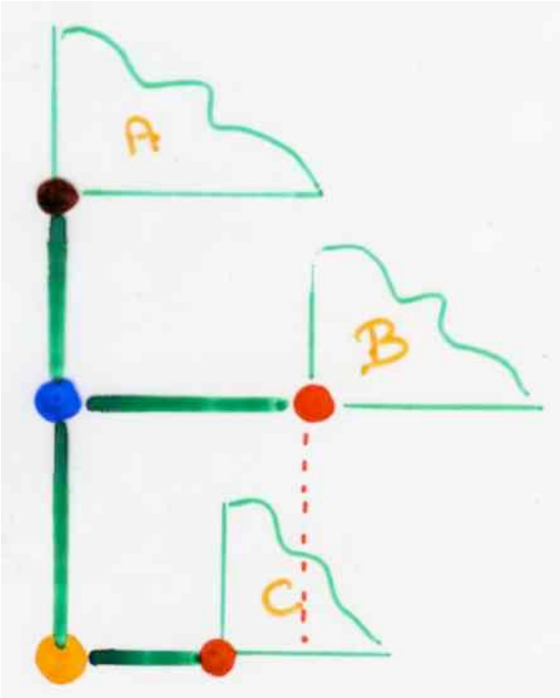
Then the cells of the whole rectangle

$$\begin{array}{ccc} \alpha & \text{———} & \beta \\ | & & | \\ \gamma & \text{———} & \delta \end{array}$$

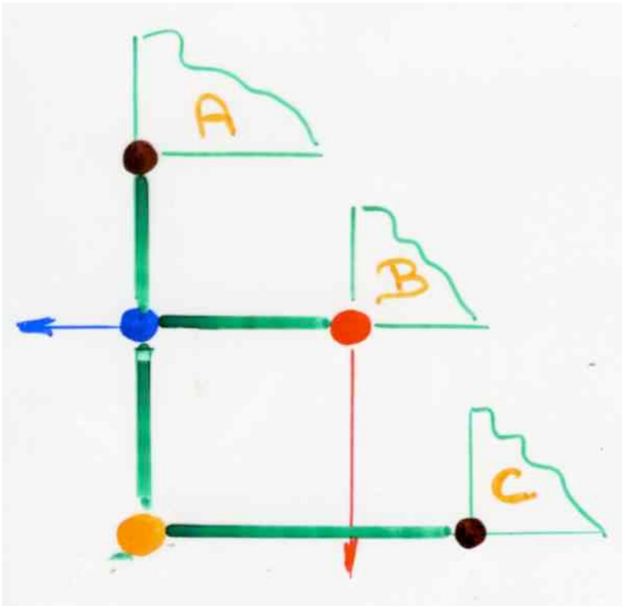
are empty (except α, β, γ)

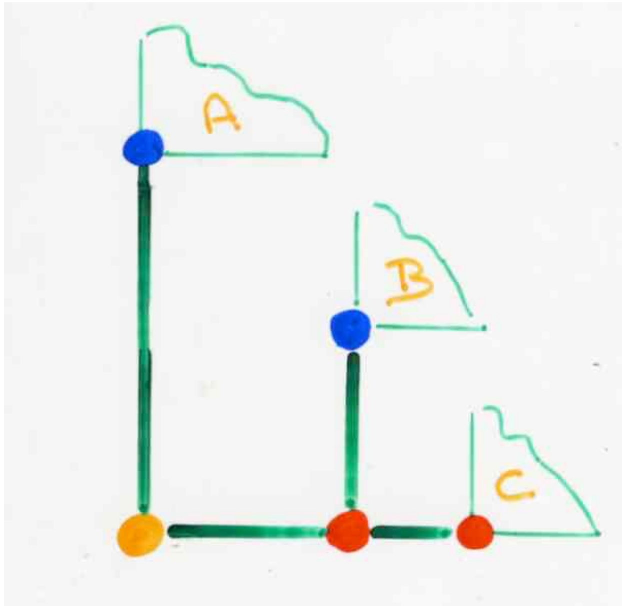
Moreover we have the following configuration of blue cells and lines, with red cells and lines :

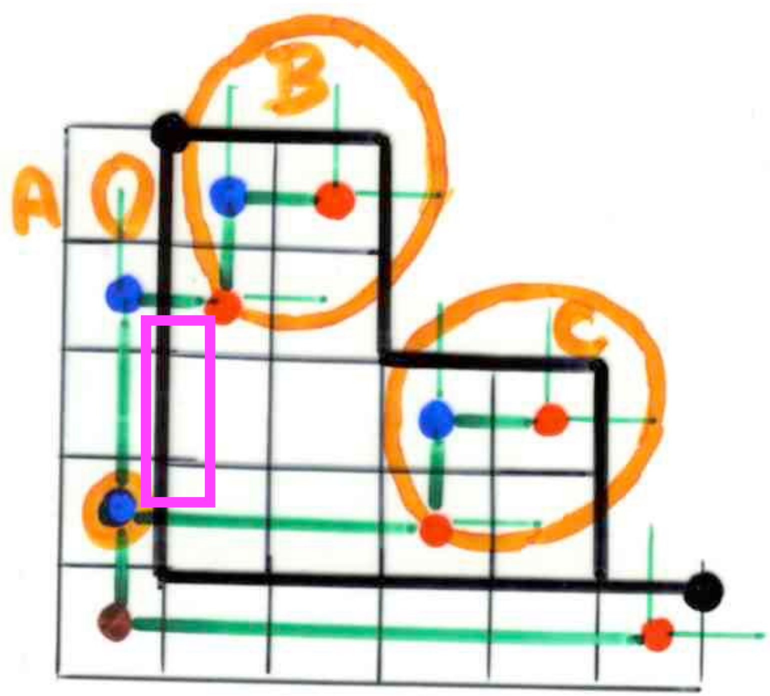

consecutive
red lines
α
β
consecutive
blue
lines
γ
δ

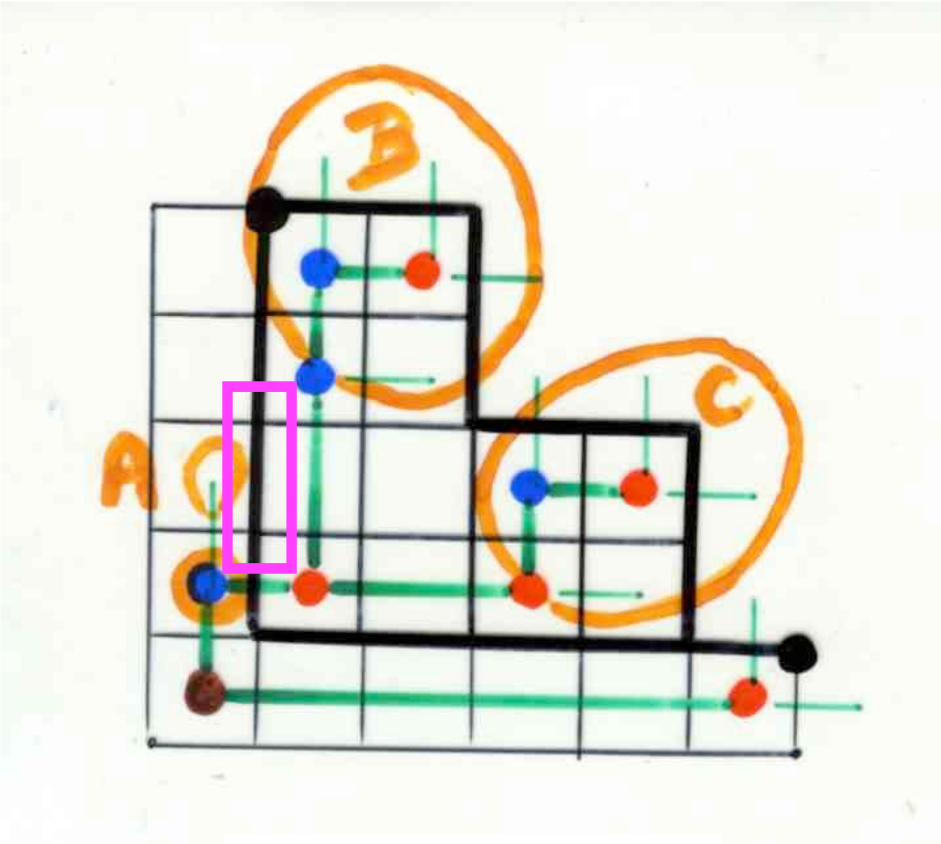


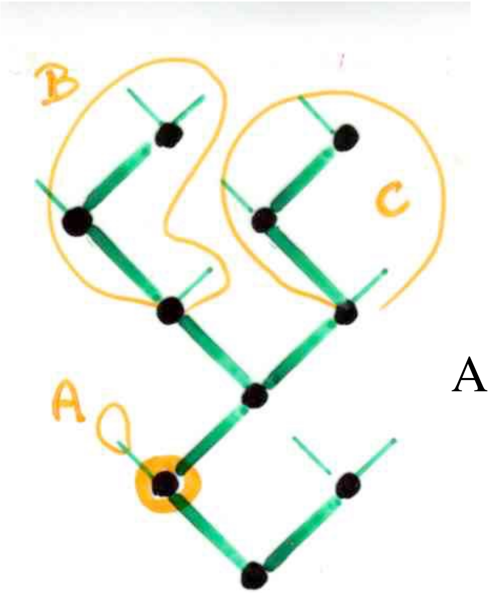
impossible !

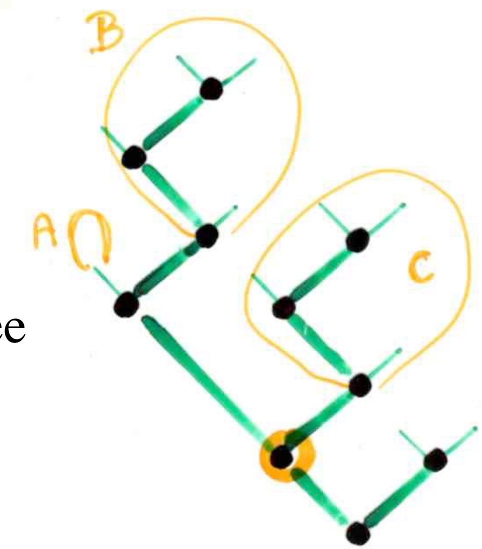








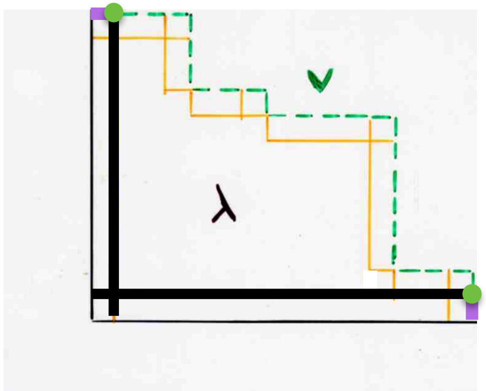




A rotation in the binary tree
corresponds exactly
to a certain Γ-move
in the associated
Catalan alternative tableau.

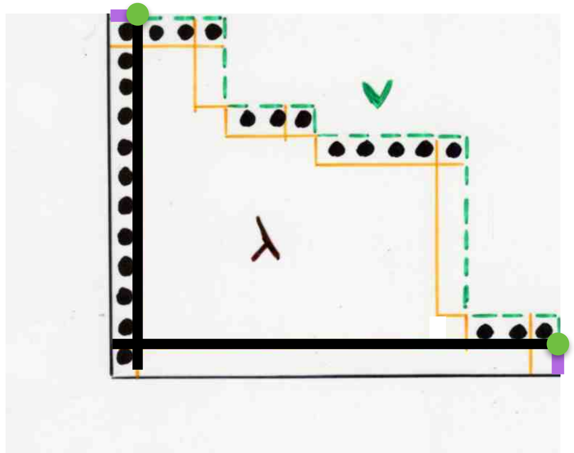
The main theorem

Main theorem



Ferrers diagram λ with profile v

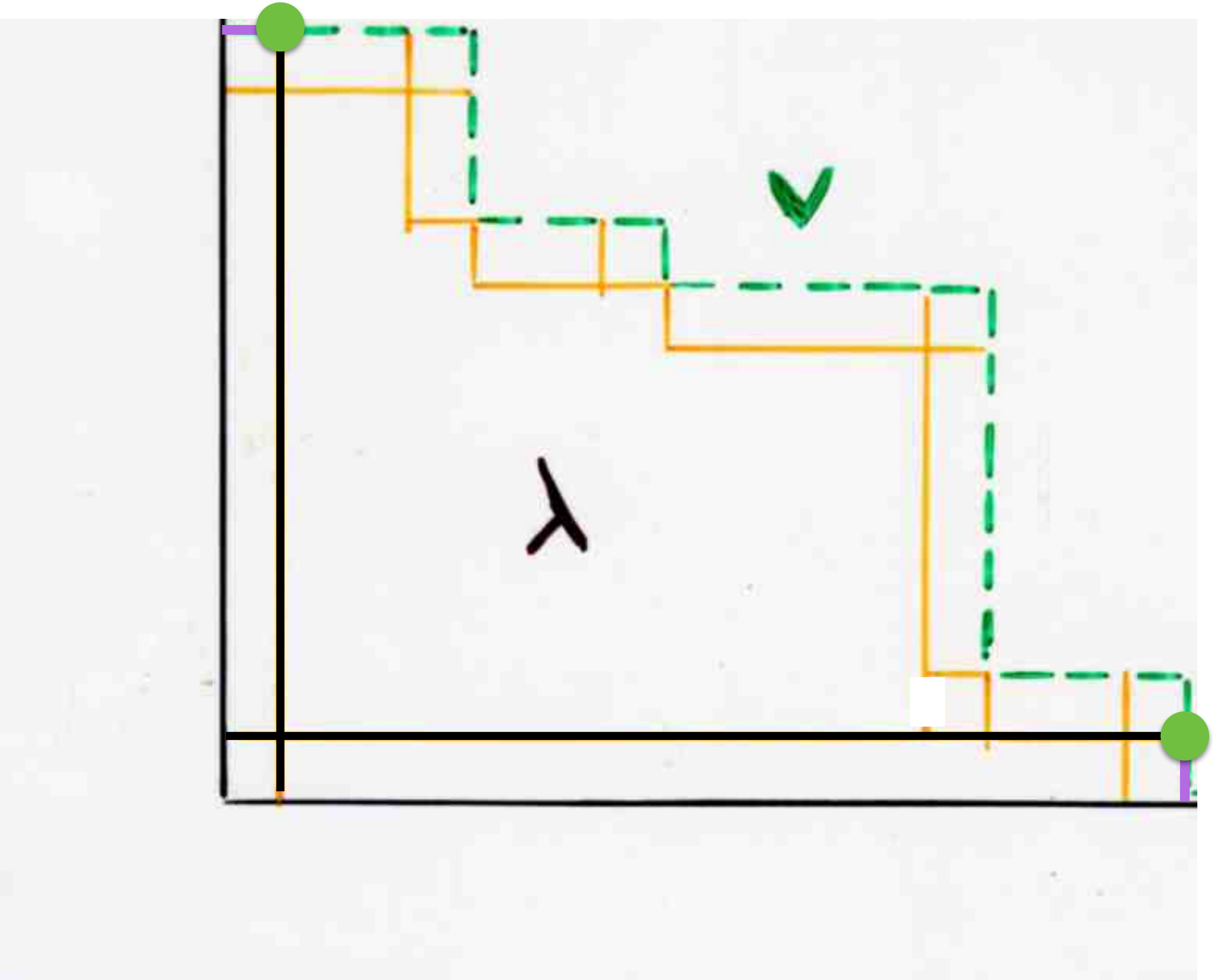
Let $X(\lambda) = X(v)$ be the cloud

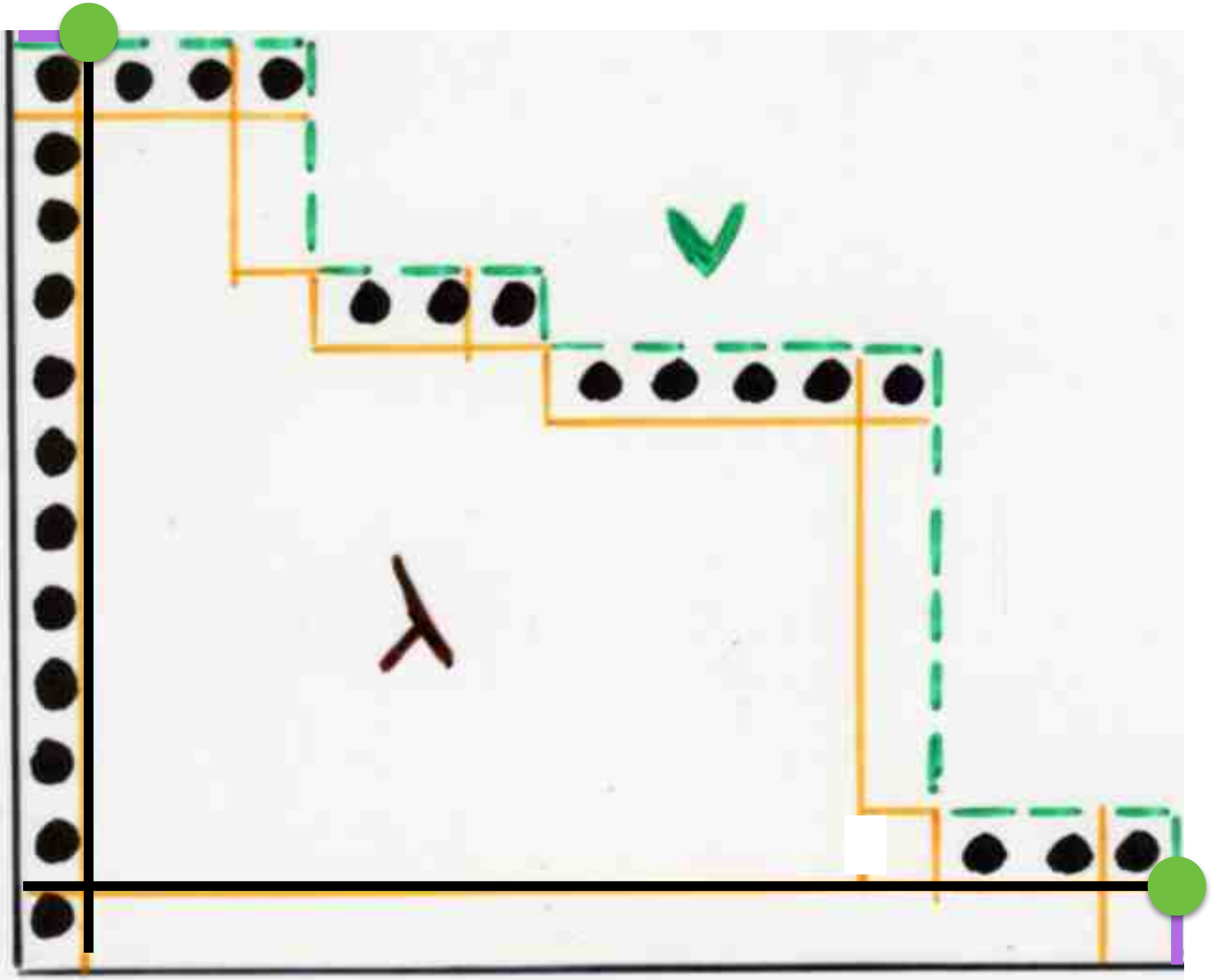


The set of binary trees having a given canopy v is an interval of the Tamari lattice.

This interval Int(v) is a maule:

$$\mathrm{Int}(v) = \mathrm{Maule}(X(v))$$


ν
λ

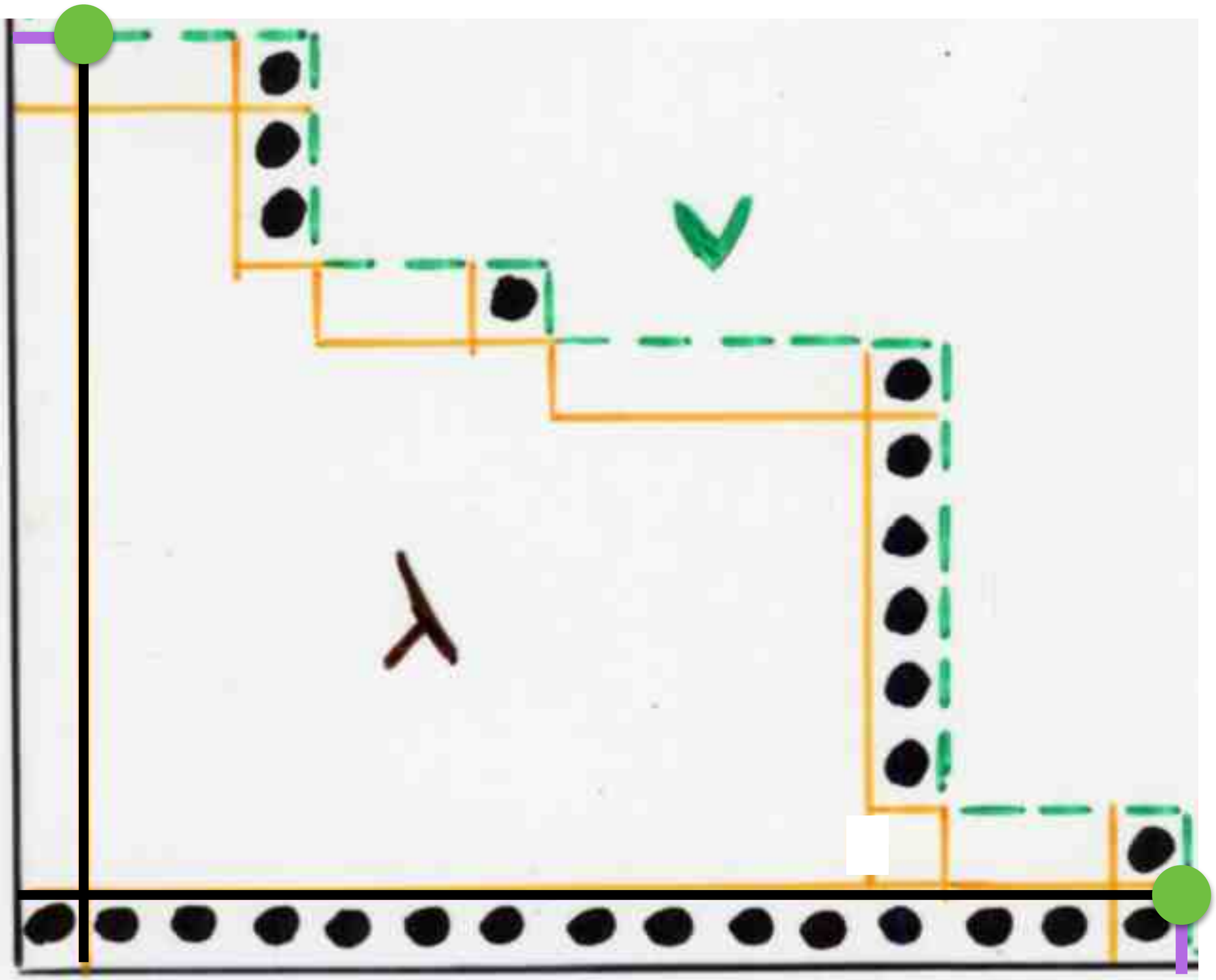


$X(v)$

$X(\lambda)$

minimum element of the maule

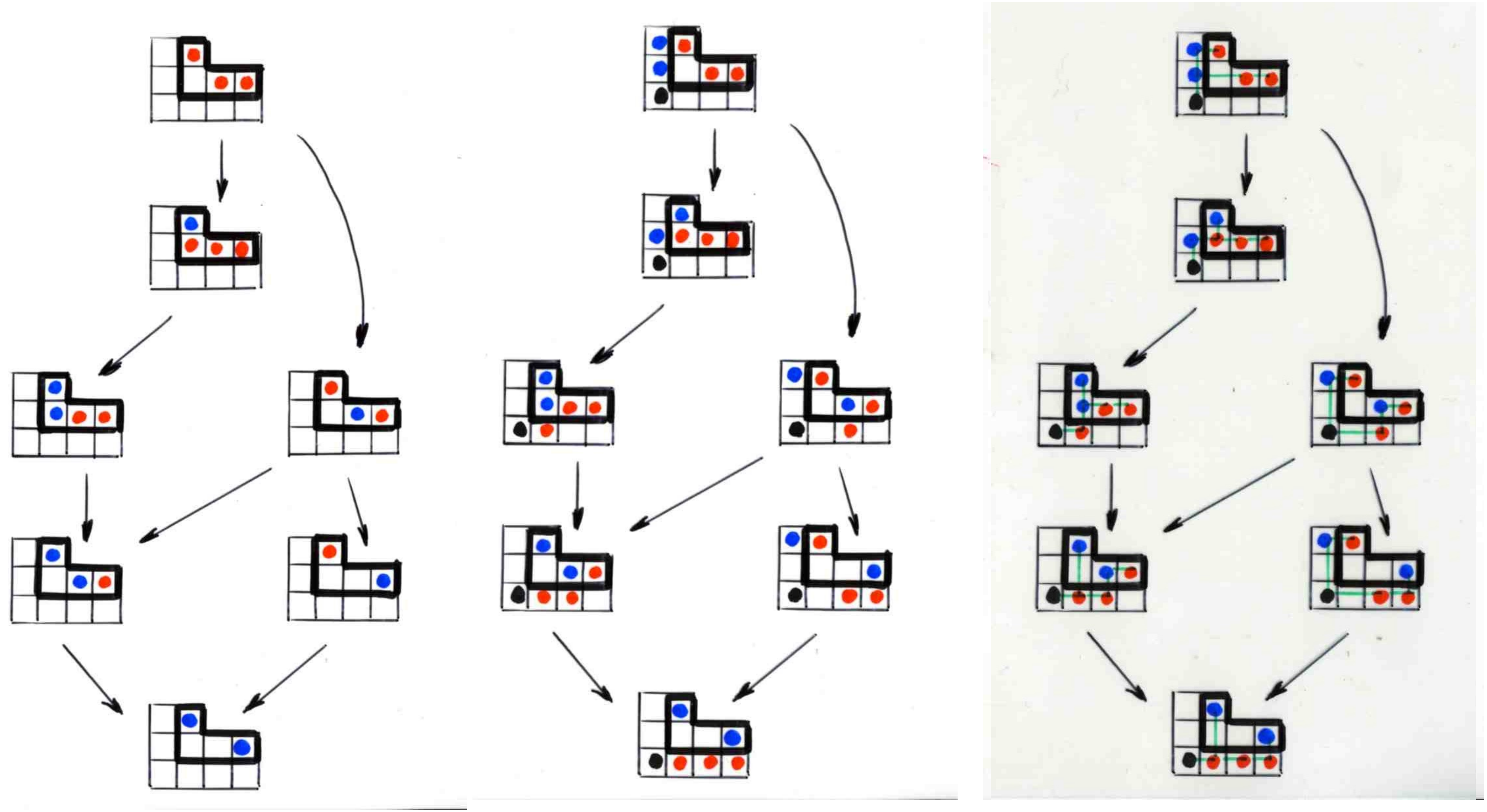
$\mathrm{Maule}(X(v))$



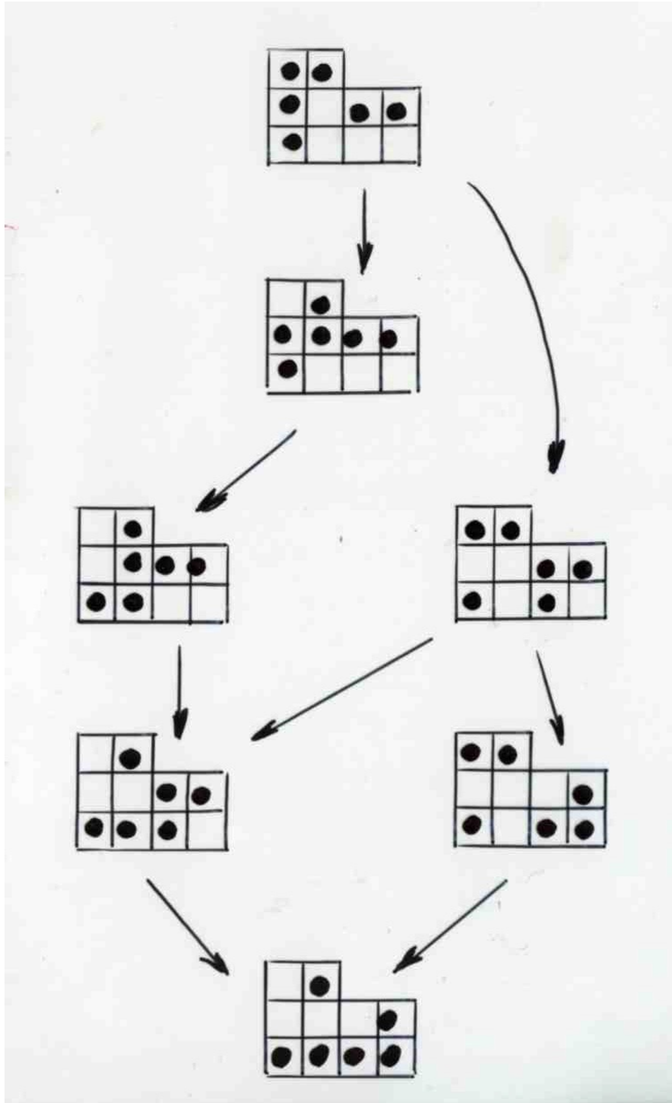
maximum element of the maule

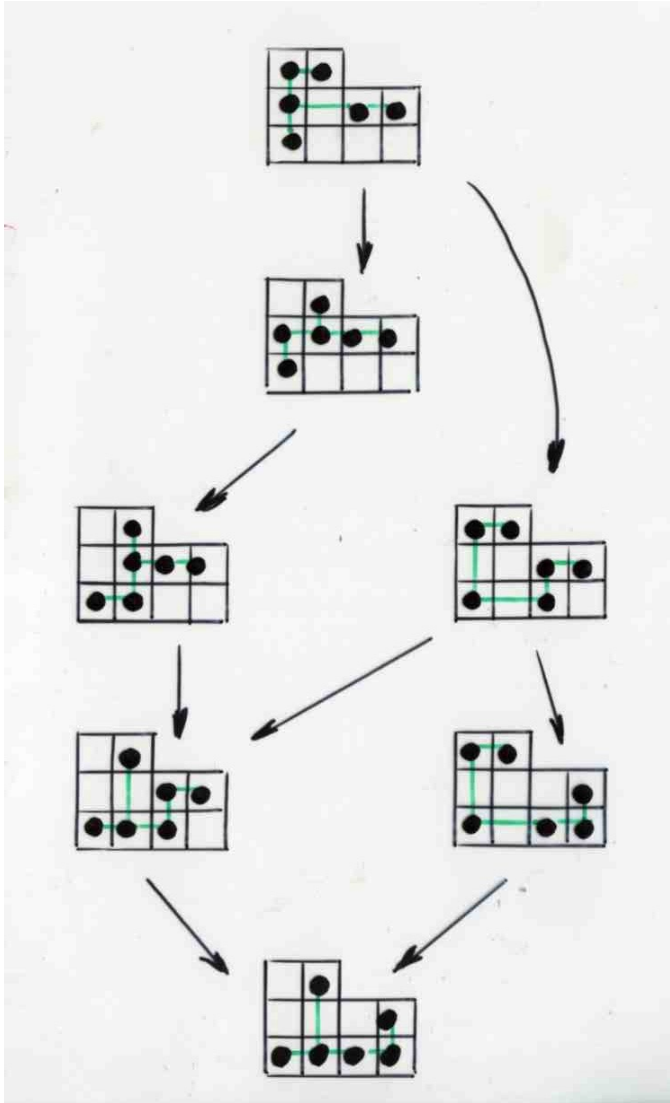
Maule$(X(v))$

an example



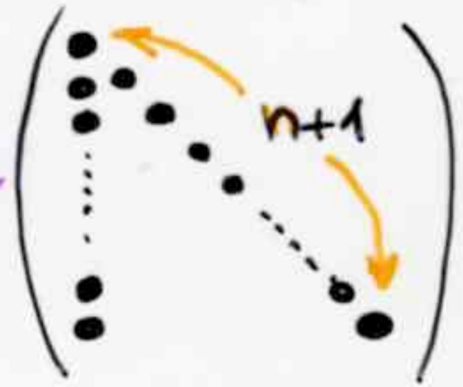
an example

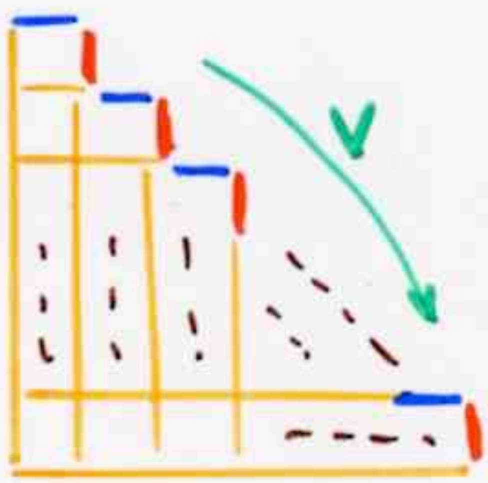




Proposition

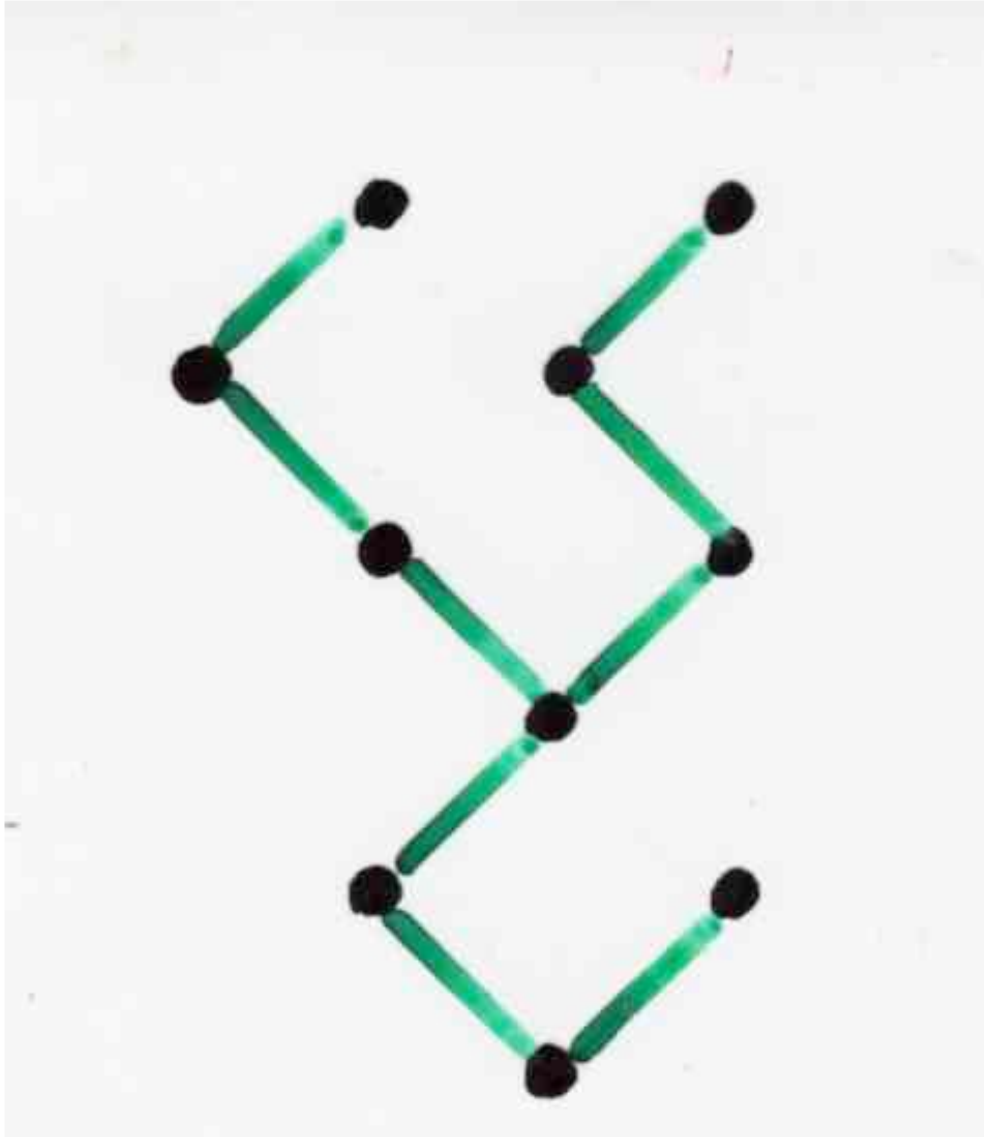
Tamari(n) = Maule

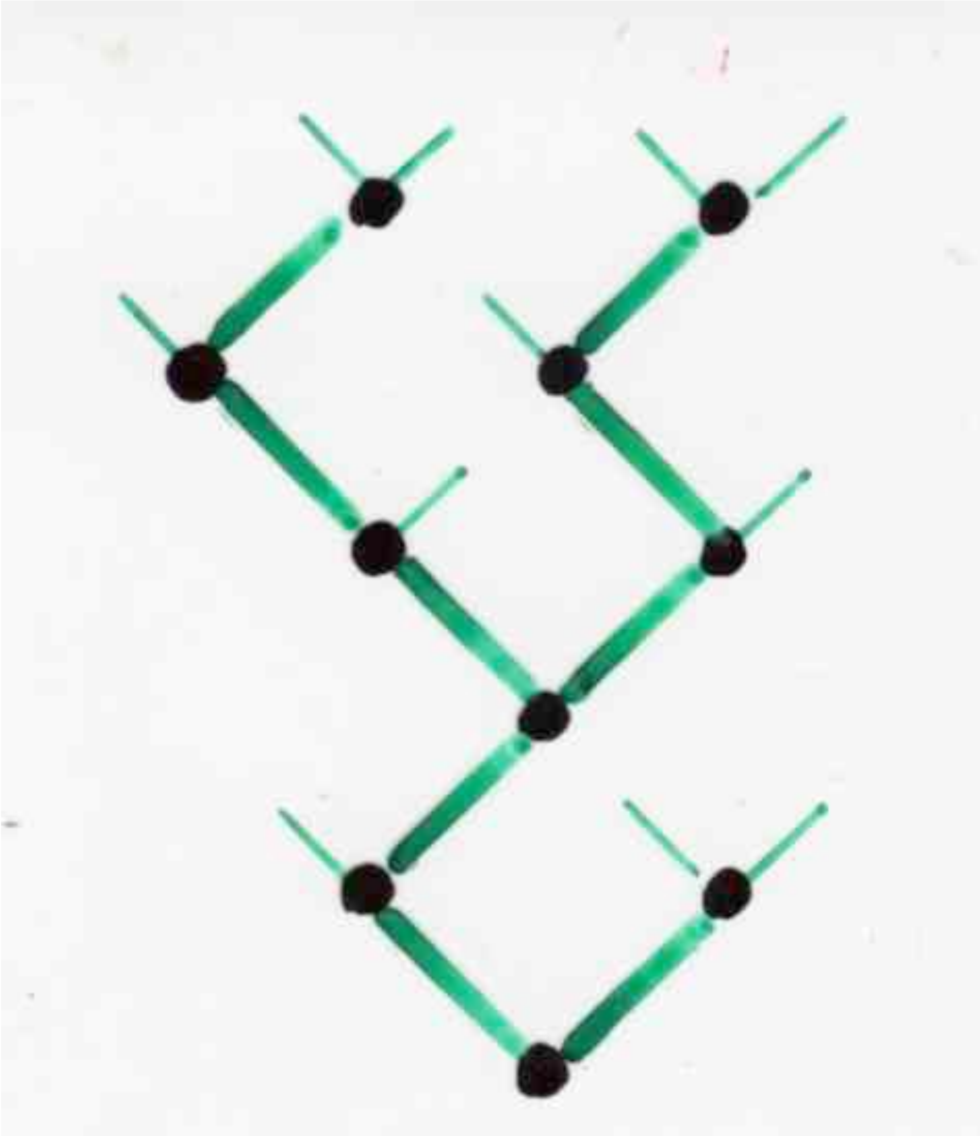


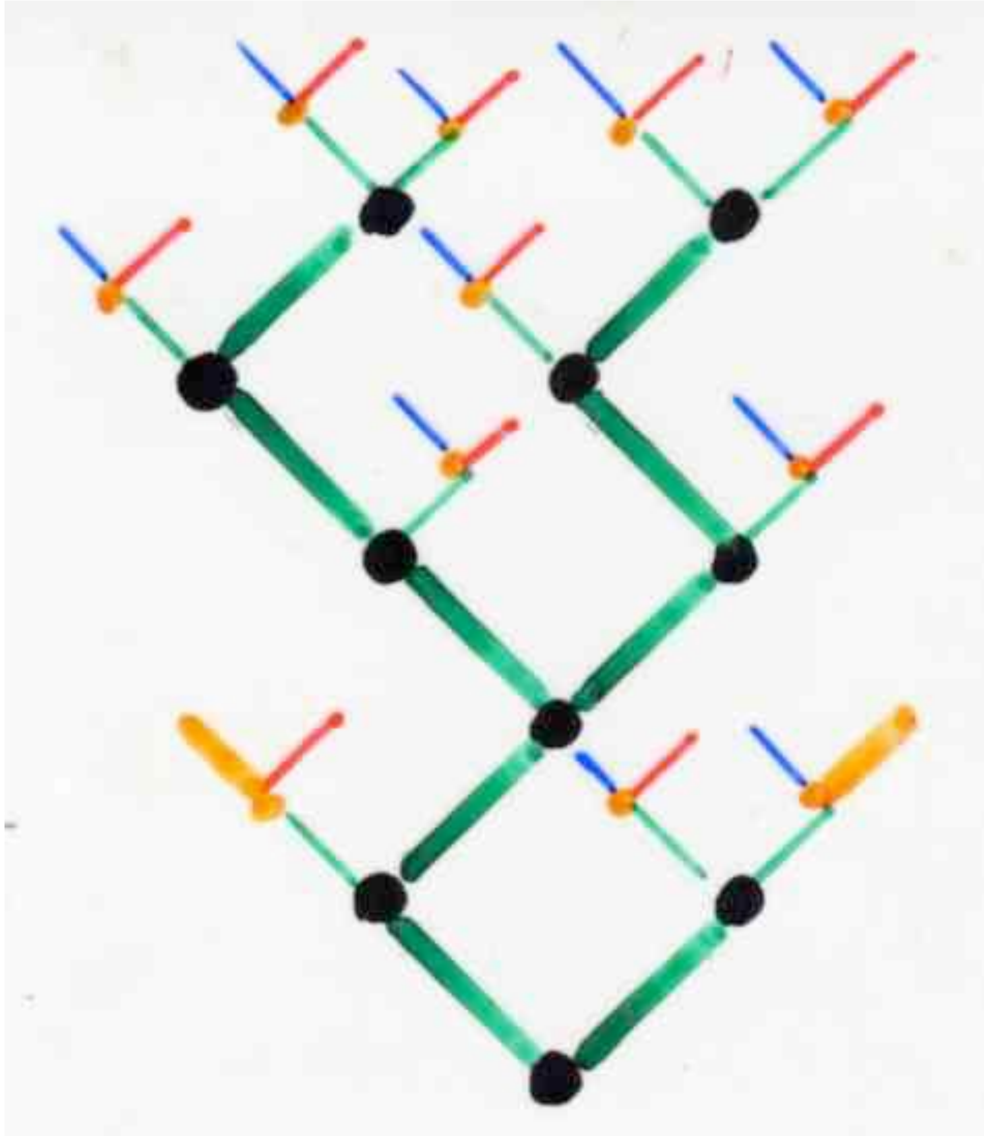


staircase
Catalan
alternative
tableaux

alternating
canopy

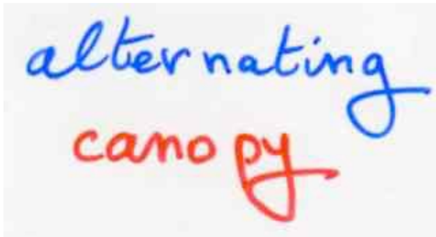






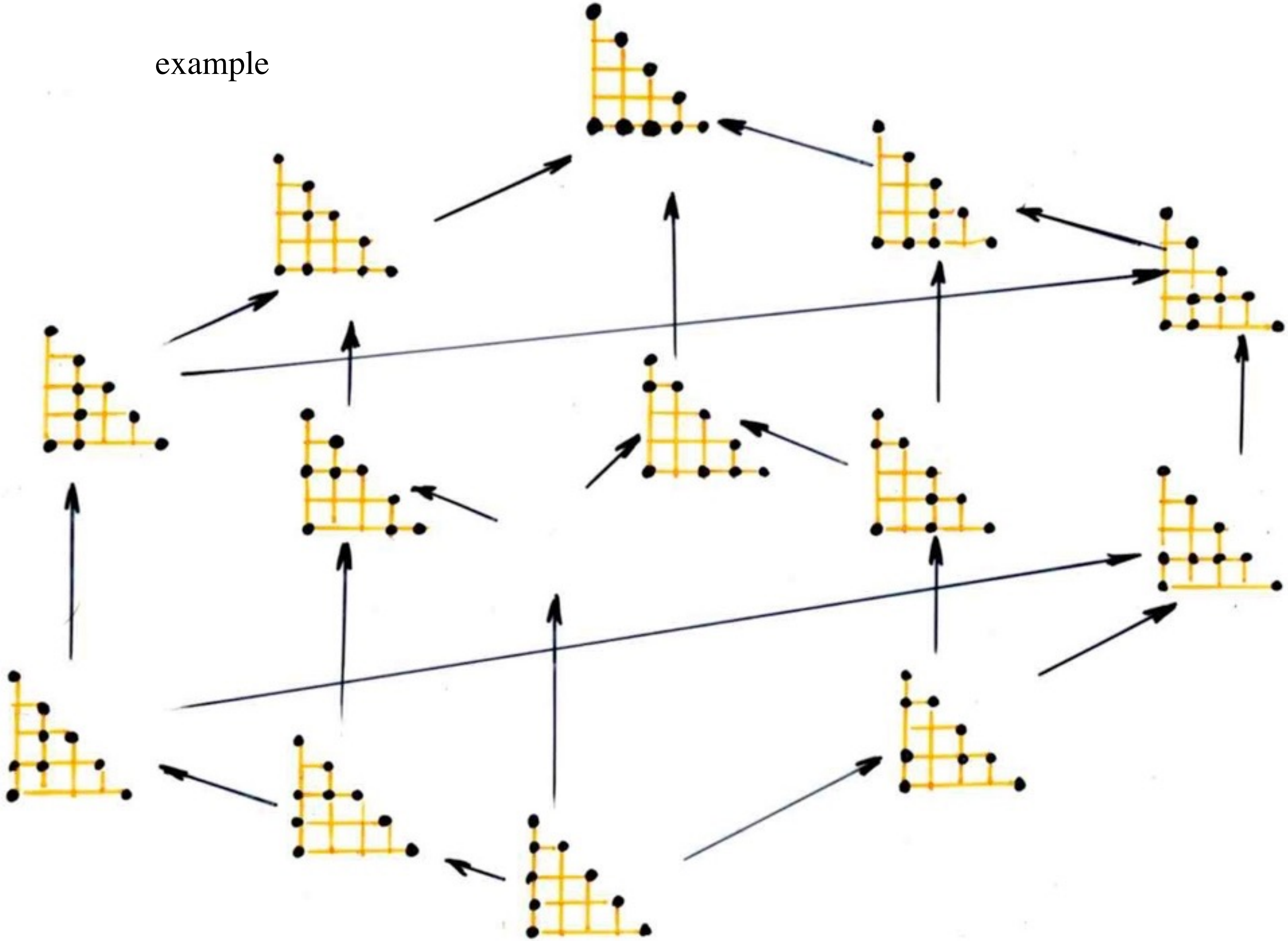
double extension
of the binary tree

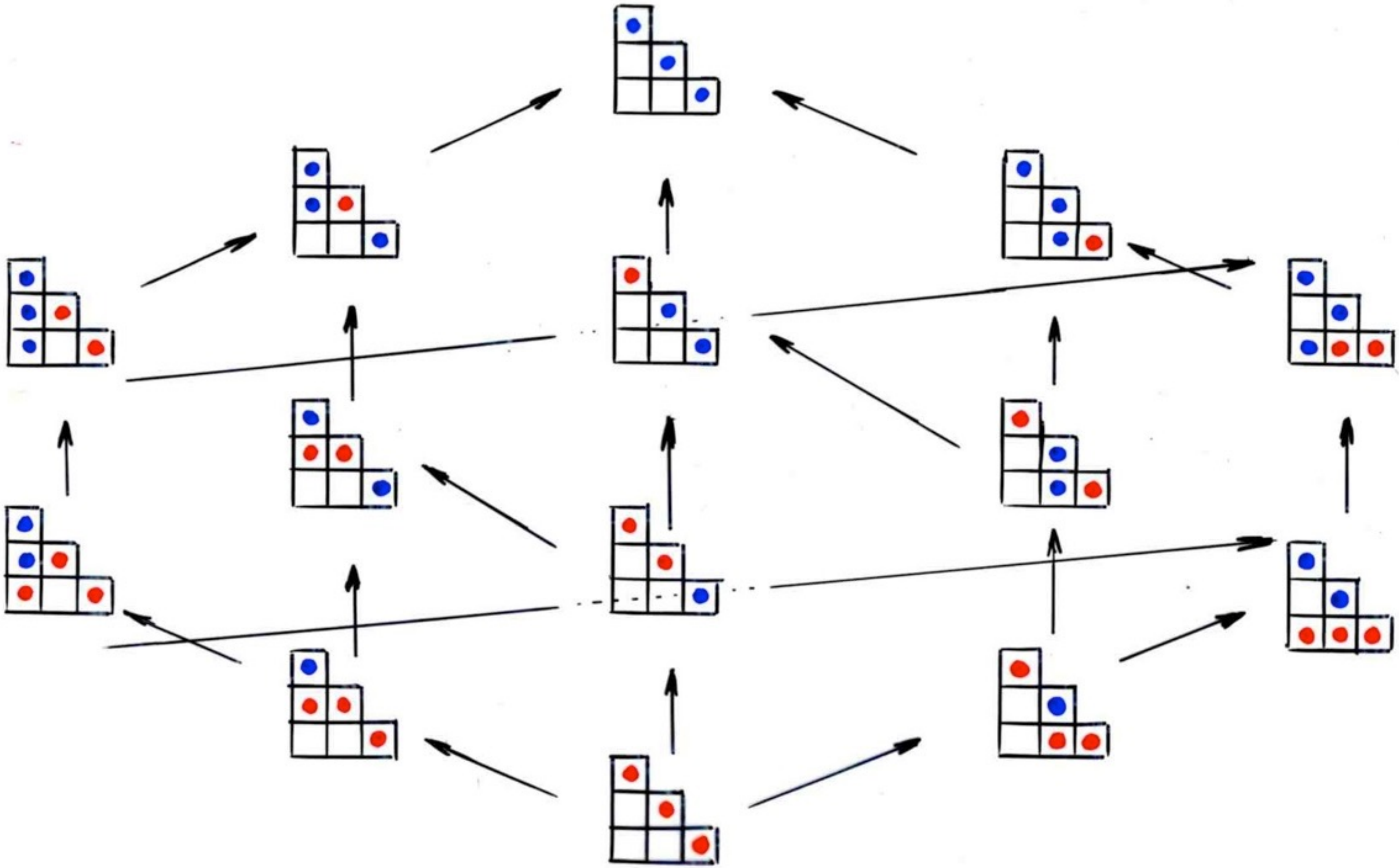
every binary tree is
in bijection
with a (complete) binary tree
having an alternating canopy
(i.e. the corresponding Ferrers
diagram has a staircase shape)

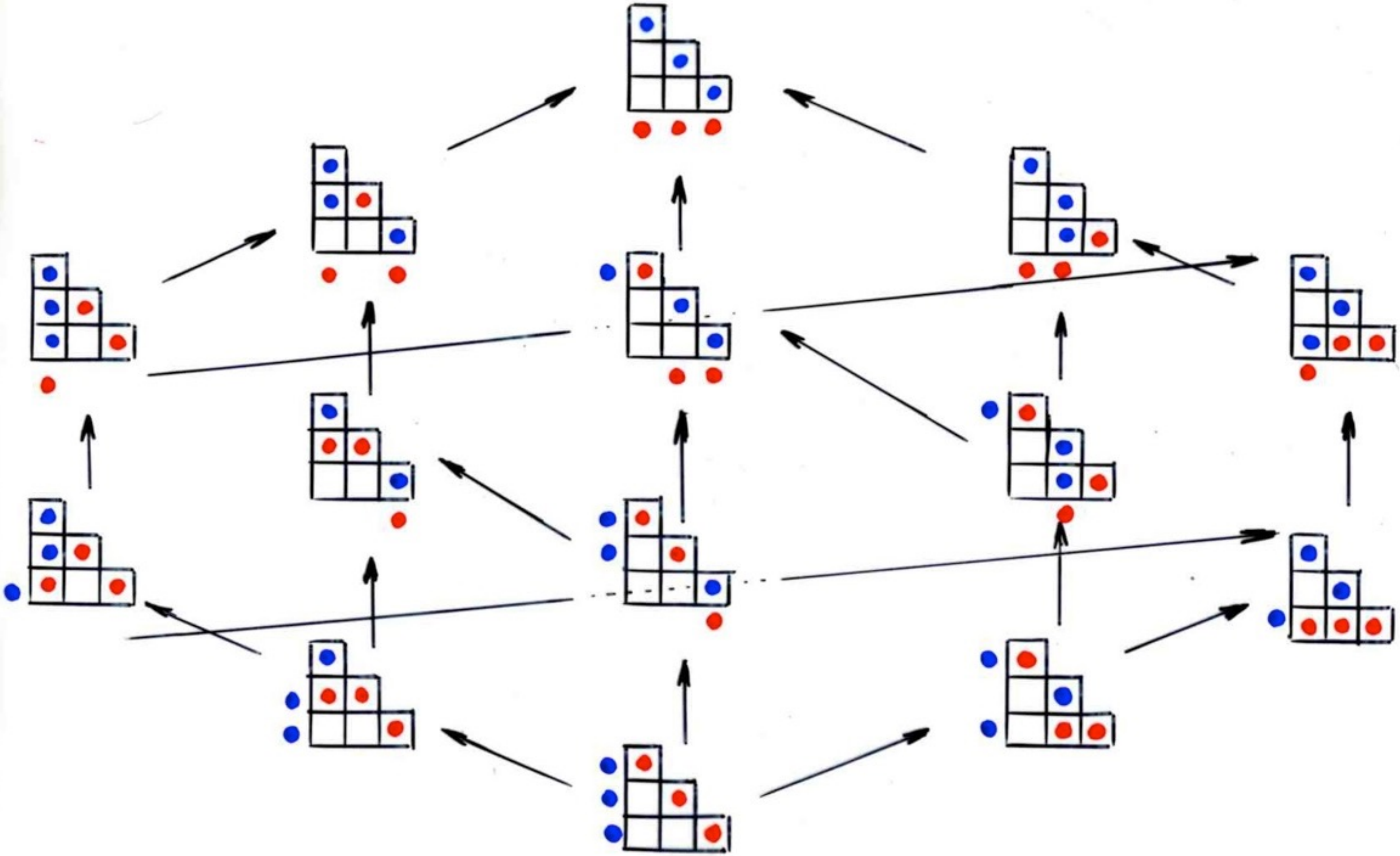


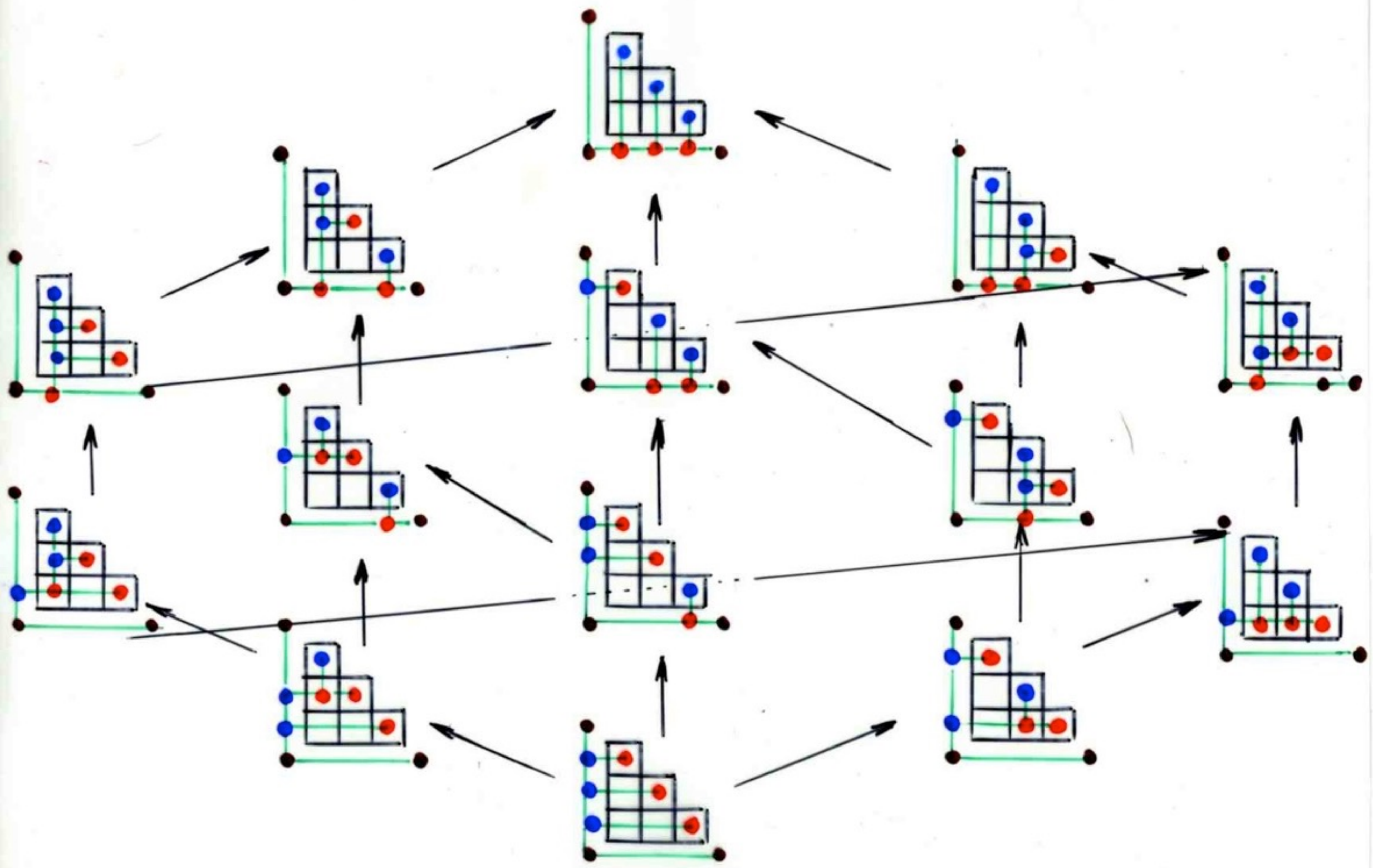
end of the proof
of the main theorem !

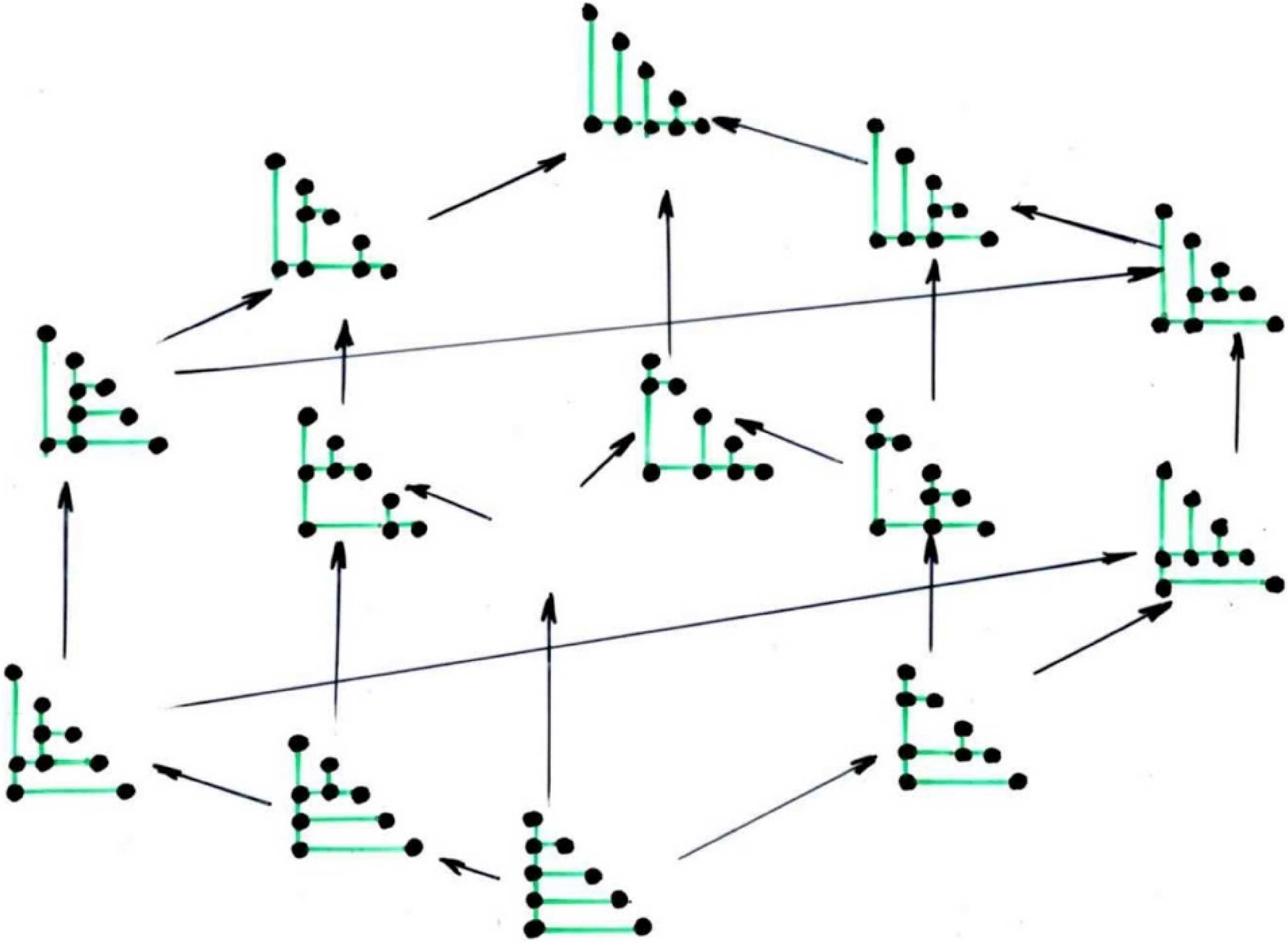
example

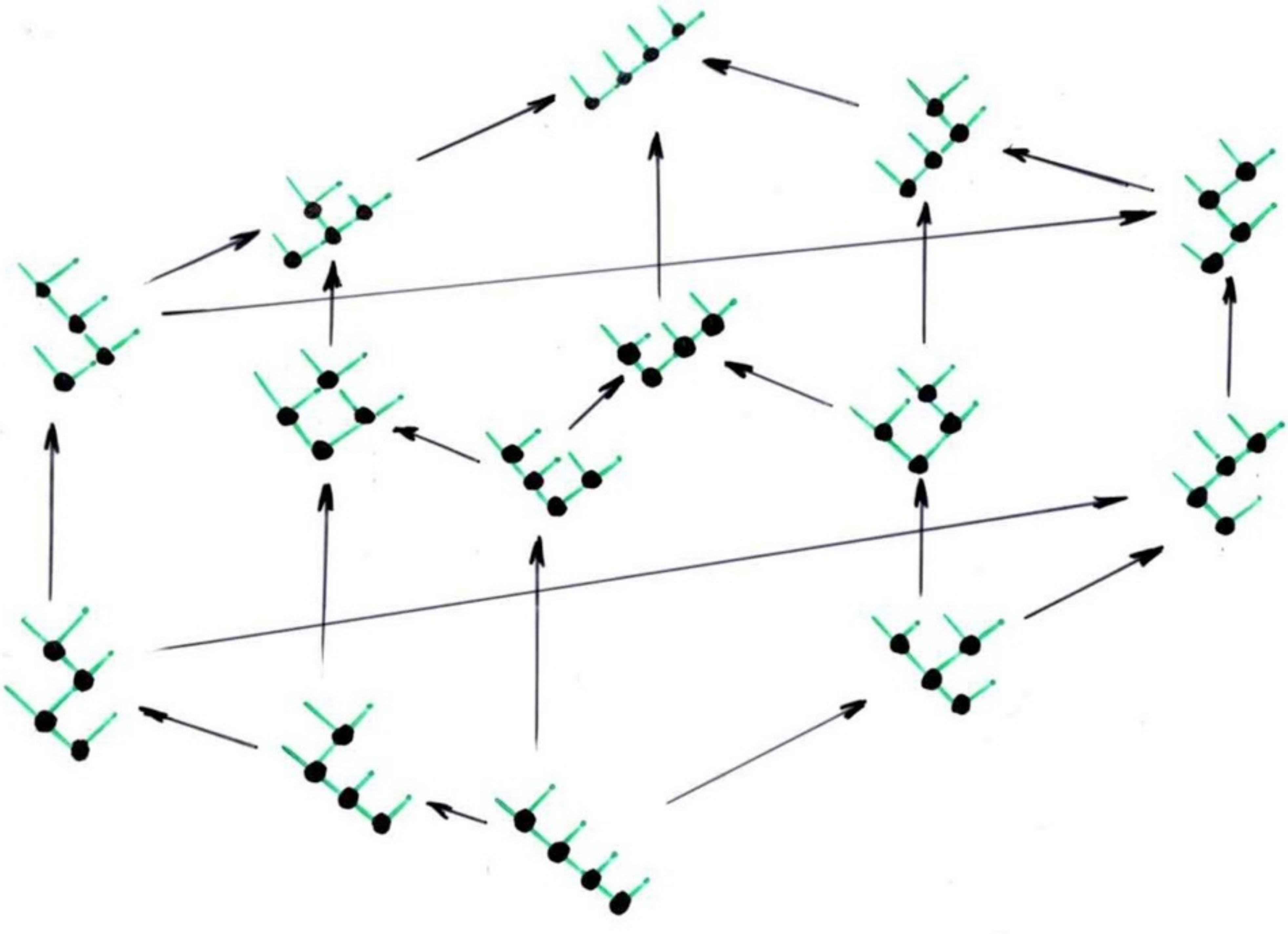








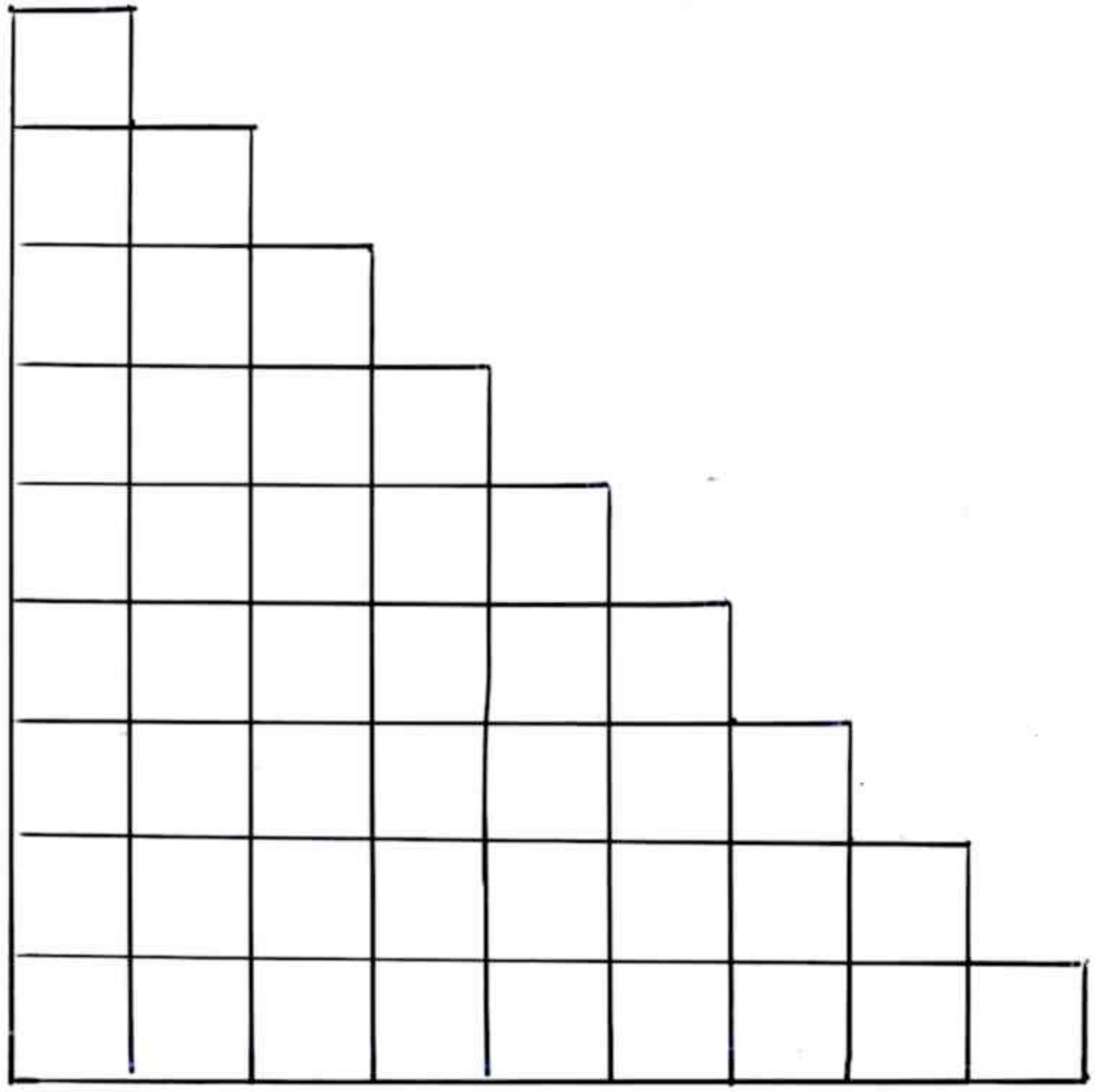


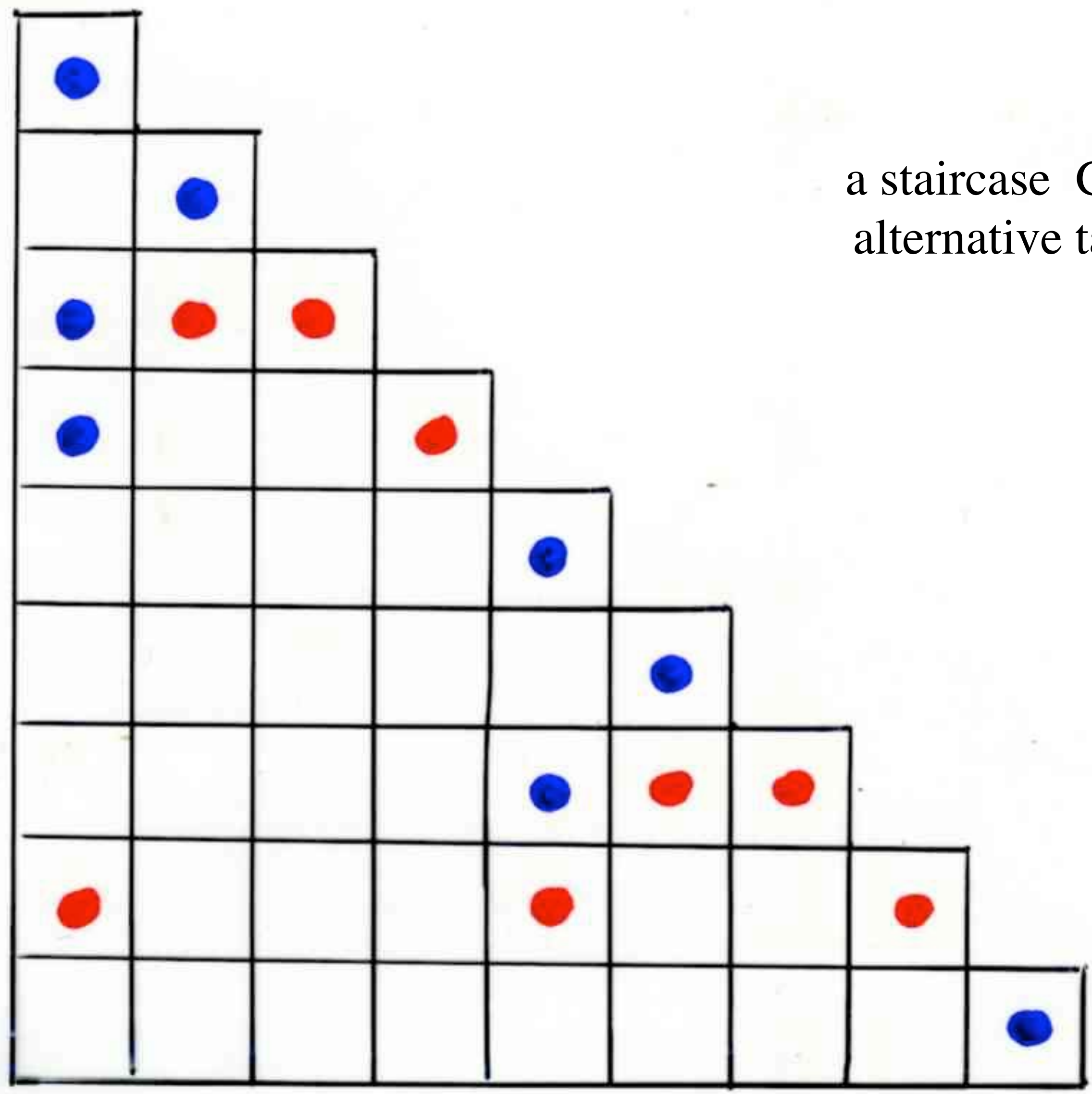


bijection

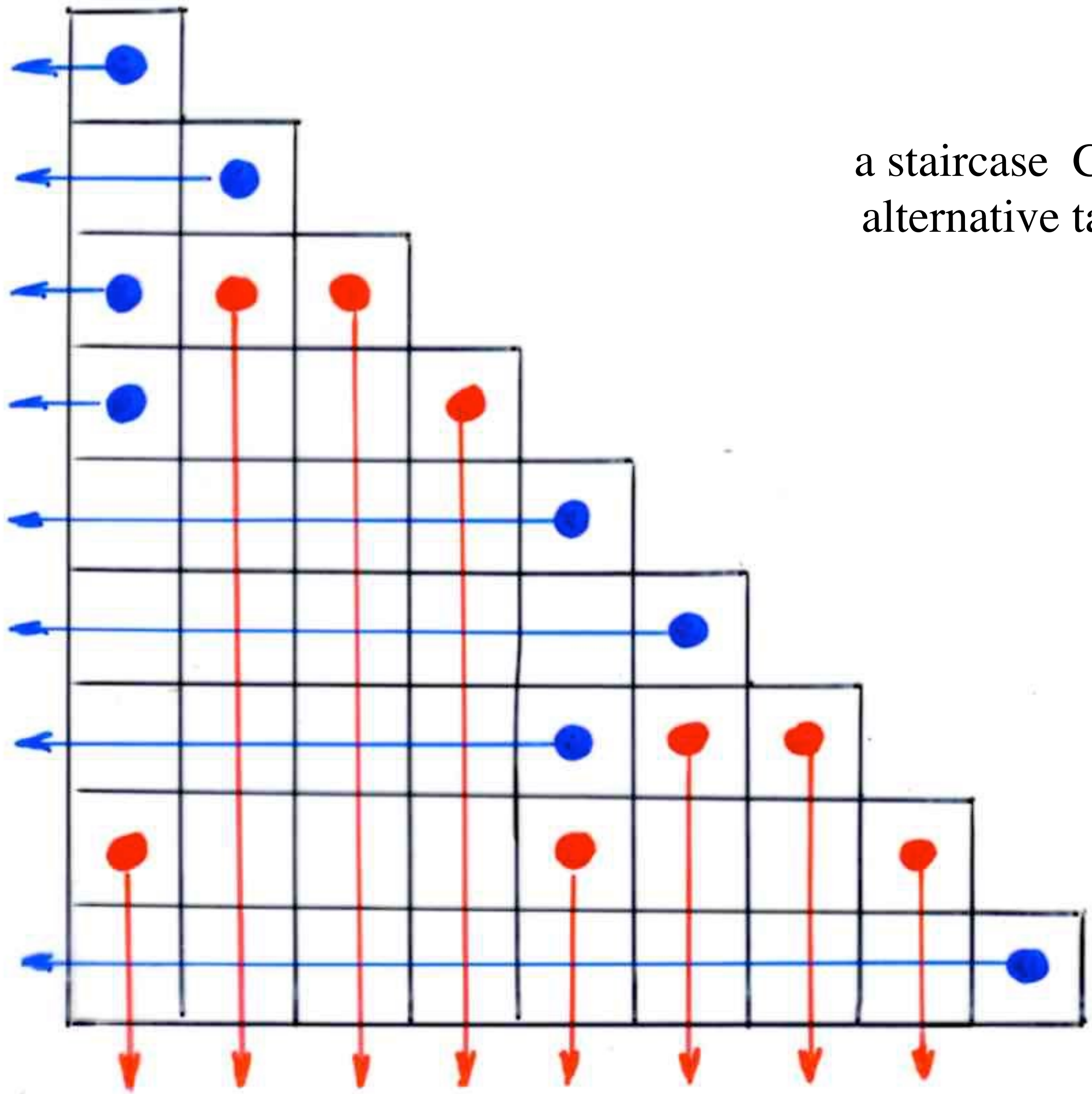
Catalan alternative tableaux

staircase Catalan alternative tableaux

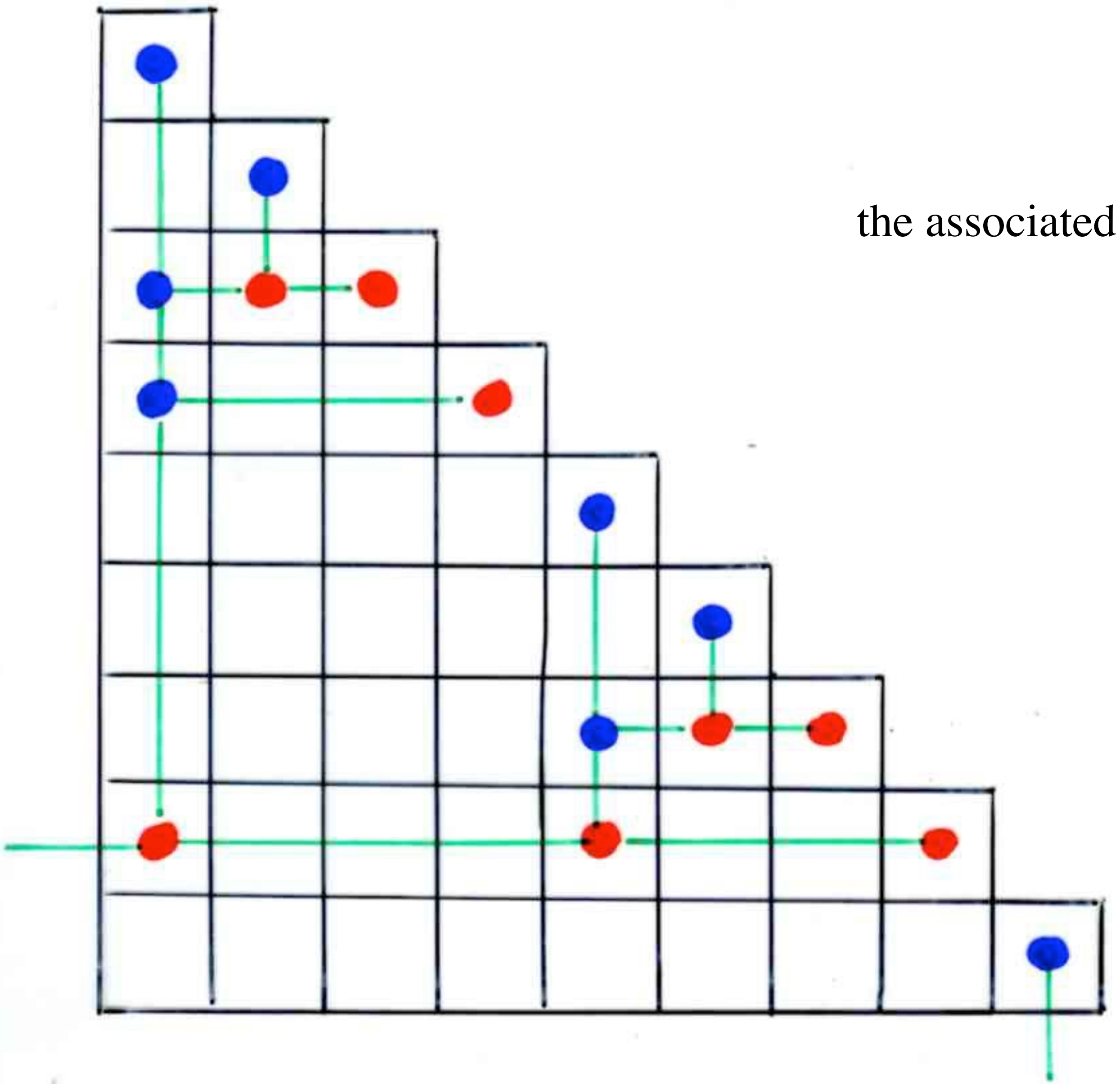




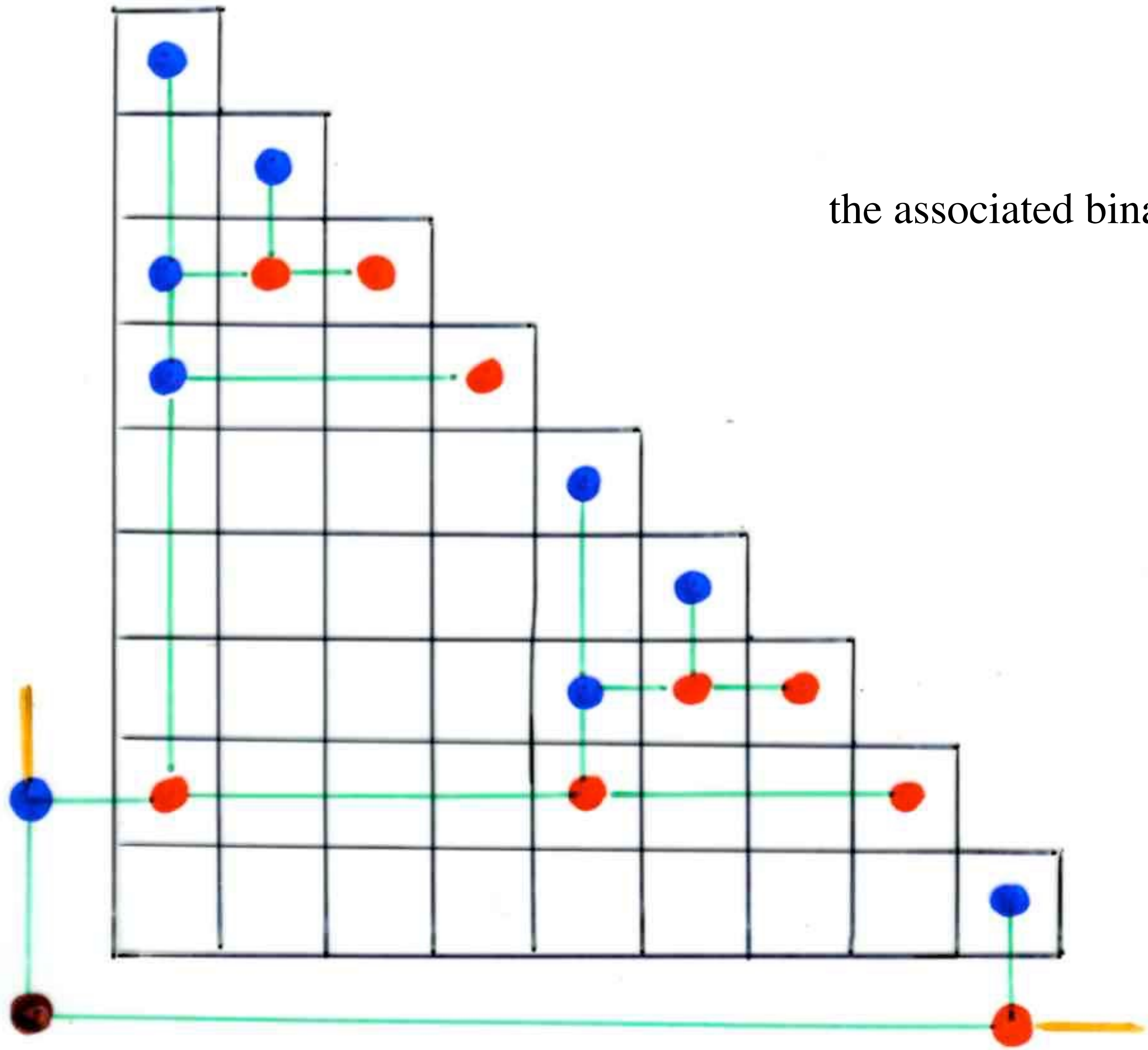
a staircase Catalan
alternative tableau



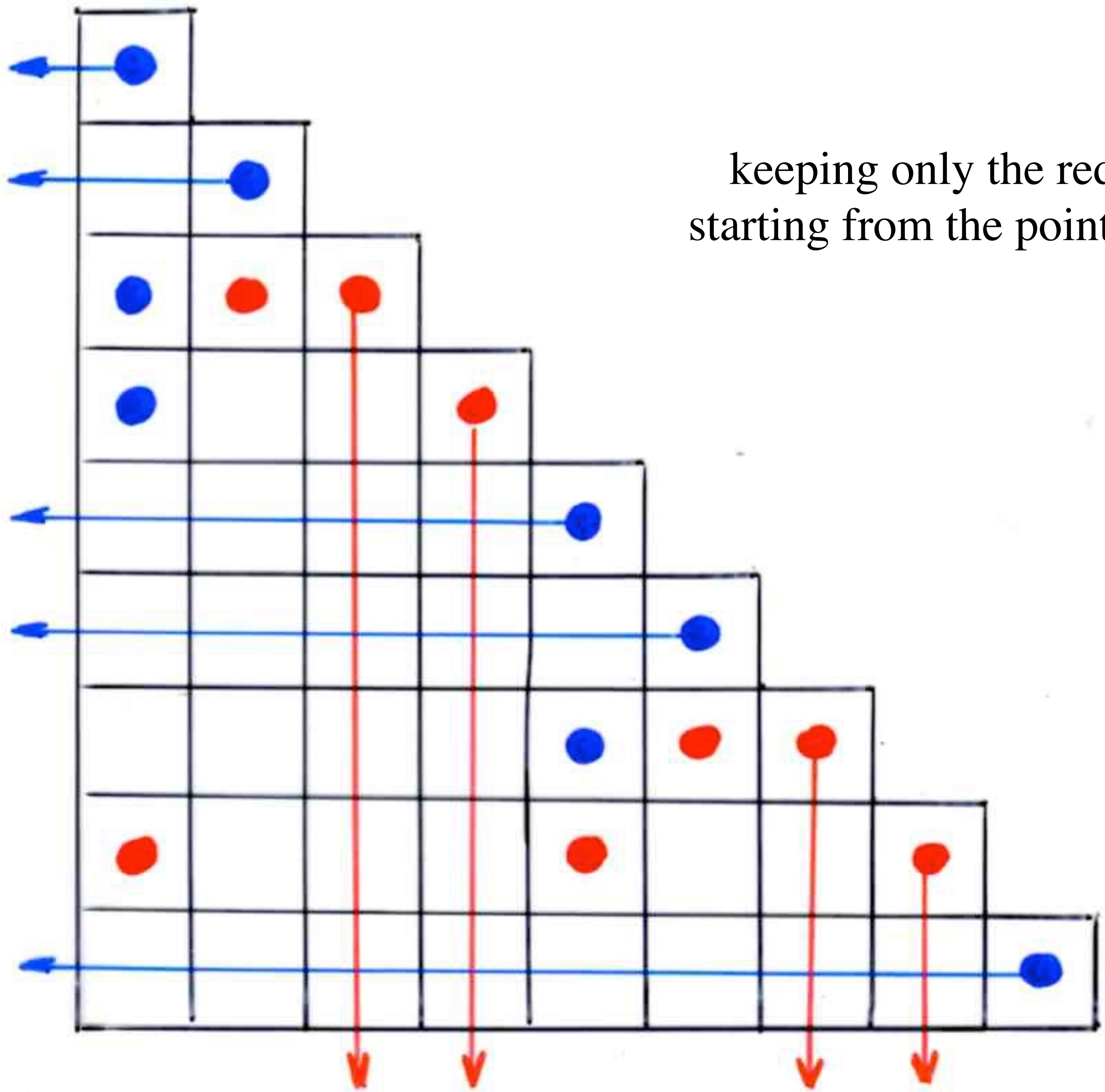
a staircase Catalan
alternative tableau



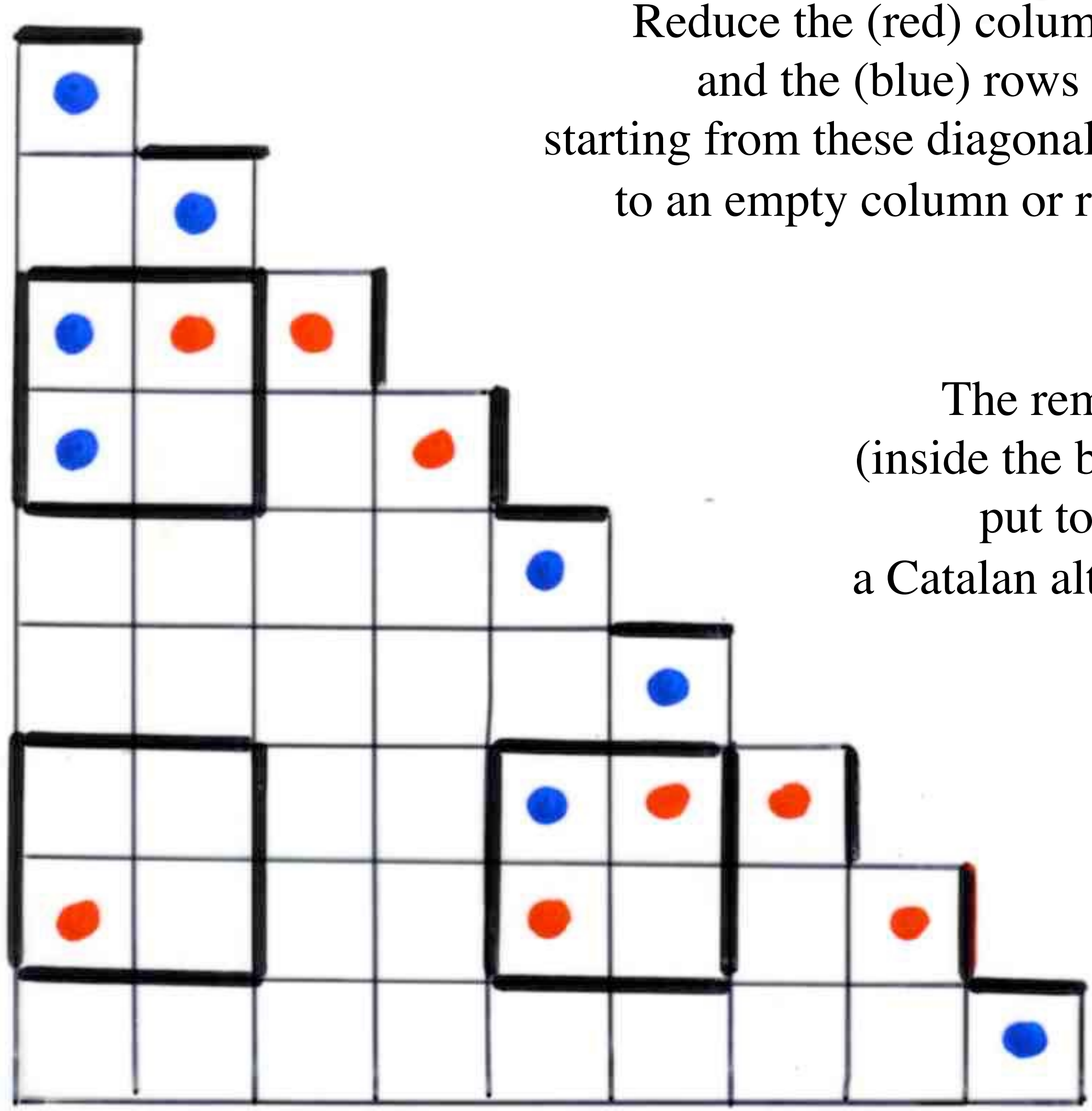
the associated binary tree



the associated binary tree

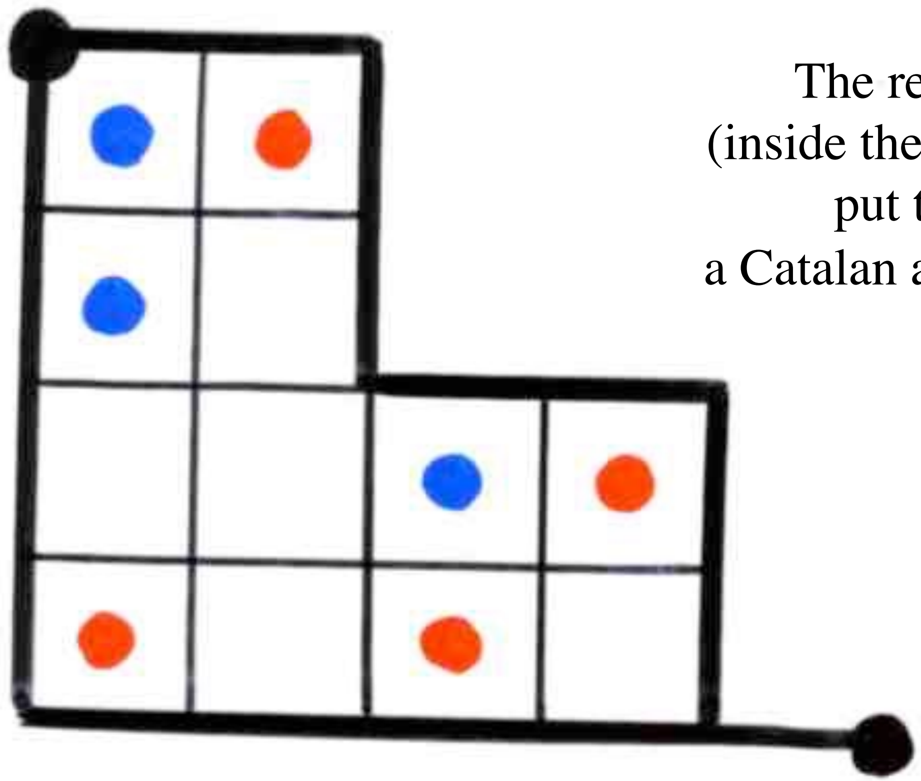


keeping only the red and blue lines
starting from the points on the diagonal

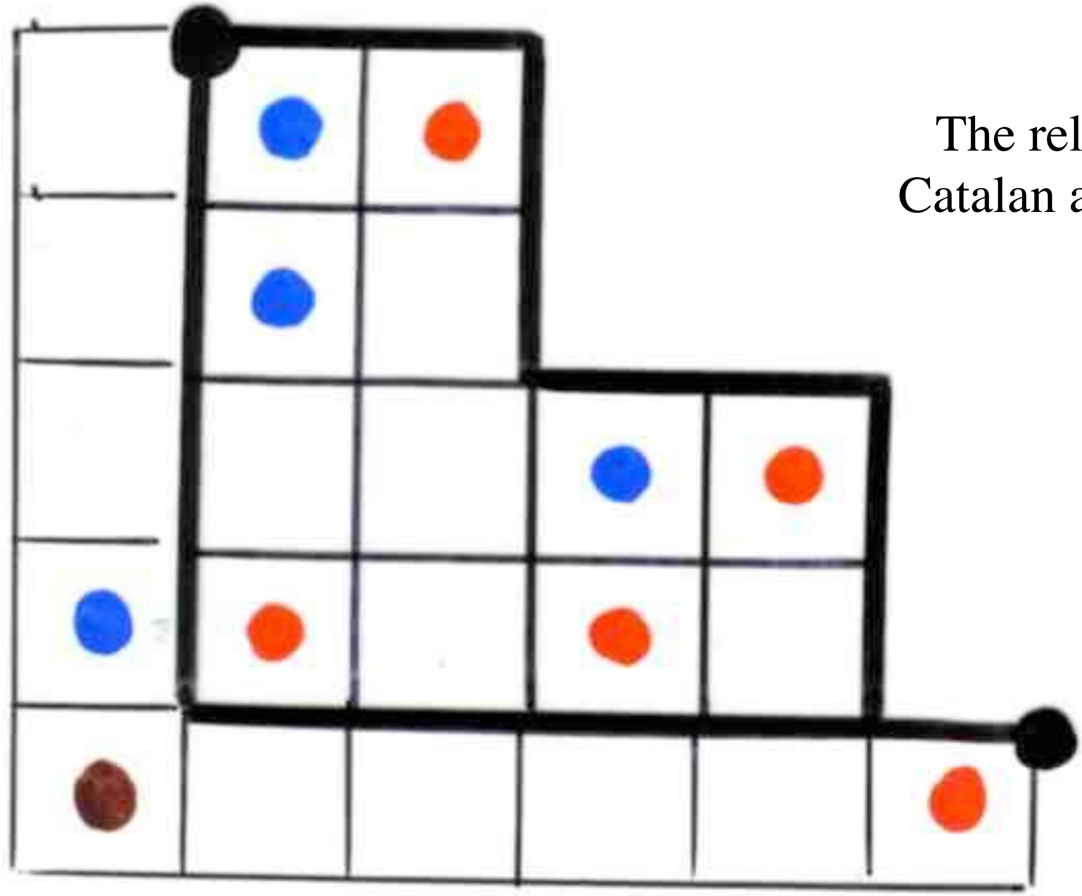


Reduce the (red) columns
and the (blue) rows
starting from these diagonal points
to an empty column or row.

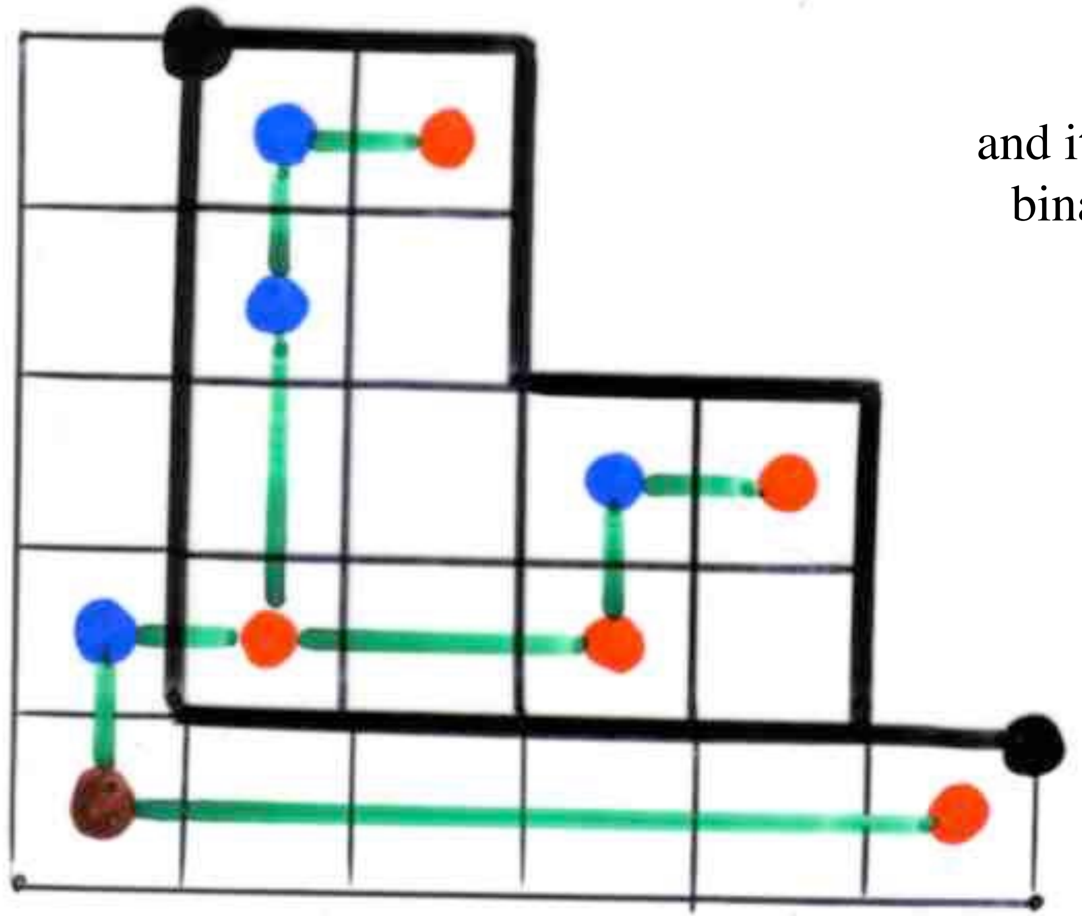
The remaining parts
(inside the black rectangles),
put together give
a Catalan alternative tableau.



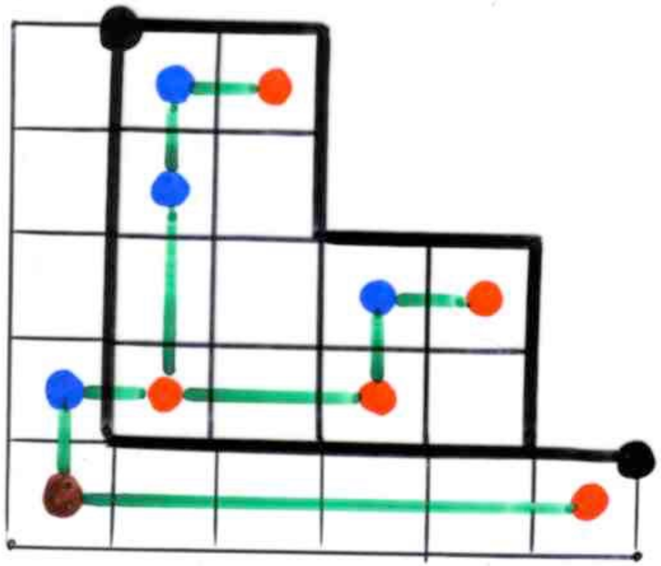
The remaining parts
(inside the black rectangles),
put together give
a Catalan alternative tableau.



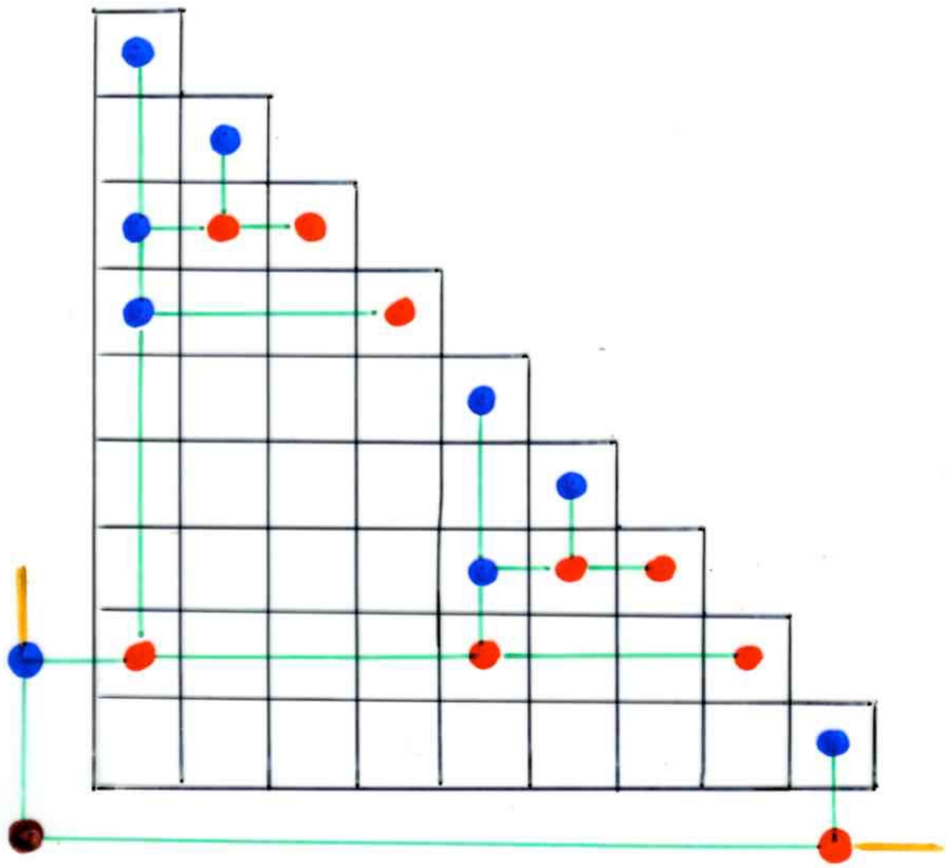
The related augmented
Catalan alternative tableau



and its associated
binary tree B.

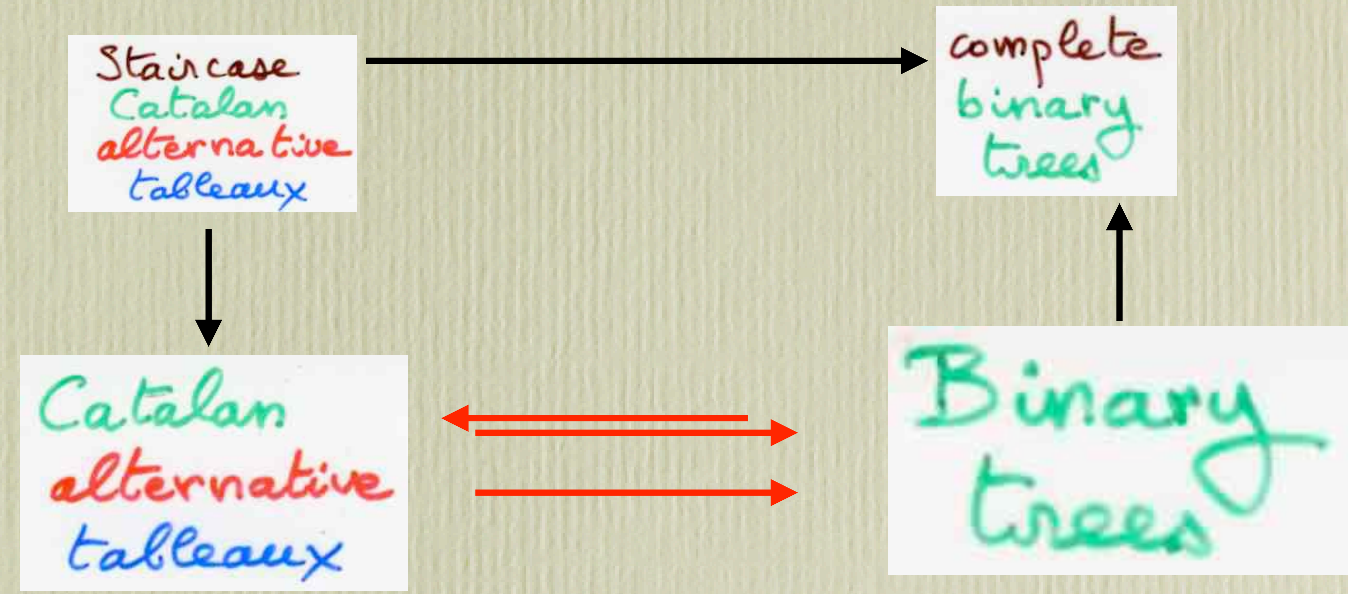


the associated
binary tree B.



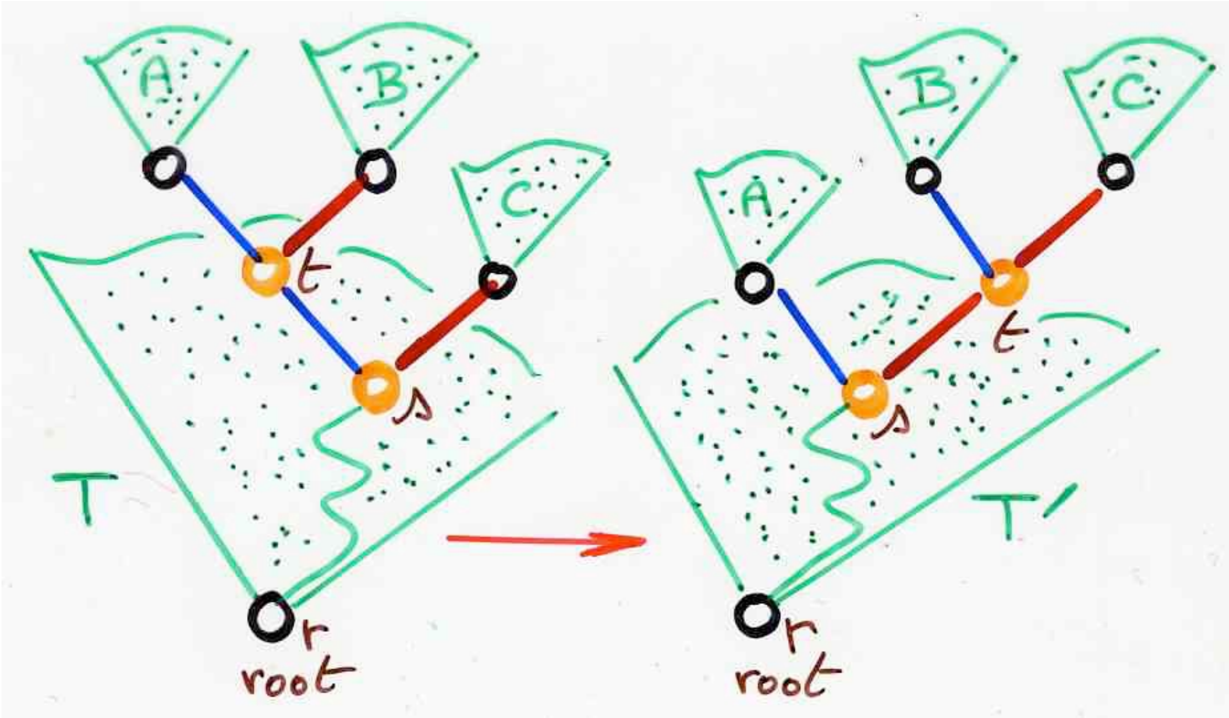
the binary tree associated to the staircase
Catalan alternative tableau
is the extension of the binary tree B

The canopy of B is the word in blue and red obtained by following downward the diagonal of the staircase Catalan alternative tableau.



commutative diagram !

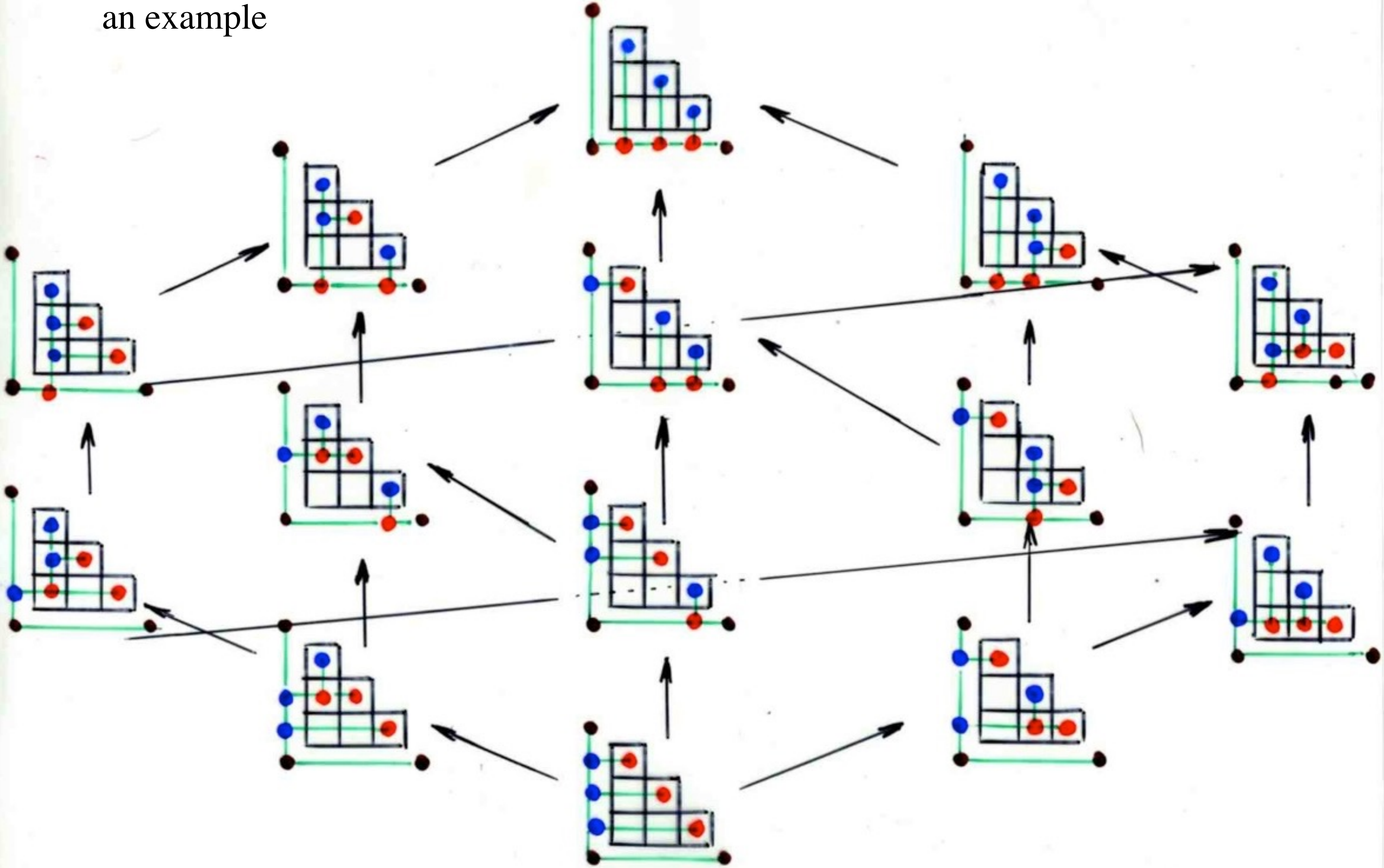
Canopy and rotation in binary trees



In the rotation the canopy of T is invariant if and only if the binary subtree B is not reduced to a single vertex. If B is reduced to a single vertex, the canopy of T' is deduced from the canopy of T by changing one edge to the right (red) into an edge to the right (blue).

In the associated Catalan staircase alternating tableau (see slide 153), this corresponds to a Γ-move where the rectangle is touching the diagonal.

an example



end of part I of the set of slides

new website
(in construction): www.viennot.org

old website: www.xavierviennot.org/xavier